1 Introduction

In the line of Topa (RES(2001)), Calvo-Armengol and Jackson (AER (2004), JET(2007)).

Topa is mostly applied, Calvo -Armengol and Jackson are theoretical. Both share the feature that, in the model, employment can be lost at an exogeneous rate, but can be found with probability that depends positively on the proximity of employed agents on a network.

Within this class of models, and under the assumption that a policy maker can directly affect such probabilities, we ask what is the optimal way to do so.

2 Calvo-Armengol and Jackson

The model of which in the AER paper is nested in that of the JET paper (see bottom of p. 30 in the JET paper for details).

2.1 AER paper

All jobs are identical

$N < \infty$ identical agents, located on a network (i.e. an arbitrary $N-$by-N matrix with entries $g_{ij} = g_{ji} = 1$ if agent i and agent j are connected and equal to zero ow)

Time is discrete:

at the beginning of each period, each agents hears of a job with probability a

if an agent is unemployed, and hears of a job, (s)he accepts the job, which (s)he can immediately loose wp b

if an agent is employed, and (s)he hears of a job, (s)he picks up at random one of her/his connected unemployed agents (with equal probability) and passes on the information (else, the information is lost). (S)he can loose the job wp b.

Let $s_i(t) = 1$ denote agent i being employed at time t and $s_i(t) = 0$ denote agent i being unemployed at time $\dot{t}$.

Let p_i be the expected number of offers that agent i gets. This is defined on p. 30 of the JET paper and on p. 429 of the AER paper as follows: let $p_{ij}(t)$ be the probability of the joint event that i hears of a job and this job ends up in agent j's hands, modeled as:

$$p_{ij}(t) = \begin{cases} a & \text{if } s_i(t) = 0 \text{ and } i = j \\ \frac{a}{\sum\limits_{k:s_k(t)=0} g_{ik}} & \text{if } s_i(t) = 1, s_j(t) = 0 \text{ and } g_{ij} = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $p_i(t) = \sum\limits_{j:\text{s.t. } g_{ij}=1} p_{ji}(t)$.

2.2 Remarks

1. My understanding is the in the JET paper an economy is a triple (g, a, b) with an initial (random) initial condition. The state-space is $s \in \{0, 1\}^N$ and transition probabilities are as follows:

$$\begin{aligned}\Pr[s_i(t+1) &= 0 \mid s_i(t) = 1] = b \\ \Pr[s_i(t+1) &= 1 \mid s_i(t) = 0] = p_i(t)(1-b)\end{aligned}$$

   where b is exogeneous and p_i depends on a and on $\dot{g}$.

   This stochastic process defines a finite Markov chain with time-dependent transition probabilities. Starting from $s = \mathbf{0}$ (all unemployed) transitions are defined by a product measure with parameter a. Starting from $s = \mathbf{1}$ (all employed) transitions are defined by a product measure with parameter b. Starting from any other s, transition probabilities are as above.

2. While the properties of strong and weak associations are easy to understand (I call them monotonicity), I need to understand the application of Freidlin and Wentzell techniques (why does the length of the period, T, come into play?)

3. Microfoundation: the only strategic element that is introduced in these models is to endogeneize the decision to drop out from the labour market (in the JET paper this is slightly more explicit). Let δ_i be the discount factor at which individuals discount future wages (In the JET model, the variables sketched above depend parametrically on wages) and c_i the expected discounted cost of staying in. It is assumed that this decision is taken by all agents independently, only once, and it is assumed that the underlying game is supermodular (i.e. that as more players decide to stay in, the decision to stay in is increasingly favoured). The focus is on the maximal equilibria.

4. It is clear that in this setup there are two forces through which an unemployed worker can get an offer: 1. the exogeneous a, which I think is at work in every period (or maybe only at the beginning of time - not sure) and 2. the neighbour's effect (the probability of getting a job is increasing (or not-decreasing) in the number of employed neighbours). Everybody looses the job with exogeneous probability b.

5. The whole model is close to being a reduced form model.

6. To look at policy implications, it is assumed that the policy maker can reduce the cost of staying in the labour market. It is then asked for which agents in the market, the policy maker should do so. This is dealt with by an example.

3 Our project

3.1 The model

A large, but finite number of agents, N, located on a one-dimensional lattice.

Interaction is local. Agent x's (nearest) neighbourhood consists of $\{x-1, x+1\} \equiv N(x)$

Agents can be in one of two states: Employed ($\eta(x) = 1$) or Unemployed ($\eta(x) = 0$)

An Employed agent becomes Unemployed at rate 1

An Unemployed agent becomes Employed at rate λ times the number of Employed agents in her/his neighbourhood. Flip rates are then:

$$
\begin{aligned}
c(\eta(x) &= 1 \to \eta(x) = 0) = 1 \\
c(\eta(x) &= 0 \to \eta(x) = 1) = \lambda \sum_{y \in N(x)} \eta(y)
\end{aligned}
$$

By construction, the only absorbing state for this process is when every agent is Unemployed ($\eta :\mid \eta \mid= 0$). Since the state space is finite, this latter state will be reached with probability one:

$$
\Pr[\lim_{t\to\infty} \mid \eta_t \mid = 0] = 1
$$

Let the process be started at $t = 0$ with all agents Employed ($\eta_{t=0} :\mid \eta_{t=0} \mid= N$) and let $\sigma_N = \inf\{t \geq 0 :\mid \eta_t \mid= 0\}$ be its absorption time.

Durrett et al (1988) (1989) show that this finite contact process shows phase transition (finite analogue to the phase transition of the infinite contact process studied by Liggett (1985) and others):

Theorem 1 (Durrett et al. (1988) and (1989)) *There exists a $\lambda_c \in (0, \infty)$ such that, in probability:*

$$
\begin{aligned}
\frac{\sigma_N}{\log N} &\to_{N\to\infty} f(\lambda) \text{ if } \lambda < \lambda_c \\
\frac{\log \sigma_N}{N} &\to_{N\to\infty} g(\lambda) \text{ if } \lambda > \lambda_c
\end{aligned}
$$

where $f(\lambda)$ and $g(\lambda)$ are deterministic functions.

Remark 2 (Intuition) *In the sub-critical case, the rate at which an agent becomes employed is weaker than the rate at which an agent becomes unemployed. Hence the absorption time is essentially the time it takes for all agents to become unemployed. For ex. if $\lambda = 0$, σ_N is the max of N iid random variables with mean one and $\frac{\log \sigma_N}{\log N} \to 1$ in prob.*

In the super-critical case, instead the process is absorbed only when all agents become unemployed simultaneously, and this grows exponentially with N.

Consider a piecewise homogeneous contact process constructed as follows. Partitition the N agents into K contiguous groups, of relative size $\alpha_1, \alpha_2, ..., \alpha_K$ with $\sum_{i=1}^{K} \alpha_i = 1$. Within group i the process evolves with parameter λ_i. The process is then defined by the triple $(K, \{\alpha_i\}_{i=1,..,K}, \{\lambda_i\}_{i=1,..,K})$.

3.2 The Question

We consider the optimization problem faced by a policy maker who aims at maximizing the absorption time (i.e. maximize the 'longevity' of employment), by distributing a fixed amount of resources which affects directly the flip rates among agents.

Since employment survives much longer when the process is supercritical (i.e. $\lambda > \lambda_c$), ideally the policy maker would like to allocate resources in such a way to make the whole process supercritical. However, the budget constraint could be tight and the constrained optimal choice may involve a trade-off between making a small part of the population (highly) supercritical and possibly spilling over to the contiguous parts, or distributing resourses evenly so that all rates are increased by a smaller amount.

Problem 3 *Let $\lambda_0 \geq 0$ be the nominal rate given to each point and $\rho \geq 0$ the additional rate to be distributed. The problem the policy maker faces is:*

$$\max_{(K,\lambda,\alpha)} \lim \inf_{N\to\infty} \left(\frac{\log E[\sigma_N]}{N}\right)$$

$$\begin{aligned} \textit{subject to } \sum_{i=1}^{K} \alpha_i \lambda_i &\leq \lambda_0 + \rho \\ \lambda_i &\geq \lambda_0 \quad \forall i = 1, ..., K \end{aligned}$$

3.3 Conjectured Answer

Estimating the rates of the piece-wise homogeneous process should be feasible and probably intuitive.

If ρ is large and/or λ_0 is above the threshold, the concavity of the log implies that the additional rate has the greatest effect when it is spread uniformly over the population.

In any other situation the trade-off between the size and the rate of the supercritical partition becomes significant and it could be optimal to invest the resources to make one small part highly supercritical and leave the remaining parts subcritical.

3.4 Remarks

- On the unfortunate trap

This model has the feature that, in the finite case, it admits only one absorbing state, where all agents are unemployed. An alternative specification is the following:

$$\begin{aligned} c(\eta(x) &= 1 \to \eta(x) = 0) = \sum_{y \in N(x)} (1 - \eta(y)) \\ c(\eta(x) &= 0 \to \eta(x) = 1) = \lambda \sum_{y \in N(x)} \eta(y) \end{aligned}$$

Here the probability of loosing a job is not exogeneously given, but rather it depends on the state (specifically on the number of unemployed agents in one's neighbourhood). The asymptotics of this model are different (in that now also a configuration where everybody is employed is absorbing. However the dynamics, which essentially depends on λ is not so dissimilar.

For $\lambda = 1$, this model is the Voter model (Liggett (1985) et al. for the infinite population case and Cox (1989) for the finite case).

For $\lambda > 1$, this is a kind of biased Voter model. No references at hand, but I believe something is known about this model as well.

This types of processes show 'consensus' in the sense that agents tend to agree with each other. Technically this means that one can focus on the border measures (i.e. the way in which a border between a segment of employed and a segment of unemployed evolves).

- On ergodicity

  The model above is non ergodic, in the sense that it admits more than one invariant measure. In fact, also in the finite case, to make it ergodic, it is enough to perturb transitions in such a way as to guarantee that every flip can occur with positive probability. But characterizing analytically the invariant measure is typically non-feasible (unless these flip rates guarantee that the process is reversible). Also, it is not clear how to formulate the 'policy question': maximize the amount of time that the system spends in the all employed trap?