Single and Multi-Period Optimal Inventory Control Models with Risk-Averse Constraints

Dali Zhang and Huifu Xu
School of Mathematics
University of Southampton
Southampton, SO17 1BJ, UK

Yue Wu
School of Management
University of Southampton
Southampton, SO17 1BJ, UK

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Abstract

This paper presents some convex stochastic programming models for single and multi-period inventory control problems where the market demand is random and order quantities need to be decided before demand is realized. Both models minimize the expected losses subject to risk aversion constraints expressed through Value at Risk (VaR) and Conditional Value at Risk (CVaR) as risk measures. A sample average approximation method is proposed for solving the models and convergence analysis of optimal solutions of the sample average approximation problem is presented. Finally, some numerical examples are given to illustrate the convergence of the algorithm.

Keywords: Inventory control; Conditional Value at Risk constraints; Sample average approximation; Stochastic programming; Convex programming

1 Introduction

Inventory control is one of the main subjects in supply chain area. Inventory control models are almost invariably stochastic optimization problems with objectives being either expected costs or expected profits or risks. In practice, a retailer may want to find an optimal decision which achieves a minimal expected cost or a maximal expected profit with low risk of deviating from the objective. In this paper, we present an inventory model that aims to maximize expected profit or minimize the expect cost subject to specified risk-averse constraints. Specifically, we consider an inventory system where demand is random and decision on inventory replenishment have to be made before the demand is realized. The objective is to minimize the expected loss and meanwhile restrict the risks of loss exceeding certain level to a specified level. Mathematically, this is a stochastic optimization problem with probabilistic constraints.

Before proceeding to a detailed discussion of the proposed model, we present some literature review on the related inventory models. Arrow, Harris and Marschak [3] and Dvoretzky, Keefer and Walfowitz [11] first consider single period inventory control problem and propose a well known newsvendor model. A popular generalization of this model is to consider a multi-period supply chain problem where the selling horizon is extended from one period to multiple periods, and a decision on order quantity in each period is made before the demand is realized. The key difference between single-period model and multi-period model is that the multi-period model may involve stock leftovers from previous periods, which makes the optimal choice of order quantities more complicated. The well known (s, S) policy is consequently proposed, where an order is placed to bring the inventory level up to S when and only when its inventory level falls below the level s. Using a dynamic programming approach, Scarf [22] shows that (s, S) policy is optimal for finite horizon dynamic inventory systems when order cost function is linear and holding cost function is convex. Further along the direction, Song and Zipkin [26], Chen and Song [8] model demand level as a state of a continuous Markov chain, and show that state-dependent (s, S) policies are optimal for a multi-period problem under a fluctuating demand environment.

Most models in the above literature consider policies which aim at minimizing the expected losses. Due to the lack of the risk-aversion measure, an optimal inventory control policy resulting from these models may incur significant losses with a positive probability. To alleviate or avoid such risks, one may incorporate some risk measures as constraints in an inventory model, and this is one of the main objectives of this paper.

Risk management and its measure are prevailing topics in finance and economics. These concepts are first introduced by Markowitz [12], where the variance of random returns or losses is used as a measure of risks. Another risk measure, which is popular in financial industry, is so called Value at Risk (VaR). VaR can be traced back as early as 1920s and it has been extensively used in analysis of portfolio optimization. See recent work by Jorion [14], Basak and Shapiro [7] and the references therein. Although VaR has been widely used as a standard benchmark for risk measure, it lacks some important mathematical properties such as subadditivity and convexity, which are important to the development of numerical methods and regarded as coherent properties by Artzner, Delbaen, Eber and Heath [5]. Consequently, a coherent risk measure, Conditional Value at Risk (CVaR), which is defined as the expected value of of tail distributions of returns or losses, is proposed. Rockafellar and Uryasev [19], Krokhmal, Palmquist and Uryasev [15] investigate CVaR models and reformulate them as convex optimization problems. More recently, Alexander, Coleman and Li [2] apply the CVaR model to a portfolio optimization problem and solve it by using a Monte Carlo method.

Risk management is not a new topic in inventory control either. Over the past few years, a number of inventory models have been proposed to deal with highly uncertain demand and fluctuating factors of environment. For instance, Tapiero [27] considers a classic inventory model which minimizes VaR. Choi, Li and Yan [9] propose a mean-variance risk measure for a supplier-retailer chain problem; Jammernegg and Kischka [13] consider CVaR in a newsvendor problem. More recently, Ahmed, Cakmak and Shapiro [1] investigate a coherent risk measure for both a single period newsvendor problem and a multi-period inventory control problem. The objectives of all these models are to minimize the risks during a whole selling horizon without considering the expected losses.

In this paper, we present models which include both expected losses and risk-averse. We start with a single period model and develop it into a multi-period case. The objective functions of the models are to minimize the expected losses/costs under the constraint that the risk of the losses/costs is controlled within a specified level. The rational behind this is that in some practical industrial problems, warehouses and retailers often seek a strategy that minimizes the expected losses with a small risk of excessive losses.

As far as we are concerned, the main contributions of this paper can be summarized as follows. We propose an inventory model that addresses both expected values (losses, costs) and risk-aversion. We show that the model can be reformulated as a convex stochastic programming problem. We propose a sample average approximation method to solve the model and show that, with probability 1, an accumulation point of an approximate solution converges to its true counterpart as sample size increases.

The rest of the paper is organized as follows. In Section 2, we introduce our model for a single-period inventory control problem. In Section 3, we discuss the model for a multi-period problem. In Section 4, we propose sample average approximation to solve the proposed model and analyze convergence. Finally, in Section 5, we report some preliminary numerical test results.

2 The single period case

2.1 Introduction to a newsvendor model

We start our discussion by considering a single period inventory control problem, where a retailer sells a seasonal/fashional product. At the beginning of a selling season, the retailer has to make a decision on its order quantity u before market demand is observed. We assume that the order is delivered by some external supplier without any delay.

The market demand is assumed to be a random variable denoted by $\xi(\omega)$, where $\xi : \Omega \to \Xi \subset \mathbb{R}$ is defined on probability space $(\Omega, \mathcal{F}, P)$ with a density function $p(\cdot)$ and a cumulative distribution function $P(\cdot)$. Because market demand does not go to infinity or negative in practice, we assume that the support set of ξ is $[0, \bar{d}]$. Note that in our model, we do not assume that $p(\cdot)$ or $P(\cdot)$ are analytically obtainable, instead we assume that a sample of $\xi(\omega)$ can be obtained from time to time.

Let s denote the unit selling price in the market, c denote the unit purchase cost, and v denote the net salvage value for each unit of leftover, which are all constant. Here, we assume that $v < c < s$. Let $\pi(u, \xi)$ be the total profit when the demand is ξ. Since the total purchase cost is cu, the quantity sold is $\min\{u, \xi\}$ and the leftover quantity is $(u - \min\{u, \xi\})$, the retailer's profit at demand scenario ξ is

$$\pi(u, \xi) = s \min\{u, \xi\} + v(u - \min\{u, \xi\}) - cu.$$

Our focus here is on loss, therefore we consider $J(u, \xi) := -\pi(u, \xi)$. In the case when $J(u, \xi) \leq 0$, the retailer obtains a profit of $-J(u, \xi)$. The loss function $J(u, \xi)$ can be rewritten as

$$J(u, \xi) = (s - v)(u - \xi)_+ - (s - c)u,$$

where let us use $(z)_+$ denote the max-function $\max\{0, z\}$.

The retailer's decision problem is to choose an optimal order quantity u to minimize the expected loss before the realization of market demand. If the retailer is risk neutral, then the corresponding optimization problem is

$$\min_{u \in \mathcal{U}} E[J(u, \xi)], \tag{2.1}$$

where $\mathcal{U} = [0, \bar{u}]$ and $\bar{u}$ is the maximum order quantity.

However, if the retailer is risk-averse, then he may be concerned that the actual realization of $J(u, \xi)$ in the model might significantly exceed this expected value with some positive probability and consequently the retailer may want the probability of $J(u, \xi)$ exceeding certain level γ to fall below θ, that is,

$$P\left(J(u, \xi(\omega)) > \gamma\right) \leq \theta, \tag{2.2}$$

where $\theta \in (0, 1)$ is usually chosen to be small. Consequently, we may add the condition (2.2) to (2.1) as a constraint of a risk aversion measure and we have

$$\begin{array}{ll} \min\limits_{u \in \mathcal{U}} & E[J(u, \xi(\omega))] \\ \text{s.t.} & P\left(J(u, \xi(\omega)) > \gamma\right) \leq \theta. \end{array} \tag{2.3}$$

By adjusting the values of γ and θ, the retailer can balance the expected return and risk aversion. Let $\beta = 1 - \theta$. The probabilistic constraint in (2.3) can be rewritten as $P\left(J(u, \xi) \leq \gamma\right) \geq \beta$ and hence (2.3) can be written as

$$\begin{array}{ll} \min\limits_{u \in \mathcal{U}} & E[J(u, \xi)] \\ \text{s.t.} & P\left(J(u, \xi) \leq \gamma\right) \geq \beta. \end{array} \tag{2.4}$$

In this model, the retailer's decision is to choose an optimal order quantity u which minimizes the expected loss with a confidence of β that the loss does not exceed level γ. In what follows, we simplify the chance constraint for the case when the cumulative distribution function $P(\cdot)$ of ξ is obtainable.

Proposition 2.1 *Let P^{-1} be the inverse of P. Then the constraint in (2.4) can be reformulated as*

$$\frac{-\gamma}{s-c} \le u \le \frac{\gamma + (s-v)P^{-1}(1-\beta)}{c-v},$$

and consequently (2.4) can be written as

$$\begin{array}{ll} \min\limits_{u\in\mathcal{U}} & E[J(u,\xi)] \\ \text{s.t.} & \frac{-\gamma}{s-c} \le u \le \frac{\gamma+(s-v)P^{-1}(1-\beta)}{c-v}. \end{array} \tag{2.5}$$

We omit a proof as it is elementary. The proposition indicates that if P^{-1} is computable, then we can simplify problem (2.4) to a one stage stochastic programming problem with a deterministic linear constraint. This reformulation can be extended to the case when ξ has a discrete distribution and hence $P(\cdot)$ is a step function. To see this, let

$$P^{-1}(\alpha) := \arg\min_{x\in\mathbb{R}} \{P(\xi(\omega) < x) \ge \alpha\}.$$

It is easy to verify that Proposition 2.1 still holds.

We give a simple example to illustrate Proposition 2.1.

Example 2.1 *Consider the case that the demand ξ has a uniform distribution with its support set $[\underline{d}, \bar{d}]$. If a retailer knows the distribution, then its order quantity will not fall below $\underline{d}$ or exceed $\bar{d}$. Therefore we may set $\mathcal{U} = [\underline{d}, \bar{d}]$. With this distribution, the expected loss can be written as*

$$\begin{aligned} E[(s-v)(u-\xi)_+ - (s-c)u] &= \int_{\underline{d}}^{u} \{(s-v)(u-\xi)\} \frac{1}{\bar{d}-\underline{d}} d\xi - (s-c)u \\ &= \frac{1}{2(\bar{d}-\underline{d})}(s-v)(u-\underline{d})^2 - (s-c)u. \end{aligned}$$

We then obtain a deterministic optimization problem

$$\begin{array}{ll} \min\limits_{u\in[\underline{d},\bar{d}]} & \dfrac{1}{2(\bar{d}-\underline{d})}(s-v)(u-\underline{d})^2 - (s-c)u \\ s.t. & \dfrac{-\gamma}{s-c} \le u \le \dfrac{\gamma + (s-v)P^{-1}(1-\beta)}{c-v}. \end{array}$$

The optimal solution to the problem is

$$u^*_{CCP} = \max\left(\frac{-\gamma}{s-c}, \min\left(\frac{(c-v)\underline{d} + (s-c)\bar{d}}{s-v}, \frac{\gamma + (s-v)[\underline{d} + (\bar{d}-\underline{d})(1-\beta)]}{c-v}\right)\right).$$

In what follows, we further discuss the risk aversion constraint. For a given parameter $\alpha \le \gamma$ and decision variable u, the scenarios of the losses $J(u,\xi)$ can be divided into two sets:

$$S_1(u,\alpha) := \{\omega | J(u,\xi) \le \alpha\}$$

which is that the losses falls below level α and

$$S_2(u,\alpha) := \{\omega | J(u,\xi) > \alpha\}$$

which corresponds to the losses exceeding the level α. If the probability measure of $S_1(u,\gamma)$ is greater than or equal to β, then the constraint in (2.4) is satisfied. Let

$$\Gamma(u,\alpha) := P\left(J(u,\xi(\omega)) \leq \alpha\right) = \int_{J(u,\xi)\leq\alpha} p(\xi) d\xi.$$

Since

$$S_1(u,\alpha_1) \subset S_1(u,\alpha_2), \qquad \text{for } \alpha_1 \leq \alpha_2, \quad u \in \mathcal{U},$$

we have

$$\Gamma(u,\alpha_1) \leq \Gamma(u,\alpha_2), \qquad \text{for } \alpha_1 \leq \alpha_2, \quad u \in \mathcal{U}.$$

Therefore, if u is a feasible solution to (2.4), then

$$\Gamma(u,\alpha^+) \geq \Gamma(u,\gamma) \geq \beta, \quad \forall \alpha^+ \geq \gamma, \quad \forall u \in \mathcal{U}.$$

Define

$$\alpha_\beta(J(u,\xi)) := \inf\{\alpha \in \mathbb{R} : \Gamma(u,\alpha) \geq \beta\}, \tag{2.6}$$

where the parameter β is a prescribed confidence level . Then, the constraint in (2.4) is equivalent to $\alpha_\beta(J(u,\xi)) \leq \gamma$. Furthermore, because the order quantity u is the only decision variable, we use $\alpha_\beta(u)$ as an abbreviation of $\alpha_\beta(J(u,\xi))$. By the monotonicity and right-continuity of $\Gamma(u,\alpha)$ in α, (2.6) can be written as

$$\alpha_\beta(u) := \min\{\alpha \in \mathbb{R} : \Gamma(u,\alpha) \geq \beta\}.$$

This leads to the following optimization problem,

$$\begin{aligned} \min_{u\in\mathcal{U}} \quad & E[J(u,\xi)] \\ \text{s.t.} \quad & \alpha_\beta(u) \leq \gamma. \end{aligned} \tag{2.7}$$

The quantity $\alpha_\beta(J(u,\xi))$ is known as the Value at Risk (VaR) in the literature of financial risk management ([19, 20]).

2.2 A single-period model with CVaR constraint

Model (2.7) achieves our goals except that it is not a convex program in that $\alpha_\beta(u)$ is not a convex function in general. This is a significant disadvantage from numerical persepective because finding an optimal solution to a nonconvex program could be very difficult. Moreover, for some special cumulative distribution functions of $\xi(\omega)$, the equation, $\Gamma(u,\xi) = \beta$, may have more than one solution, for some $\beta \in (0, 1)$ and this might result in discontinuity of $\alpha_\beta(u)$: a jump may occur with a slight increase of α. These disadvantages motivate us to introduce a better measure of risk, that is, the conditional Value at Risk measure. In what follows, we develop this idea from CVaR. Let $\beta \in (0, 1)$. Define

$$\phi_\beta(J(u,\xi)) := [\text{expectation of the } \beta\text{-tail distribution of } J(u,\xi)].$$

Mathematically,

$$\phi_\beta(J(u,\xi)) = \frac{1}{1-\beta}\int_{J(u,\xi)\geq\alpha_\beta(u)} J(u,\xi)p(\xi)d\xi. \tag{2.8}$$

In this paper, we will use $\phi_\beta(u)$ to denote $\phi_\beta(J(u,\xi))$ for the simplicity of notation. From (2.8), it can be easily verified that $\phi_\beta(u)$ is bounded. Moreover,

$$\phi_\beta(u) \quad \geq \quad \frac{1}{1-\beta}\int_{J(u,\xi)\geq\alpha_\beta(u)} \alpha_\beta(u)p(\xi)d\xi = \alpha_\beta(u).$$

Therefore, $\{u|\phi_\beta(u) \leq \gamma\}$ is a subset of $\{u|\alpha_\beta(u) \leq \gamma\}$. Based on the risk measure $\phi_\beta(u)$, we introduce a new inventory optimization model,

$$\begin{aligned} \min_{u\in\mathcal{U}} \quad & E[J(u,\xi)] \\ \text{s.t.} \quad & \phi_\beta(u) \leq \mu, \end{aligned} \tag{2.9}$$

where μ can be chosen to balance the effect from the inequality, $\phi_\beta(u) \geq \alpha_\beta(u)$.

Note that, the definition of risk measure $\phi_\beta(u)$ is the same as the definition of conditional Value at Risk in [19, 20]. The definition of CVaR is introduced for inventory management by Jammernegg and Kischka [13]. Ahmed, Cakmak and Shapiro[1] investigate risk optimization model with a CVaR objective function.

CVaR is defined as the expected value of $J(u,\xi)$ when $J(u,\xi) \geq \alpha_\beta(u)$. It has been proved in [20] that

$$\phi_\beta(u) = \min_{\alpha\in\mathbb{R}}\left(\alpha + \frac{1}{1-\beta}E[(J(u,\xi)-\alpha)_+]\right).$$

Let

$$F_\beta(u,\alpha) := \alpha + \frac{1}{1-\beta}E[(J(u,\xi)-\alpha)_+].$$

It can be easily verified that $F_\beta(u,\alpha)$ is convex and continuously differentiable with respect to u at almost every point. Consequently, we can reformulate (2.9) as

$$\begin{aligned} \min_{u\in\mathcal{U}} \quad & E[J(u,\xi)] \\ \text{s.t.} \quad & \min_{\alpha\in\mathbb{R}} F_\beta(u,\alpha) \leq \mu. \end{aligned} \tag{2.10}$$

In what follows, we discuss the feasible value of α. Because of boundedness of both the order quantity u and the demand ξ, the loss function $J(u,\xi)$ is also a bounded function on $[0,\bar{u}]\times[0,\bar{d}]$

and let us denote the ceiling of $J(u,\xi)$ by $\bar{J}$. For any α_1 and α_2, by assuming that $\alpha_1 > \alpha_2 \geq \bar{J}$, we have

$$F_\beta(u,\alpha_1) = \alpha_1 + \frac{1}{1-\beta} E[(J(u,\xi)-\alpha_1)_+] = \alpha_1 > F_\beta(u,\alpha_2) = \alpha_2 > \bar{J}.$$

$F_\beta(u,\alpha)$ is an increasing function in α on $(\bar{J},+\infty)$.

On the other hand, because $J(u,\xi)$ is bounded by $\underline{J} := -(s-c)\bar{u}$, then for any $\alpha^{'} \leq \underline{J}$ and fixed u, we have

$$F_\beta(u,\alpha') = \alpha' + \frac{1}{1-\beta} E[(J(u,\xi)-\alpha')_+] = E[J(u,\xi)],$$

which is a constant. So the problem of $\min_{\alpha\in\mathbb{R}} F_\beta(u,\alpha)$ is equivalent to $\min_{\alpha\in[\underline{J},\bar{J}]} F_\beta(u,\alpha)$. Let us take the set $\mathcal{A} := [\underline{J},\bar{J}]$ as the feasible set of α, which is a compact and convex set.

Note that (2.10) is a two-level optimization problem. This following proposition states that (2.10) can be reformulated as a one level optimization problem.

Proposition 2.2 *(2.9) is equivalent to*

$$\begin{aligned} \min_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \quad & E[J(u,\xi)] \\ \text{s.t.} \quad & F_\beta(u,\alpha) \leq \mu. \end{aligned} \tag{2.11}$$

in the sense that (u^*,α^*) *is an optimal solution of (2.11) if and only if* u^* *is an optimal solution of (2.9) and* $F_\beta(u^*,\alpha^*) = \phi_\beta(u^*)$.

We omit a proof because this is analogous to [15, Theorem 4] in the context of portfolio optimization.

3 The multi-period case

3.1 Introduction to a multi-period model

In this section, we extend our proposed model to the multi-period case. Assume that the selling season is a $\hat{t}$-period horizon. At the beginning of each period, $t \in \{1,\cdots,\hat{t}\}$, the retailer first observes its inventory level left from the last period, y_{t-1}, and then places an order to replenish the inventory level, where u_t denotes the order quantity. After the inventory is replenished to $y_{t-1}+u_t$, demand in the t-th period, ξ_t, is realized. As in the single period model, for each t, the demand ξ_t is assumed to take its value in $[0,\bar{d}]$. The inventory level in the t-th period can be expressed in a recursive way as

$$y_t = y_{t-1} + u_t - \xi_t. \tag{3.12}$$

Note that in the multi-period case, we *only* take the *cost* in each period into account as in [23, 26], which means that we do not include the profit from the selling in each period. Our problem can be regarded as a cost minimization problem with risk aversion constraints. As in [23, 26], we only consider the costs in the whole selling horizon. We use the following notations.

- $\{1, \cdots, \hat{t}\}$, the time horizon of the multi-period inventory problem,
- $\{t, \cdots, t'\}$, the horizon of the inventory problem from t-th period to t'-th period,
- u_t, the ordering units at period t,
- $\mathbf{u}_{t|t'} = \{u_t, u_{t+1}, \cdots, u_{t'}\}$, $1 \leq t < t' \leq \hat{t}$, the sequence of order quantities from t-th period to t'-th period,
- $p_t(\cdot)$, the probability density function (pdf) of ξ_t, where $\xi_t \leq \bar{d} < +\infty$, where $\bar{d}$ is the upper bound of possible demand,
- $P_t(\cdot)$, the cumulative distribution function (cdf) corresponding to $p_t(\cdot)$,
- $\xi_{t|t'} = \{\xi_t, \xi_{t+1}, \cdots, \xi_{t'}\}$, $1 \leq t < t' \leq \hat{t}$, the sequence of demands from the t-th period to t'-th, with its conditional density function as $p_{t|t'}(\cdot|\xi_{1|t-1})$ and distribution function as $P_{t|t'}(\cdot|\xi_{1|t-1})$, where the "conditional" means that the trajectory of $\xi_{1|t-1}$ is realized,
- $c_t(u_t)$, the cost of ordering u_t units at period t,
- $f_t(u_t, \xi_t; y_{t-1})$, the holding cost for the leftover and the backorder penalty cost in period t.

Assumption 3.1 *For $t = 1, \cdots, \hat{t}$, $c_t(u_t)$ is convex and continuously differentiable. $f_t(u_t, \xi_t; y_{t-1})$ is locally Liptschitz continuous and convex with respect to u_t and y_{t-1}.*

Note that, in this model, we do not assume that for every $t < \hat{t}$, the demand ξ_{t+1} is independent of the demands at preceding periods, $\{\xi_1, \xi_2, \cdots, \xi_t\}$. Using the notations above, we can formulate the overall costs incurred in the time horizon $\{t, \cdots, \hat{t}\}$ as follows:

$$J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; y_{t-1}) := \sum_{i=t}^{\hat{t}} \left(c_i(u_i) + f_i(u_i, \xi_i; y_{i-1}) \right).$$

Let the initial inventory level y_0 be fixed. Because $y_t = y_{t-1} + u_t - \xi_t$, $J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; y_{t-1})$ essentially depends on $\xi_1, \ldots, \xi_{\hat{t}}$ and $u_1, \ldots, u_{\hat{t}}$, therefore we can write it as $J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$, where $\mathbf{u}_{1|t-1}, \xi_{1|t-1}$ and y_0 are treated as parameters. Moreover, we can derive the following recursive formula as

$$J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; y_{t-1}) = J_{t+1}(\mathbf{u}_{t+1|\hat{t}}, \xi_{t+1|\hat{t}}; y_{t-1} + u_t - \xi_t) + c_t(u_t) + f_t(u_t, \xi_t; y_{t-1}),$$

for $t \in \{1, 2, \ldots, \hat{t}\}$. Or equivalently,

$$J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) = J_{t+1}(\mathbf{u}_{t+1|\hat{t}}, \xi_{t+1|\hat{t}}; \mathbf{u}_{1|t}, \xi_{1|t}; y_0) + c_t(u_t) + f_t(u_t, \xi_t; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0),$$

where, based on the recursive form, $y_t = y_{t-1} + u_t - \xi_t$, function $f_t(u_t, \xi_t; y_{t-1})$ can be reformulated as $f_t(u_t, \xi_t; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$.

3.2 A minimal cost model with VaR constraints

We assume that, before the beginning of the first selling season, the retailer will make a decision on order quantities of all seasons in the selling horizon, denoted by $\mathbf{u}_{1|\hat{t}}$, and send them to the external suppliers or manufacturers. This is the case for many retailers in practice particularly when suppliers and or manufacturers are overseas who need to plan production schedules well before the start of the selling horizon because of long transportation time. In this model, the retailer's aim is to minimize the expected cost with an acceptable risk in each selling horizon. We denote by $E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)]$ the expected cost in this selling horizon $\{1, 2, \dots, \hat{t}\}$.

Let $\alpha > 0$ be a specified positive number. The probability that the overall cost over the horizon $\{t, \dots, \hat{t}\}$ exceeds level α is

$$\begin{aligned}\Gamma(\mathbf{u}_{t|\hat{t}}, \alpha; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) &:= P_{t|\hat{t}}\big(J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) \leq \alpha | \xi_{1|t-1}\big) \\ &= \int_{J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) \leq \alpha} p_{t|\hat{t}}(\xi_{t|\hat{t}} | \xi_{1|t-1}) d\xi_{t|\hat{t}},\end{aligned}$$

where the probability, $\Gamma(\mathbf{u}_{t|\hat{t}}, \alpha; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$, depends on the realization of the random variable $\xi_{1|t-1}$ and the value of $\mathbf{u}_{1|\hat{t}}$. Let $\beta \in (0, 1)$ be a confidence level. Following the discussion in Section 2, the VaR of $J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$ can be defined as

$$\begin{aligned}\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) &:= \min\{\alpha \in \mathbb{R} : \Gamma(\mathbf{u}_{t|\hat{t}}, \alpha; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) \geq \beta\} \\ &= \min\{\alpha \in \mathbb{R}^+ \cup \{0\} : \Gamma(\mathbf{u}_{t|\hat{t}}, \alpha; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) \geq \beta\},\end{aligned} \tag{3.13}$$

where the second equality is from the nonnegativity of the cost function. Note that, for each $t \in \{1, \dots, \hat{t}\}$, the VaR function, $\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$, depends on the random variable $\xi_{1|t-1}$ and the value of $\mathbf{u}_{1|\hat{t}}$. By assumption, the decision variables, $\mathbf{u}_{1|\hat{t}}$, are decided before the start of the selling horizon. Let $\mathbf{u}_{1|\hat{t}}$ be given. Then the value of the risk, $\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$, in each horizon $\{t, \dots, \hat{t}\}_{t>1}$, is dependent of the random variable $\xi_{1|t-1}$.

We assume that the retailer's aim is to control the expected VaR to some thresholds γ_t, $t = 1, \dots, \hat{t}$. Using the definition in (3.13), we consider the following minimal expected cost inventory control model subject to the expected VaR as defined by (3.13):

$$\begin{aligned}\min_{\mathbf{u}_{1|\hat{t}} \in \mathcal{U}} \quad & E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)] \\ \text{s.t.} \quad & E[\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)] \leq \gamma_t, \quad \text{for } t \in \{1, 2, \dots, \hat{t}\}.\end{aligned} \tag{3.14}$$

To simplify the (3.14), we will show a monotonic property of $E[\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)]$ in the following proposition.

Proposition 3.1 *Let y_0 be fixed. For $u_i \in \mathcal{U}$, $i = 1, 2, \dots, \hat{t}$, we have*

$$E\big[\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)\big] \geq E\big[\alpha_\beta(\mathbf{u}_{t+1|\hat{t}}; \mathbf{u}_{1|t}, \xi_{1|t}; y_0)\big]. \tag{3.15}$$

Proof. From the discussion in [5], we have that for any $\beta \in (0,1)$ and density function $p_{t|\hat{t}}(\cdot|\xi_{1|t-1})$, VaR, $\alpha_\beta(\cdot)$, satisfies the Translation invariance (see Axiom **T** in [5] or A3 in [1]), which means that for any random outcome Z and constant a, $\alpha_\beta(Z+a) = \alpha_\beta(Z) + a$. Consequently, for any $t = 1, 2 \ldots, \hat{t}$, from Translation invariance, we have

$$\begin{aligned}
& E[\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)] \\
= \;& E\big[E[\alpha_\beta(J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)) \mid \xi_{1|t}]\big] \\
= \;& E\big[E[\alpha_\beta\big(J_{t+1}(\mathbf{u}_{t+1|\hat{t}}, \xi_{t+1|\hat{t}}; \mathbf{u}_{1|t}, \xi_{1|t}; y_0) + c_t(u_t) + f_t(u_t, \xi_t; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)\big) \mid \xi_{1|t}]\big] \\
= \;& E\big[E[\alpha_\beta\big(J_{t+1}(\mathbf{u}_{t+1|\hat{t}}, \xi_{t+1|\hat{t}}; \mathbf{u}_{1|t}, \xi_{1|t}; y_0)\big) + c_t(u_t) + f_t(u_t, \xi_t; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) \mid \xi_{1|t}]\big] \\
= \;& E\big[E[\alpha_\beta\big(J_{t+1}(\mathbf{u}_{t+1|\hat{t}}, \xi_{t+1|\hat{t}}; \mathbf{u}_{1|t}, \xi_{1|t}; y_0)\big) \mid \xi_{1|t}] + E[c_t(u_t) + f_t(u_t, \xi_t; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) \mid \xi_{1|t}]\big] \\
\geq \;& E\big[E[\alpha_\beta(\mathbf{u}_{t+1|\hat{t}}; \mathbf{u}_{1|t}, \xi_{1|t}; y_0) \mid \xi_{1|t}]\big] \\
= \;& E\big[\alpha_\beta(\mathbf{u}_{t+1|\hat{t}}; \mathbf{u}_{1|t}, \xi_{1|t}; y_0)\big],
\end{aligned}$$

where the third equality follows from the Translation invariance and the inequality is due to the nonnegativity of the cost function $c_t(u_t) + f_t(u_t, \xi_t; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$ for each u_t and ξ_t, $t \in \{1, \ldots, \hat{t}\}$. ■

Observe that when $t = 1$, the VaR of the whole horizon $\{1, 2, \ldots, \hat{t}\}$, $\alpha_\beta(\mathbf{u}_{1|\hat{t}}; \mathbf{u}_{1|0}, \xi_{1|0}; y_0)$ is deterministic, where $\xi_{1|0} := 0$ and $\mathbf{u}_{1|0} := 0$. For any t and $\theta \geq 0$, define the feasible set for each constraint on $\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$ as

$$S_t(\theta) := \{\mathbf{u}_{1|\hat{t}} : E\big[\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)\big] \leq \theta\}.$$

By Proposition 3.1 and the deterministic property of $\alpha_\beta(\mathbf{u}_{1|\bar{0}}; \mathbf{u}_{1|0}, \xi_{1|0}; y_0)$, for any $\theta \geq 0$ and $t \in \{1, 2, \ldots, \hat{t}\}$, we have

$$S_t(\theta) \subset S_{t-1}(\theta) \subset \ldots \subset S_1(\theta) = \{\mathbf{u}_{1|\hat{t}} : \alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) \leq \theta\}.$$

Using Proposition 3.1, we can simplify the model (3.14).

Proposition 3.2 *Suppose that $\gamma_i, i = 1, \cdots, \hat{t}$, is a constant γ. Then model (3.14) is equivalent to the following:*

$$\begin{aligned}
\min_{\mathbf{u}_{1|\hat{t}} \in \mathcal{U}} \quad & E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)] \\
\text{s.t.} \quad & \alpha_\beta(\mathbf{u}_{1|\hat{t}}; \mathbf{u}_{1|0}, \xi_{1|0}; y_0) \leq \gamma,
\end{aligned} \tag{3.16}$$

where $\xi_{1|0} := 0$ and $\mathbf{u}_{1|0} := 0$.

Note that VaR constraint is equivalent to a chance constraint problem. Hence, we can rewrite the constraints in (3.16) as

$$P_{1|\hat{t}}(J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0) \leq \gamma) \geq \beta.$$

Here we use the idea of *efficient trajectory* due to [16] to our model. Based on the distribution of the demand in the whole selling season, we introduce a $\beta-$efficient demand trajectories DT^{β} for the horizon $\{1,\ldots,\hat{t}\}$ by the following definition.

Definition 3.1 *Let β be a confidence level in $(0,1)$. For any $\mathbf{u}_{1|\hat{t}} \in \mathcal{U}$, a $\beta-$efficient demand trajectory of $\mathbf{u}_{1|\hat{t}}$, denoted by $DT^{\beta}(\mathbf{u}_{1|\hat{t}})$, is a sample path of the demands in the multi-period horizon,*

$$DT^{\beta}(\mathbf{u}_{1|\hat{t}}) := \{\xi_1^{\beta}, \xi_2^{\beta}, \ldots, \xi_{\hat{t}}^{\beta}\},$$

which satisfies the following conditions,

(i) For joint probability distribution function $P_{1|\hat{t}}(\cdot)$,

$$P_{1|\hat{t}}\big(J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0) \leq J_1(\mathbf{u}_{1|\hat{t}}, DT^{\beta}(\mathbf{u}_{1|\hat{t}}); y_0)\big) \geq \beta. \tag{3.17}$$

(ii) There is no other trajectory $DT_1^{\beta}(\mathbf{u}_{1|\hat{t}})$ such that

$$J_1\big(\mathbf{u}_{1|\hat{t}}, DT_1^{\beta}(\mathbf{u}_{1|\hat{t}}); y_0\big) < J_1\big(\mathbf{u}_{1|\hat{t}}, DT^{\beta}(\mathbf{u}_{1|\hat{t}}); y_0\big),$$

and

$$P_{1|\hat{t}}\big(J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0) \leq J_1(\mathbf{u}_{1|\hat{t}}, DT_1^{\beta}(\mathbf{u}_{1|\hat{t}}); y_0)\big) \geq \beta.$$

From the two conditions, we can take the trajectory $DT^{\beta}(\mathbf{u}_{1|\hat{t}})$ as a minimum such that

$$J_1(\mathbf{u}_{1|\hat{t}}, DT^{\beta}(\mathbf{u}_{1|\hat{t}}); y_0) = \min_{\gamma \geq 0} \big\{P_{1|\hat{t}}\big(J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}, ; y_0) \leq \gamma\big) \geq \beta\big\}.$$

For any $\mathbf{u}_{1|\hat{t}}$, let us define $U_{\beta}(\mathbf{u}_{1|\hat{t}})$ as the set of all $\beta-$efficient demand trajectories of the order sequence, $\mathbf{u}_{1|\hat{t}} \in \mathcal{U}$. If $DT_1^{\beta}(\mathbf{u}_{1|\hat{t}})$ and $DT_2^{\beta}(\mathbf{u}_{1|\hat{t}})$ are both the elements in $U_{\beta}(\mathbf{u}_{1|\hat{t}})$, then it can be easily verified that

$$J_1(\mathbf{u}_{1|\hat{t}}, DT_1^{\beta}(\mathbf{u}_{1|\hat{t}}); y_0) = J_1(\mathbf{u}_{1|\hat{t}}, DT_2^{\beta}(\mathbf{u}_{1|\hat{t}}); y_0).$$

From the discussion above on $\beta-$efficient demand trajectories, a robust and convex optimization problem can be introduced to replace the problem (3.16):

$$\begin{array}{ll} \min\limits_{\mathbf{u}_{1|\hat{t}} \in \mathcal{U}} & E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)] \\ \text{s.t.} & J_1(\mathbf{u}_{1|\hat{t}}, DT^{\beta}(\mathbf{u}_{1|\hat{t}}); y_0) \leq \gamma, \\ & DT^{\beta}(\mathbf{u}_{1|\hat{t}}) \in U_{\beta}(\mathbf{u}_{1|\hat{t}}). \end{array} \tag{3.18}$$

Proposition 3.3 *The problem (3.18) has a unique optimal solution, if the following conditions hold:*
(i) The joint distribution function $P_{1|\hat{t}}(\xi_{1|\hat{t}})$ of the demand is discrete.
(ii) For any $\gamma \geq 0$, $S_1(\gamma) \neq \phi$.

The proof of Proposition 3.3 can be illustrated in the following way. First, from the discrete distribution function $P_{1|\hat{t}}(\xi_{1|\hat{t}})$ and the boundedness of the demand variable ξ_t, for $1 \le t \le \hat{t}$, we have the cardinality of every set $U_\beta(\mathbf{u}_{1|\hat{t}})$ to be finite. On the other hand, from the assumption of the multi-period inventory model, the cost function $J_1(\mathbf{u}_{1|\hat{t}}, DT^\beta(\mathbf{u}_{1|\hat{t}}); y_0)$ is a convex function. Summarizing, we obtain, from the convexity of the finite constraints, the existence and uniqueness of the optimal solution.

3.3 A minimal cost model with CVaR constraints

Analogous to the discussion in Section 3.1, we may consider a CVaR model for the multi-period inventory control problem. Let $\beta \in (0,1)$ be a constant. For any scenario $\xi_{1|t-1}$ and decision sequence $\mathbf{u}_{1|\hat{t}}$, we define a risk measure on $\{t, \cdots, \hat{t}\}$ as

$$\phi_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) :=$$
$$\frac{1}{1-\beta}\int_{J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) \ge \alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)} J_t(\mathbf{u}_{t|\hat{t}}, \xi_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0) p_{t|\hat{t}}(\xi_{t|\hat{t}}|\xi_{1|t-1}) d\xi_{t|\hat{t}},$$

where $\phi_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$ depends on $\mathbf{u}_{1|\hat{t}}$ and $\xi_{1|t-1}$. Similarly as the VaR in the multi-period case, before the selling horizon, the retailer cannot predict the value, $\phi_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$. In the retailer's decision making problem, the constraint on the expected CVaR is used in our model. For every $t \in \{1, 2, \ldots, \hat{t}\}$ and the fixed thresholds sequence $\{\mu_t\}_{t\ge 1}$, by using the defined CVaR, we can consider the following minimal expected cost inventory control model with CVaR constraints as follows:

$$\begin{array}{ll} \min\limits_{\mathbf{u}_{1|\hat{t}} \in \mathcal{U}} & E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)] \\ \text{s.t.} & E\big[\phi_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)\big] \le \mu_t, \quad \text{for } t \in \{1, 2, \cdots, \hat{t}\}. \end{array} \tag{3.19}$$

The model is varied from model (3.14) by replacing the VaR constraints with CVaR constraints. Here we set an upper bound μ_t for the CVaR over the selling period $\{t, \cdots, T\}$. As in the literature [23, 26], we assume that the initial inventory level y_0 is a fixed nonnegative constant. To simplify the CVaR constraints in (3.19), we set a constant bound for the CVaR, that is, $\mu_1 = \cdots = \mu_{\hat{t}} = \mu$. The following result is a corollary of Proposition 3.1.

Corollary 3.1 *Let $y_0 \ge 0$ be fixed. For $u_i \in \mathcal{U}$, $i = 1, \cdots, \hat{t}$, we have*

$$E\big[\phi_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)\big] \ge E\big[\phi_\beta(\mathbf{u}_{t+1|\hat{t}}; \mathbf{u}_{1|t}, \xi_{1|t}; y_0)\big]. \tag{3.20}$$

As a coherent risk measure, CVaR satisfies Axioms Translation invariance (see A3.[1]). Corollary 3.1 can be obtained by a similar proof as in Proposition 3.1. Here, to illustrate the monotonic properties in Proposition 3.1 and Corollary 3.1, a simple example is given as following.

Example 3.1 *Consider a multi-period inventory control model with $\hat{t} = 10$. The demand ξ_t is independent of each other, and follows a truncated normal distribution, which is a normal* *distribution function,* $\mathcal{N}(10, 2)$*, truncated by the interval* $[0, 20]$*. The order cost and the leftover and backorder cost in each period is* $c_t(u_t) := \eta^t u_t$ *and* $f_t(u_t, \xi_t; y_{t-1}) := \eta^t(y_{t-1}+u_t-\xi_t)^2$*, where* η *is a discount factor and set 0.9. Here, we present the diagrams of* $\alpha_\beta(\mathbf{u}_{\mathbf{t}|\hat{\mathbf{t}}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$ *and* $\phi_\beta(\mathbf{u}_{\mathbf{t}|\hat{\mathbf{t}}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)$ *at each period t in Figure 1 and Figure 2.*

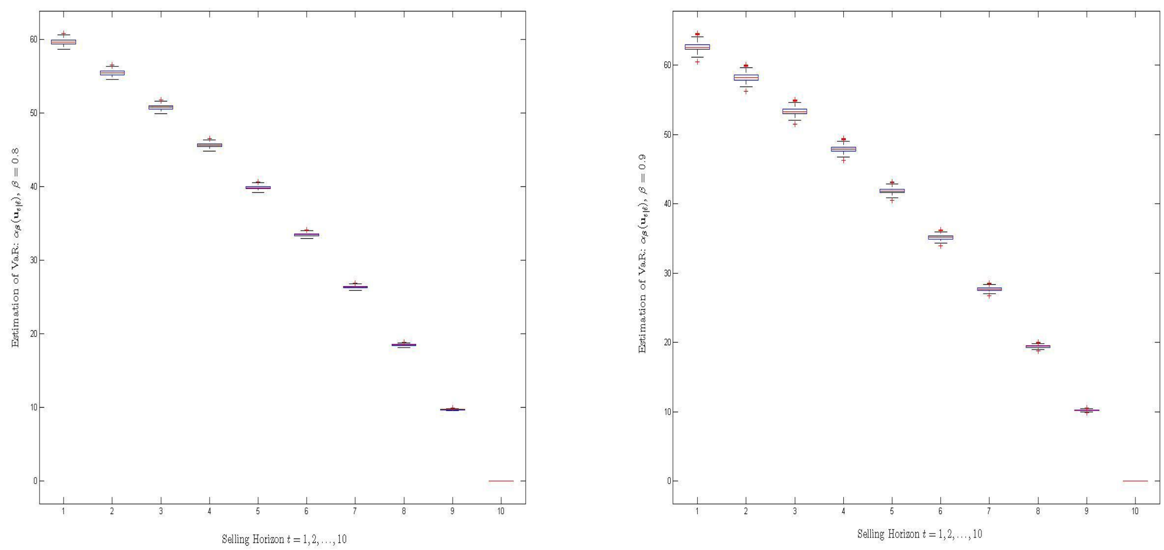

Figure 1: Estimation of VaR: $E\big[\alpha_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)\big]$, $\beta = 0.8, 0.9$ and $\hat{t} = 10$.

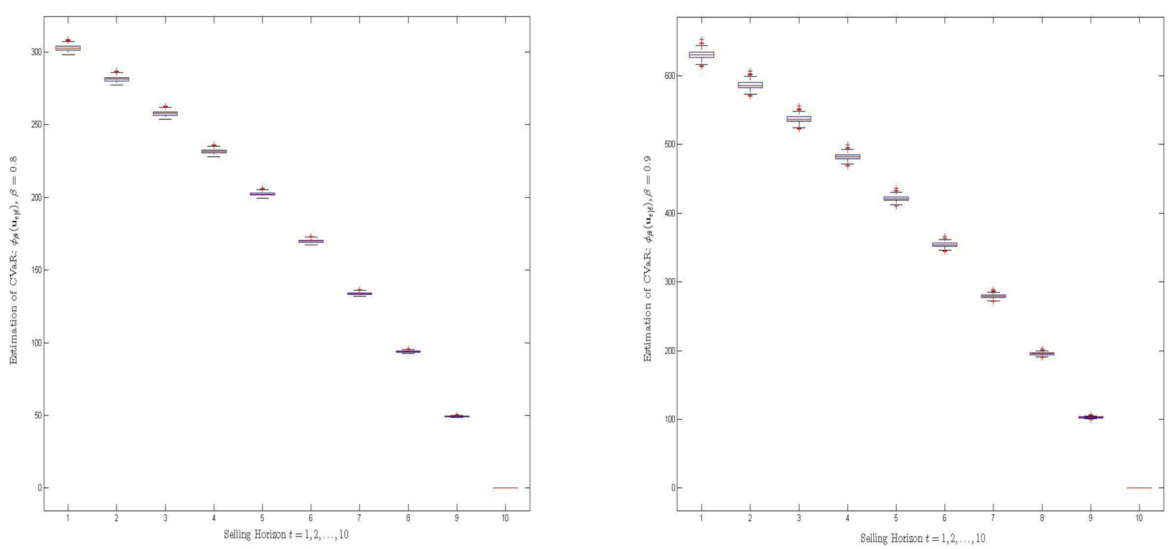

Figure 2: Estimation of CVaR: $E\big[\phi_\beta(\mathbf{u}_{t|\hat{t}}; \mathbf{u}_{1|t-1}, \xi_{1|t-1}; y_0)\big]$, $\beta = 0.8, 0.9$ and $\hat{t} = 10$.

Note that, both of Figures 1 and 2 show that the estimation of the expected VaR and CVaR in the different periods of the selling horizon, where the expected values, $E\big[\alpha_\beta(\mathbf{u}_{t|\hat{t}};\mathbf{u}_{1|t-1},\xi_{1|t-1};y_0)\big]$ and $E\big[\phi_\beta(\mathbf{u}_{t|\hat{t}};\mathbf{u}_{1|t-1},\xi_{1|t-1};y_0)\big]$, are estimated by the averages of VaR and CVaR samples. From the figures, we can easily see that for different values of β, the estimated values of VaR and CVaR are all monotonically decreasing when t increases from 1 to $\hat{t}$.

By Corollary 3.1 and (2.11), we can reformulate (3.19) as:

$$\begin{array}{cl} \min\limits_{(\mathbf{u}_{1|\hat{t}},\alpha)\in\mathcal{U}\times\mathcal{A}} & E[J_1(\mathbf{u}_{1|\hat{t}},\xi_{1|\hat{t}};y_0)] \\ \text{s.t.} & \phi_\beta(\mathbf{u}_{1|\hat{t}};\mathbf{u}_{1|0},\xi_{1|0};y_0)\le\mu. \end{array} \tag{3.21}$$

From the definition of $\phi_\beta(\mathbf{u}_{t|\hat{t}};\mathbf{u}_{1|t-1},\xi_{1|t-1};y_0)$, we have that

$$\phi_\beta(\mathbf{u}_{1|\hat{t}};\mathbf{u}_{1|0},\xi_{1|0};y_0)=\frac{1}{1-\beta}\int_{J_1(\mathbf{u}_{1|\hat{t}},\xi_{1|\hat{t}};y_0)\ge\alpha_\beta(\mathbf{u}_{1|\hat{t}};\mathbf{u}_{1|0},\xi_{1|0};y_0)} J_1(\mathbf{u}_{1|\hat{t}},\xi_{1|\hat{t}};y_0)p_{1|\hat{t}}(\xi_{1|\hat{t}})d\xi_{1|\hat{t}},$$

which is totally the same as the definition of the CVaR in the single period problem. Hence, the corresponding auxiliary function is

$$F_\beta(\mathbf{u}_{1|\hat{t}},\alpha;y_0):=\alpha+\frac{1}{1-\beta}E[(J_1(\mathbf{u}_{1|\hat{t}},\xi_{1|\hat{t}};y_0)-\alpha)_+],$$

where $\phi_\beta(\mathbf{u}_{1|\hat{t}};\mathbf{u}_{1|0},\xi_{1|0};y_0)=\min_{\alpha\in\mathcal{A}}F_\beta(\mathbf{u}_{1|\hat{t}},\alpha;y_0)$. Following Proposition 2.2, the optimization problem (3.21) can be rewritten as following, which is the main model that we propose for the multi-period inventory control problem,

$$\begin{array}{cl} \min\limits_{(\mathbf{u}_{1|\hat{t}},\alpha)\in\mathcal{U}\times\mathcal{A}} & E[J_1(\mathbf{u}_{1|\hat{t}},\xi_{1|\hat{t}};y_0)] \\ \text{s.t.} & F_\beta(\mathbf{u}_{1|\hat{t}},\alpha;y_0)\le\mu. \end{array} \tag{3.22}$$

Remark 3.1 *Model (3.22) has at least two advantages in comparison with (3.21). One is that the objective and the constraint functions are both convex. This makes it easier to obtain an optimal solution. The other is that (3.22) is one level stochastic optimization problem as opposed to two levels in (3.21).*

4 Sample average approximation method

In this section, we discuss numerical methods for solving the inventory problems. Our focus will be on model (3.22) because of its nice feature as outlined in Remark 3.1. The numerical methods depend essentially on the information about the demands. If the retailer has full information on the distribution of market demands and the expected values in the problem (3.22) can be integrated out analytically, then (3.22) is a deterministic minimization problem and we can use any appropriate nonlinear programming code to solve it.

In this section, we consider the case that the distribution of market demand is unknown but it can be obtained by sampling, e.g, from the historical data or computer simulation. Consequently we consider sample average approximation (SAA) method for solving our proposed inventory problems. The basic idea of SAA is to approximate the expected value by its sample average. The method is also known as Sample Path Optimization (SPO) method. There has been extensive literature on SAA and SPO. See recent work [4, 17, 29] and the references therein. More recently, Xu and Zhang [30] proposed a smoothing SAA method for solving a single period problem (2.11).

Throughout the section, we use the following notation. $\|\cdot\|$ denotes the Euclidean norm of a vector and a compact set of vectors. When $\mathcal{M}$ is a compact set of vectors, $\|\mathcal{M}\| := \max_{M\in\mathcal{M}} \|M\|$. $d(x,\mathcal{D}) := \inf_{x'\in\mathcal{D}} \|x-x'\|$ denotes the distance from point x to set $\mathcal{D}$. For two compact sets $\mathcal{C}$ and $\mathcal{D}$,

$$\mathbb{D}(\mathcal{C},\mathcal{D}) := \sup_{x\in\mathcal{C}} d(x,\mathcal{D})$$

denotes the distance from set $\mathcal{C}$ to set $\mathcal{D}$, and $\mathbb{H}(\mathcal{C},\mathcal{D})$ denotes the Hausdorff distance between the two sets, that is,

$$\mathbb{H}(\mathcal{C},\mathcal{D}) := \max\left(\mathbb{D}(\mathcal{C},\mathcal{D}), \mathbb{D}(\mathcal{D},\mathcal{C})\right).$$

We use $B(x,\delta)$ to denote the closed ball in $\mathbb{R}^m$ with radius δ and center x, that is $B(x,\delta) := \{x' \in \mathbb{R}^m : \|x'-x\| \le \delta\}$. When δ is dropped, $B(x)$ represents a neighborhood of point x. Finally, for a set-valued mapping $\mathcal{A} : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, we use $\overline{\lim}_{y\to x} \mathcal{A}(y)$ to denote the outer limit of the mapping at point x.

4.1 The SAA model and optimality conditions

Observe that demands in the multi-period model can be represented by a random vector with each component denoting a demand at a single period. Let $\xi^i_{1|\hat{t}}$, $i = 1, 2, \ldots, N$, be a sample of $\xi_{1|\hat{t}}$, let the initial inventory level y_0 be fixed. The SAA of (3.22) is,

$$\begin{array}{cl} \min\limits_{(\mathbf{u}_{1|\hat{t}},\alpha)\in\mathcal{U}\times\mathcal{A}} & \frac{1}{N}\sum_{i=1}^N J_1(\mathbf{u}_{1|\hat{t}}, \xi^i_{1|\hat{t}}; y_0) \\ \text{s.t.} & \frac{1}{N}\sum_{i=1}^N g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi^i_{1|\hat{t}}; y_0) \le 0, \end{array} \tag{4.23}$$

where

$$g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi^i_{1|\hat{t}}; y_0) := \alpha + \frac{1}{1-\beta}\Big(J_1(\mathbf{u}_{1|\hat{t}}, \xi^i_{1|\hat{t}}; y_0) - \alpha\Big)_+ - \mu.$$

We call (3.22) true problem and (4.23) its sample average approximation. Observe that both the objective function and the constraint functions are piecewise smooth with respect to u_t, for $t = \{1, 2 \ldots, \hat{t}\}$.

In what follows, we investigate the convergence of (4.23) in the sense that if we obtain an optimal solution to (4.23) for every N, whether the sequence of such optimal solutions converge to an optimal solution of the true problem as sample size increases. There are two ways to do so, one is to investigate the convergence of the sample average of the function both in the objective and constraints, the other is to consider first order necessary conditions that an optimal solution must satisfy. The first approach involves the approximation of feasible region and this incurs some technical difficulties from unboundedness of the feasible region due to α. The second approach requires some nonsmooth analysis because the underlying functions are not differentiable. It requires some results related to sample average random set-valued mappings which have been established in the past few years [25, 30]. In what follows, we use the second approach.

Observe first that under Assumption 3.1, both $J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)$ and $g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}; y_0)$ are locally Liptschitz continuous with respect to $\mathbf{u}_{1|\hat{t}}$ and α. However, these functions are not necessarily differentiable and hence we need to consider subdifferential of the functions in order to derive a first order necessary condition.

Definition 4.1 *Let $h : \mathbb{R}^n \times \Xi \to \mathbb{R}$ be a continuous function which is convex with respect to x. Let $\xi \in \Xi$ be fixed. The convex subdifferential [18] of h at a point x_0 with respect to x is*

$$\partial_x h(x_0, \xi) := \{\eta \in \mathbb{R}^n \mid h(x, \xi) - h(x_0, \xi) \geq \eta^T (x - x_0)\}$$

where $a^T b$ denotes the scalar product of two vectors.

Using the notion of convex subdifferentials, we can write down the generalized Karush-Kuhn-Tucker (GKKT) condition of the SAA problem (4.23) as follows:

$$\left\{ \begin{array}{l} 0 \in \frac{1}{N} \sum_{i=1}^N \partial_{(\mathbf{u}_{1|\hat{t}}, \alpha)} J_1(\mathbf{u}_{1|\hat{t}}, \xi^i_{1|\hat{t}}; y_0) + \lambda \frac{1}{N} \sum_{i=1}^N \partial_{(\mathbf{u}_{1|\hat{t}}, \alpha)} g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi^i_{1|\hat{t}}; y_0) \\ \qquad + \mathcal{N}_{\mathcal{U} \times \mathcal{A}}(\mathbf{u}_{1|\hat{t}}, \alpha), \\ 0 \leq \lambda^N \perp -\frac{1}{N} \sum_{i=1}^N g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi^i_{1|\hat{t}}; y_0) \geq 0, \end{array} \right. \tag{4.24}$$

where the sum of sets is in the sense of Minkowski, $\mathcal{N}_{\mathcal{U} \times \mathcal{A}}(\mathbf{u}_{1|\hat{t}}, \alpha)$ denotes the normal cone of $\mathcal{U} \times \mathcal{A}$ at $(\mathbf{u}_{1|\hat{t}}, \alpha)$, that is,

$$\mathcal{N}_{\mathcal{U} \times \mathcal{A}}(\mathbf{u}_{1|\hat{t}}, \alpha) := \{z \in \mathbb{R}^{\hat{t}+1} : z^T((\mathbf{u}'_{1|\hat{t}}, \alpha') - (\mathbf{u}_{1|\hat{t}}, \alpha)) \leq 0, \quad \forall (\mathbf{u}'_{1|\hat{t}}, \alpha') \in \mathcal{U} \times \mathcal{A}\}, \ \forall (\mathbf{u}_{1|\hat{t}}, \alpha) \in \mathcal{U} \times \mathcal{A}.$$

A point $(\mathbf{u}^N_{1|\hat{t}}, \alpha^N)$ satisfying (4.24) is said to be a *stationary point* and λ^N is called the corresponding Lagrange multiplier. Since problem (4.23) is a convex program, a feasible solution is an optimal solution if and only if it is a stationary point. In what follows, we make a blanket assumption that the SAA problem has an optimal solution, which means (4.24) has at least one solution. In what follows, we study the convergence of $\{(\mathbf{u}^N_{1|\hat{t}}, \alpha^N)\}$ as the sample size N increases. We need the following assumption.

Assumption 4.1 *The Lipschitz module of $J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)$ is bounded by an integrable function $\kappa(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}})$.*

Under Assumptions 3.1 and 4.1, $E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)]$ is well defined, convex and locally Liptschitz continuous with respect to $\mathbf{u}_{1|\hat{t}}$. Likewise, $E[g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}; y_0)]$ is well defined, convex and locally Liptschitz continuous with respect to $\mathbf{u}_{1|\hat{t}}$ and α. Therefore, the convex subdifferentials of

$E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)]$ and $E[g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}; y_0)]$, denoted respectively by $\partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)]$ and $\partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} E[g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}; y_0)]$, are well-defined. Consequently, we can write down the GKKT conditions of the true problem (3.22) in terms of convex subdifferentials:

$$\begin{cases} 0 \in \partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} E[J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)] + \lambda \partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} E[g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}; y_0)] + \mathcal{N}_{\mathcal{U}\times\mathcal{A}}(\mathbf{u}_{1|\hat{t}}, \alpha), \\ 0 \leq \lambda \perp -E[g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}; y_0)] \geq 0, \end{cases} \tag{4.25}$$

where $\mathcal{N}_{\mathcal{U}\times\mathcal{A}}(\mathbf{u}_{1|\hat{t}}, \alpha)$ denotes the normal cone of $\mathcal{U} \times \mathcal{A}$ at $(\mathbf{u}_{1|\hat{t}}, \alpha)$.

Lemma 4.1 *([28, Proposition 2.10]) Let $h : \mathbb{R}^n \times \Xi \to \mathbb{R}$ be a random continuous function which is convex with respect to x. Let $\xi \in \Xi$ be fixed. Assume that $E[h(x, \xi)]$ is finite in a neighborhood of x. Then*

$$\partial E[h(x, \xi)] = E[\partial_x h(x, \xi)],$$

where $E[\partial_x h(x, \xi)]$ denotes set of integral of measurable selections from $\partial_x h(x, \xi)$ [6], that is, $E[\partial_x h(x, \xi)] = \{E[\eta] : \eta$ is a measurable selection from $\partial_x h(x, \xi)\}$.

Using Lemma 4.1, we can rewrite (4.25) as

$$\begin{cases} 0 \in E[\partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)] + \lambda E[\partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}; y_0)] + \mathcal{N}_{\mathcal{U}\times\mathcal{A}}(\mathbf{u}_{1|\hat{t}}, \alpha), \\ 0 \leq \lambda \perp -E[g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}; y_0)] \geq 0. \end{cases} \tag{4.26}$$

4.2 Convergence analysis

To proceed the convergence analysis of stationary points of (4.24), we need the following intermediate result which is established in [25, 29].

Lemma 4.2 *Let $V \subset \mathbb{R}^m$ be a compact set, and $\Phi(v, \xi) : V \times \mathbb{R}^k \to 2^{\mathbb{R}^m}$ be a compact set-valued mapping which is upper semi-continuous with respect to v on V for every fixed ξ. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\xi : \Omega \to \Xi \subset \mathbb{R}^k$ be a random vector and $\xi^1, \cdots, \xi^N$ be an i.i.d sample of ξ. Let*

$$\Phi_N(v) := \frac{1}{N} \sum_{i=1}^{N} \Phi(v, \xi^i).$$

Suppose that: (a) probability measure P of our considered space $(\Omega, \mathcal{F}, P)$ is nonatomic; (b) there exists an integrable function $\sigma(\xi)$ such that $\|\Phi(v, \xi)\| \leq \sigma(\xi)$. Then for any $\delta > 0$

$$\overline{\lim_{N\to+\infty}} \Phi_N(v) \subset E[\Phi_\delta(v, \xi)], w.p.1$$

uniformly for $v \in V$, where

$$\Phi_\delta(v) := \bigcup_{w \in B(v,\delta)} \Phi(w, \xi),$$

and $E[\Phi_\delta(v, \xi)]$ denotes set of integral of measurable selections from $\Phi_\delta(v, \xi)$ [6].

Using the above lemma, we can establish the convergence of the optimal solution of SAA optimization problem (4.23) to the true problem (3.22) as the sample size N tends to infinity.

Theorem 4.1 *Let* $\{(\mathbf{u}^N_{1|\hat{t}}, \alpha^N)\}$ *be a sequence of optimal solutions of the SAA problem (4.23) and* $\{(\mathbf{u}^*_{1|\hat{t}}, \alpha^*)\}$ *be an accumulation point. Suppose that there exists a compact subset* $\mathcal{C}$ *of* $\mathcal{U}\times\mathcal{A}$ *such that w.p.1 the whole sequence* $\{(\mathbf{u}^N_{1|\hat{t}}, \alpha^N)\}$ *is contained in* $\mathcal{C}$ *and the Lagrange multiplier* $\{\lambda^N\}$ *which corresponds to* $\{(\mathbf{u}^N_{1|\hat{t}}, \alpha^N)\}$ *is bounded. Suppose also that the probability space* $(\Omega, \mathcal{F}, P)$ *is nonatomic. Then w.p.1, there exists* $\lambda^* \geq 0$ *such that*

$$\begin{cases} 0 \in E[\partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} J_1(\mathbf{u}^*_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)] + \lambda^* E[\partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} g_\beta(\mathbf{u}^*_{1|\hat{t}}, \alpha^*, \xi_{1|\hat{t}}; y_0)] + \mathcal{N}_{\mathcal{U}\times\mathcal{A}}(\mathbf{u}^*_{1|\hat{t}}, \alpha^*), \\ 0 \leq \lambda^* \perp -E[g_\beta(\mathbf{u}^*_{1|\hat{t}}, \alpha^*, \xi_{1|\hat{t}}; y_0)] \geq 0, \end{cases} \tag{4.27}$$

and $\{(\mathbf{u}^*_{1|\hat{t}}, \alpha^*)\}$ *is the optimal solution of the true problem (3.22).*

Proof. The second part of the conclusion is trivial in that a stationary point of a convex program is an optimal solution. It therefore suffices to show (4.27). The stationary point $(\mathbf{u}^N_{1|\hat{t}}, \alpha^N)$ and corresponding Lagrange multiplier λ^N of (4.24) satisfies the following:

$$\begin{cases} 0 \in \frac{1}{N}\sum_{i=1}^N [\partial_{(\mathbf{u}^N_{1|\hat{t}},\alpha^N)} J_1(\mathbf{u}^N_{1|\hat{t}}, \xi^i_{1|\hat{t}}; y_0)] + \lambda^N \frac{1}{N}\sum_{i=1}^N \partial_{(\mathbf{u}^N_{1|\hat{t}},\alpha^N)} g_\beta(\mathbf{u}^N_{1|\hat{t}}, \alpha^N, \xi^i_{1|\hat{t}}; y_0) \\ \qquad +\mathcal{N}_{\mathcal{U}\times\mathcal{A}}(\mathbf{u}^N_{1|\hat{t}}, \alpha^N), \\ 0 \leq \lambda^N \perp -\frac{1}{N}\sum_{i=1}^N [g_\beta(\mathbf{u}^N_{1|\hat{t}}, \alpha^N, \xi^i_{1|\hat{t}}; y_0)] \geq 0. \end{cases} \tag{4.28}$$

Let us define the set-valued mappings

$$\Phi_1(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}) := \partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} J_1(\mathbf{u}_{1|\hat{t}}, \xi_{1|\hat{t}}; y_0)$$

and

$$\Phi_2(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}}) := \partial_{(\mathbf{u}_{1|\hat{t}},\alpha)} g_\beta(\mathbf{u}_{1|\hat{t}}, \alpha, \xi^i_{1|\hat{t}}; y_0).$$

Then we can rewrite the first equation in (4.28) as

$$0 \quad \in \quad \frac{1}{N}\sum_{i=1}^N \Big[\Phi_1(\mathbf{u}^N_{1|\hat{t}}, \alpha^N, \lambda^N, \xi^i_{1|\hat{t}}) + \lambda^N \Phi_2(\mathbf{u}^N_{1|\hat{t}}, \alpha^N, \lambda^N, \xi^i_{1|\hat{t}})\Big] + \mathcal{N}_{\mathcal{U}\times\mathcal{A}}(\mathbf{u}^N_{1|\hat{t}}, \alpha^N).$$

Observe that $\Phi_i(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}})$, $i = 1, 2$, is a random compact set-valued mapping and it is upper semicontinuous with respect to $(\mathbf{u}_{1|\hat{t}}, \alpha)$ on set $\mathcal{U} \times \mathcal{A}$. Moreover, under Assumptions 3.1 and 4.1, $\Phi_i(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}})$ is bounded by an integrable function, that is, for $i = 1, 2$,

$$\|\Phi_i(\mathbf{u}_{1|\hat{t}}, \alpha, \xi_{1|\hat{t}})\| \leq \kappa(\xi_{1|\hat{t}}),$$

where $\kappa(\xi_{1|\hat{t}})$ is an integrable function. For the simplicity of notation, let us denote (u^N, α^N) by v^N and (u^*, α^*) by v^*. Let N be sufficiently large such that $\|v^N - v^*\| \leq \delta$. For $i = 1, 2$,

$$\begin{aligned} \mathbb{D}(\tfrac{1}{N}\textstyle\sum_{i=1}^N \Phi_i(v^N, \xi^i, E[(\Phi_1)_{2\delta}(v^*)]) \leq \quad & \mathbb{D}\Bigg(\frac{1}{N}\sum_{i=1}^N \Phi_i(v^N, \xi^i), \frac{1}{N}\sum_{i=1}^N (\Phi_i)_\delta(v^*, \xi^i)\Bigg) \\ & +\mathbb{D}\Bigg(\frac{1}{N}\sum_{i=1}^N (\Phi_i)_\delta(v^*, \xi^i), E[(\Phi_i)_{2\delta}(v^*)]\Bigg). \end{aligned}$$

It follows from Lemma 4.2 that the second term on the right hand side of the inequality above tends to zero w.p.1 as $N \to +\infty$. On the other hand, since $\Phi_i(v^N, \xi^i) \subset (\Phi_i)_\delta(v^*, \xi^i)$, for $v^N \in B_\delta(v^*)$, the first term on the right is zero for N sufficiently large. This shows that

$$0 \in E[(\Phi_1)_{2\delta}(v^*, \xi)] + \lambda^* E[(\Phi_2)_{2\delta}(v^*, \xi)] + \mathcal{N}_{\mathcal{U}\times\mathcal{A}}(v^*)$$

w.p.1. Driving δ to 0, we have by the Lebesgue dominated convergence theorem that

$$0 \in E[(\Phi_1)(v^*, \xi)] + \lambda^* E[(\Phi_2)(v^*, \xi)] + \mathcal{N}_{\mathcal{U}\times\mathcal{A}}(v^*),$$

which corresponds to the first equation in (4.27). To complete the proof, we need to show the second equation in (4.27) holds. Because $g_\beta(v, \xi)$ is locally Lipschitz continuous with respect to v and the Lipschitz module is integrable. By the classical strong law of large numbers [21, sections 2.6 and 6.2], $\frac{1}{N}\sum_{i=1}^N g_\beta(u, \alpha, \xi^i)$, converges to $E[g_\beta(v, \xi)]$ w.p.1 uniformly over a compact set. Consequently

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^N g_\beta(u^N, \alpha^N, \xi^i) = E[g_\beta(u^*, \alpha^*, \xi)]$$

w.p.1. The second equation of (4.27) follows. The proof is complete. ■

5 Computational results

We have undertaken some numerical tests on the single period problem (2.11) and multiple periods problem (3.22). In what follows we report some preliminary results of the tests.

The tests are carried out in Matlab 7.2 installed in a PC with Windows XP operating system. We use the Matlab built-in optimization solver *fmincon* to solve sample average approximation (4.23). Note that, to avoid the nonsmoothness of both objective function and constraint function in our Matlab program, a smoothing technique is incorporated when we solve (4.23) with *fmincon*. Specifically, we replace $\max(0, z)$ (which is also denoted by $(z)_+$) with following function:

$$f(z, \epsilon) = z + \epsilon \ln\left(1 + \exp(-\frac{z}{\epsilon})\right),$$

where ϵ is a smoothing parameter and set as 0.01 in our numerical test. This kind of smoothing technique is used in [30]. Convergence of this smoothing approximation as $\epsilon \to 0$ can be found in [30].

Example 5.1 *Consider an inventory model (2.11) with unit selling price $s = 1$, unit purchase cost $c = 0.5$, unit salvage value $v = 0$. We assume that the market demand, ξ, satisfies a truncated normal distribution with density function, denoted by $\rho(\xi; m, \sigma, 0, 2m)$, where*

$$\rho(\xi; m, \sigma, 0, 2m) = \begin{cases} 0, & \xi < 0; \\ \frac{\phi(\frac{\xi - m}{\sigma})}{\Phi(\frac{m}{\sigma}) - \Phi(-\frac{m}{\sigma})}, & 0 \le \xi \le 2m; \\ 0, & \xi > 2m, \end{cases} \quad (5.29)$$

where we use $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, to denote the density function and distribution function of a random variable with normal distribution, $\bar{\xi} \sim \mathcal{N}(m, \sigma)$. In the numerical tests, we set $m = 1$ and $\sigma = 1/3$. The optimal decision problem becomes

$$\begin{array}{ll} \min\limits_{u \times \alpha \in \mathcal{U} \times \mathcal{A}} & E[J(u,\xi)] = E[0.5(u-\xi)_+ - u] \\ s.t. & F_\beta(u,\alpha) - \mu = \alpha + \frac{1}{1-\beta} E[(0.5(u-\xi)_+ - u - \alpha)_+] - \mu \leq 0. \end{array} \tag{5.30}$$

Moreover, on the basis of the distribution of demand, we can set an upper bound and a lower bound for the order quantity, u, as -100 and 100. Similarly, based on the discussion in Section 2, we can assume

$$\alpha^* \in [-(s-c)\bar{u}, \max_{u,\xi} J(u,\xi)] \subset [-50, 50].$$

We carry out tests with $\beta = 0.5, 0.8, 0.9, 0.95$ and $\mu = -0.1, 0, 0.1$, the test results are displayed in Tables 1-3. Note that, when $\beta = 0.5$ and $\mu = -0.1, 0, 0.1$, the optimal solution of the true problem (5.30) is $u^* = 1$.

For $\beta = 0.5$, the results in Tables 1-3 show that both optimal solutions and values of SAA problems approximate their true counterparts very well as sample size increases. These results also show that, when N is sufficiently large, the increase of the sample size does not affect the optimal solutions and values of SAA problems very much. For example, in Tables 1-3, we test with SAAs of sample sizes $N = 300, 600, 1000, 3000$ and find that the optimal values of SAA problems with $N = 3000$ is almost the same as that of $N = 600, 1000$. Note that, for other β values, we do not know the precise solutions of the true problems. Figures 3 and 4 demonstrate the relationship between the SAA optimal value and the sample size. It shows that, as the sample sizes increases from 200 to 1000, the deviations of the SAA solutions are decreasing.

β	Sample Size	Optimal Solution u^*	Optimal Value
$\beta = 0.5$	$N = 300$	1.0065	-0.377
	$N = 600$	0.9987	-0.3634
	$N = 1000$	0.9969	-0.3674
	$N = 3000$	1.0031	-0.3677
$\beta = 0.8$	$N = 300$	1.0036	-0.3849
	$N = 600$	0.9988	-0.3608
	$N = 1000$	1.0044	-0.3691
	$N = 3000$	0.9964	-0.3673
$\beta = 0.9$	$N = 300$	0.8711	-0.355
	$N = 600$	0.8651	-0.3509
	$N = 1000$	0.8406	-0.3553
	$N = 3000$	0.8518	-0.3555
$\beta = 0.95$	$N = 300$	0.6170	-0.2876
	$N = 600$	0.6432	-0.3009
	$N = 1000$	0.6284	-0.2921
	$N = 3000$	0.6269	-0.2920

Table 1: Single period model with $\mu = 0$.

β	Sample Size	Optimal Solution u^*	Optimal Value
$\beta = 0.5$	$N = 300$	1.0102	-0.3755
	$N = 600$	1.0063	-0.3705
	$N = 1000$	1.0013	-0.3683
	$N = 3000$	1.0029	-0.3689
$\beta = 0.8$	$N = 300$	0.8573	-0.3548
	$N = 600$	0.8980	-0.3626
	$N = 1000$	0.8685	-0.3582
	$N = 3000$	0.8744	-0.3584
$\beta = 0.9$	$N = 300$	0.6242	-0.2899
	$N = 600$	0.6432	-0.2986
	$N = 1000$	0.6230	-0.2898
	$N = 3000$	0.6398	-0.2965

Table 2: Single period model with $\mu = -0.1$.

β	Sample Size	Optimal Solution u^*	Optimal Value
$\beta = 0.5$	$N = 300$	1.0276	-0.3800
	$N = 600$	0.9905	-0.3791
	$N = 1000$	1.0010	-0.3734
	$N = 3000$	1.0046	-0.3695
$\beta = 0.9$	$N = 300$	0.9823	-0.3688
	$N = 600$	1.0074	-0.3709
	$N = 1000$	1.0061	-0.3694
	$N = 3000$	0.9997	-0.3673
$\beta = 0.95$	$N = 300$	0.8079	-0.3417
	$N = 600$	0.8151	-0.3434
	$N = 1000$	0.8187	-0.3437
	$N = 3000$	0.8284	-0.3492

Table 3: Single period model with $\mu = 0.1$.

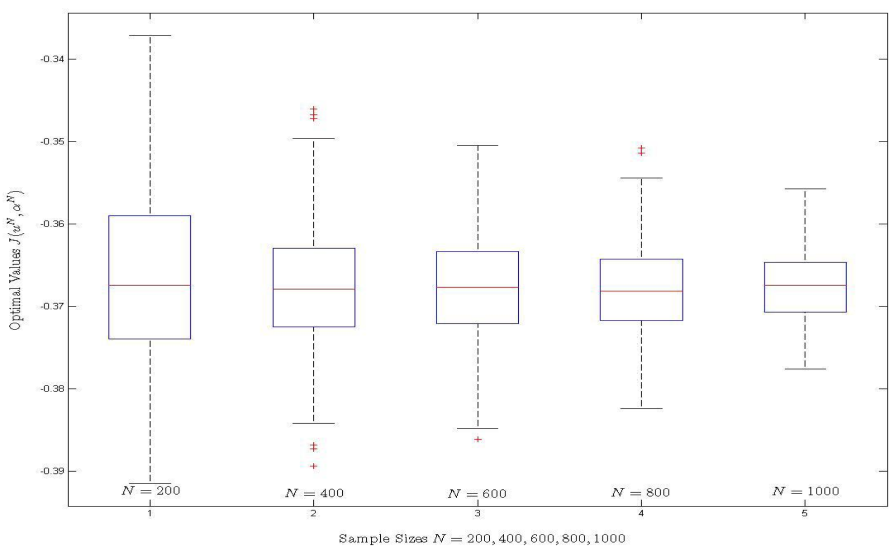

Figure 3: $E[J(u^N, \xi)]$ with $N = 200, 400, 600, 800, 1000$ ($\mu = 0$ and $\beta = 0.5$).

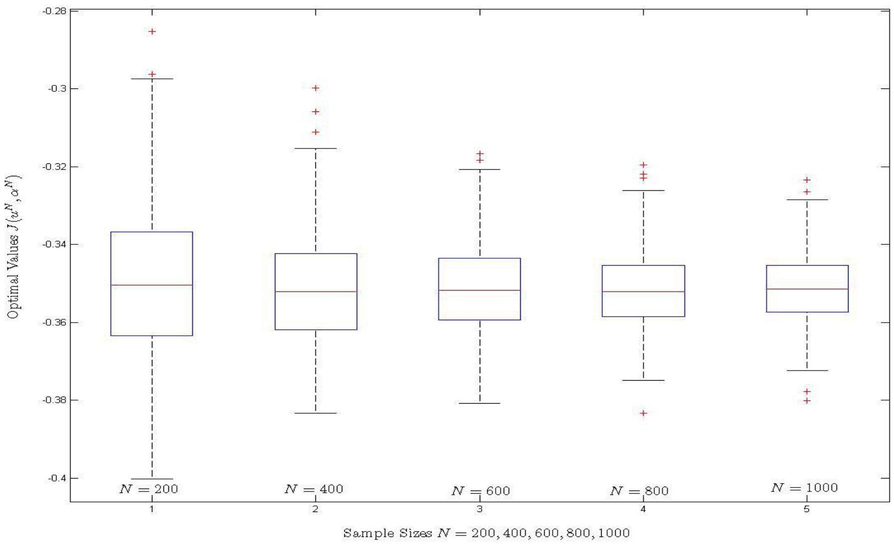

Figure 4: $E[J(u^N, \xi)]$ with $N = 200, 400, 600, 800, 1000$ ($\mu = 0$ and $\beta = 0.9$).

Example 5.2 *Consider a half-a-year order scheduling problem. In this problem, the retailer has to make its decision on its ordering at each month, denoted by $u_1, \ldots, u_6$ or $\mathbf{u}_{1|6}$ before the start of the 6 months. The initial inventory level y_0 is assumed to be 0 and demand in each month is denoted by ξ_t, $t = 1, 2, \ldots, 6$. Therefore, the inventory level in each month is $y_t = \sum_{i=1}^{t}(u_i - \xi_i)$. The ordering cost function is $c_t(u_t) := \eta^t u_t$ and the holding/backordering penalty cost function is $f_t(u_t, \xi_t; y_{t-1}) := \eta^t(u_t - \xi_t + y_{t-1})^2$, where η is a discount factor and set* 0.9*. The market demand in each period $t = 1, 2, \ldots, 6$, ξ_t, is assumed to be independent from each other and satisfies a truncated normal distributed with its density function, $\rho(\xi; m, \sigma, 0, 2m)$ as defined in (5.29) with $m = 1$ and $\sigma = 0.1$. When $\beta = 0$ and $\mu = 3.8$ and 4.0, we can obtain the exact optimal solution of the true problem, which is $\mathbf{u}^*_{1|6} = (0.95, 1, 1, 1, 1, 0.55)$. We carry out numerical test with $\beta = 0, 0.5, 0.9$ and $\mu = 3.8, 4.0$. The results are displayed in Tables 4 and 5.*

β	Sample Size	Optimal Solution u^*	Optimal Value
$\beta = 0$	$N = 300$	0.9449, 1.0088, 0.9988, 1.0061, 1.0122, 0.5489	4.6876
	$N = 600$	0.9488, 1.0006, 0.9960, 1.0042, 0.9988, 0.5548	4.6760
	$N = 1000$	0.9514, 0.9964, 0.9972, 1.0013, 1.0019, 0.5553	4.6754
$\beta = 0.5$	$N = 300$	1.0310, 0.9079, 1.0046, 1.0063, 1.0071, 0.5540	4.6764
	$N = 600$	1.0316, 0.9253, 0.9949, 0.9995, 1.0095, 0.5565	4.6793
	$N = 1000$	1.0488, 0.9024, 1.0031, 1.0024, 1.0034, 0.5506	4.6907
$\beta = 0.9$	$N = 300$	1.3395, 0.6306, 0.9946, 1.0276, 1.0020, 0.5616	4.8422
	$N = 600$	1.3239, 0.6405, 1.0246, 1.0116, 1.0105, 0.5605	4.8179
	$N = 1000$	1.3329, 0.6321, 1.0205, 1.0099, 1.0088, 0.5650	4.8298

Table 4: Multi-period model with $\mu = 3.8$.

For $\beta = 0$, the results in Tables 4 and 5 demonstrate that both optimal solutions and values

β	Sample Size	Optimal Solution u^*	Optimal Value
$\beta = 0$	$N = 300$	$0.9506, 1.0110, 1.0001, 1.0064, 1.0011, 0.5464$	4.6837
	$N = 600$	$0.9502, 1.0009, 0.9967, 0.9947, 0.9989, 0.5500$	4.6719
	$N = 1000$	$0.9506, 0.9949, 1.0032, 0.9980, 0.9984, 0.5487$	4.6693
$\beta = 0.5$	$N = 300$	$0.9393, 1.0107, 1.0099, 0.9874, 0.9940, 0.5488$	4.6621
	$N = 600$	$0.9477, 0.9986, 0.9999, 1.0020, 1.0036, 0.5443$	4.6647
	$N = 1000$	$0.9483, 0.9934, 0.9954, 0.9948, 1.0010, 0.5541$	4.6613
$\beta = 0.9$	$N = 300$	$1.1997, 0.7689, 1.0234, 1.0103, 1.0069, 0.5635$	4.7517
	$N = 600$	$1.1902, 0.7874, 1.0174, 1.0079, 1.0059, 0.5513$	4.7524
	$N = 1000$	$1.1845, 0.7712, 1.0165, 1.0110, 1.0112, 0.5687$	4.7368

Table 5: Multi-period model with $\mu = 4.0$.

of SAA problems approximate their true counterparts very well as sample size increases. These results also show that, when N is sufficiently large, the increase of the sample size does not affect the optimal solutions and values of SAA problems very much. For instance, in Tables 4 and 5, the difference between optimal values of SAA problems with $N = 1000$ and $N = 300, 600$ is very small.

6 Conclusion

In this paper, we propose new models for optimizing single and multi-period stochastic inventory control problems with different risk aversion constraints. Using some mathematical manipulation, we show that these models can be reformulated as expected loss minimization problems subject to CVaR constraints, which are convex stochastic programming problems. We apply the well known sample average approximation (SAA) method to solve the convex stochastic programming problems. One of the advantages of SAA is that it allows to use historical data as a sample to simulate the uncertainties of market demand. Moreover, we show that the optimal solution of the SAA problem converges to its counterpart in the true problem based on the generalized KKT conditions for the SAA problems and the true problems.

However, with risk aversion constraints, it is still some difficulties to formulate the multi-period models as stochastic dynamic programming, which can solve the multi-period inventory control recursively. Hence, the SAA method may not be effective for solving the problems with a large number of periods in the selling horizon. In future research, we would like to consider an effective stochastic dynamic programming model for the risk aversion constrained inventory control problems.

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