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Stability Analysis of One Stage Stochastic Mathematical Programs with Complementarity Constraints

Yongchao Liu · Huifu Xu · Gui-Hua Lin

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Abstract We study the quantitative stability of the solution sets, optimal value and M-stationary points of one stage stochastic mathematical programs with complementarity constraints when the underlying probability measure varies in some metric probability space. We show under moderate conditions that the optimal solution set mapping is upper semi-continuous and the optimal value function is Lipschitz continuous with respect to probability measure. We also show that the set of M-stationary points as a mapping is upper semi-continuous with respect to the variation of the probability measure. A particular focus is given to empirical probability measure approximation which is also known as sample average approximation (SAA). It is shown that optimal value and M-stationary points of SAA programs converge to their true counterparts with probability one (w.p.1.) at exponential rate as the sample size increases.

Keywords SMPEC · Stability · Error bound · Empirical probability measure · M-stationary point

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Y. Liu (✉) · G.-H. Lin
School of Mathematical Sciences, Dalian University of Technology, Dalian, Liaoning, China
e-mail: googleliu0717@yahoo.com.cn

G.-H. Lin
e-mail: lin_g_h@yahoo.com.cn

H. Xu
School of Mathematics, University of Southampton, Southampton, UK
e-mail: h.xu@soton.ac.uk

1 Introduction

Mathematical programs with equilibrium constraints (MPECs) is an important class of optimization problems arising frequently in applications such as engineering design, economic equilibrium and multilevel game [1, 2]. In practice, MPECs often involve some stochastic data, and this motivates one to consider stochastic MPECs (SMPECs). Since the first paper on SMPEC by Patriksson and Wynter [3], many researchers have paid attention to the class of optimization problems; see for example [4–9].

In this paper, we consider a one stage SMPEC where the underlying functions in the objective and constraints are expected value of some random functions. Birbil et al. [4] apparently first considered the problem and proposed a sample-path optimization method for solving the problem. They investigated convergence of optimal solutions and stationary points when the underlying functions are approximated by a sample-path based simulation. Lin et al. [5] proposed a smoothing penalty function method for solving the one stage SMPEC, where the complementarity constraints are reformulated as a system of non-smooth equations and then smoothed and penalized to the objective. The well-known Monte Carlo and quasi-Monte Carlo methods were subsequently used to approximate the expected value of the underlying random functions and almost sure convergence of optimal values and B-stationary points were obtained. The model was further studied by Meng and Xu [7], who demonstrated exponential rate convergence of weak stationary points obtained from solving the SAA problem and presented some numerical results based on the well-known NLP-regularization method [10]. A detailed convergence analysis of the latter (SAA combined with NLP-regularization) was given by Liu and Lin [6].

Our focus here is on the stability analysis of the one stage SMPEC. Specifically, we look into change of optimal values, optimal solutions and stationary points as the underlying probability measure varies under some appropriate metric. This kind research is numerically motivated in that, in practice, due to lack of complete information of the distribution of the random variables, it is often difficult to obtain a closed form of the expected values of the random function in the objective and constraints and subsequently numerical schemes are proposed to approximate the expected values. The stability analysis in this paper may provide a unified theoretical framework for various numerical approximation schemes of the expected values of the underlying functions in the SMPEC. Indeed, such a stability analysis has been well-known for classical stochastic programs with equality and/or inequality constraints although it is new for SMPEC; see for instance [11–13] and [14] for the recent development when this kind of stability analysis is applied to stochastic mathematical programs with dominance constraints.

An interesting question is: instead of applying the existing stability results in [12, 13], why should we carry out a separate analysis? To answer this question, we need to point out the fundamental difference between SMPEC and classical stochastic programming problems: reformulating the complementarity constraints as a system of equality or inequalities does not guarantee certain constraint qualifications [15, 16] (such as LICQ, MFCQ) which are often needed for stability analysis.

This motivates us to undertake an independent stability analysis. Our key approach is to use an error bound of the feasible set and Klatte's earlier stability result on an abstract parametric nonlinear programming [17, 18] to derive the Lipschitz continuity of optimal values and semi-continuity of optimal solutions of the problem. Moreover, by exploiting a recent breakthrough in reformulation of MPEC first order optimality conditions [19], we extend our stability analysis to stationary points, particularly the M-stationary point. As a particular case, we examine the asymptotic convergence of stationary points under empirical probability measure approximation.

The rest of the paper are organized as follows. In Sect. 2, we present some basic definitions. In Sect. 3, we study the stability of optimal solutions, optimal values and stationary points with respect to the probability measure. In Sect. 4, we focus on the empirical probability measure case. Finally a brief conclusion is given in Sect. 5.

2 Preliminaries

Throughout this paper, we use the following notation. For vectors $a, b \in \mathbb{R}^n$, $a^T b$ denotes the scalar product, $\|\cdot\|$ denotes the Euclidean norm of a vector, $\mathcal{B}$ denotes the closed unit ball in the respective space. $d(z, \mathcal{D}) := \inf_{z' \in \mathcal{D}} \|z - z'\|$ denotes the distance from a point z to a set $\mathcal{D}$. For two bounded sets $\mathcal{C}$ and $\mathcal{D}$,

$$\mathrm{dist}_{\mathrm{V}}(\mathcal{C}, \mathcal{D}) := \sup_{z \in \mathcal{C}} d(x, \mathcal{D})$$

denotes the deviation of $\mathcal{C}$ from $\mathcal{D}$ and $\mathrm{dist}_{\mathrm{H}}(\mathcal{C}, \mathcal{D}) := \max(\mathrm{dist}_{\mathrm{V}}(\mathcal{C}, \mathcal{D}), \mathrm{dist}_{\mathrm{V}}(\mathcal{D}, \mathcal{C}))$ denotes the Hausdorff distance between $\mathcal{C}$ and $\mathcal{D}$. Moreover, $\mathcal{C} + \mathcal{D}$ denotes the Minkowski addition of the two sets, that is,

$$\{C + D : C \in \mathcal{C}, D \in \mathcal{D}\}.$$

For a real-valued differentiable function $g(z)$, we use $\nabla g(z)$ to denote the transpose of Jacobian of g at point z. Finally, for a set $\{(x, y) = z : z \in Z\}$, $\Pi_x Z = \{x : \exists y \text{ such that } (x, y) \in Z\}$.

Definition 2.1 [20] Let C be a nonempty subset of $\mathbb{R}^n$. The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a *uniform P-function* over set C iff, for some $\gamma > 0$,

$$\max_{1 \le i \le n} \big[F_i(x) - F_i(y)\big](x_i - y_i) \ge \gamma \|x - y\|^2, \quad \forall x, y \in C.$$

Let X, Y be finite dimensional spaces and $T : X \rightrightarrows Y$ be a set-valued mapping. Let $(\bar{x}, \bar{y}) \in \mathrm{gph}\, T$, the graph of T. T is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ iff there exist constants $\kappa > 0, \delta > 0$ such that

$$d\big(x, T^{-1}(y)\big) \le \kappa d\big(y, T(x)\big), \quad \forall (x, y) \in (\bar{x}, \bar{y}) + \delta \mathcal{B}.$$

It is said to be *upper semi-continuous* at x in the sense of Berge iff for any $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$T(x') \subseteq T(x) + \epsilon \mathcal{B}, \quad \forall x' \in x + \delta \mathcal{B}.$$

It is said to be *Lipschitz continuous* at x iff there exist a constant L and a neighborhood U_x of x such that

$$\mathrm{dist}_{\mathrm{H}}\big(T(x'), T(x'')\big) \le L\|x' - x''\|, \quad \forall x', x'' \in U_x.$$

See [21] for more details of set-valued mapping.

Consider the standard MPEC:

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & z \in Z, \\ & 0 \le g(z) \perp h(z) \ge 0, \end{array} \tag{1}$$

where Z is a nonempty, closed and convex subset of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable functions.

Definition 2.2 [22] A feasible point z^* is said to be a *weak stationary point* of (1) iff there exist Lagrangian multiplier vectors $u^*, v^* \in \mathbb{R}^m$ such that

$$\begin{aligned} &0 \in \nabla f(z^*) - \nabla g(z^*)u^* - \nabla h(z^*)v^* + \mathcal{N}_Z(z^*), \\ &u_i^* = 0, \quad i \notin \mathcal{I}_g(z^*), \\ &v_i^* = 0, \quad i \notin \mathcal{I}_h(z^*), \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_g(z^*) &:= \big\{i : g_i(z^*) = 0,\ i = 1, \ldots, m\big\}, \\ \mathcal{I}_h(z^*) &:= \big\{i : h(z^*) = 0,\ i = 1, \ldots, m\big\}. \end{aligned}$$

Moreover,

- z^* is called *Clarke* (C-) stationary to (1) iff $u_i^* v_i^* \ge 0$ holds for each $i \in \mathcal{I}_g(z^*) \cap \mathcal{I}_h(z^*)$;
- z^* is called *Mordukhvich* (M-) stationary to (1) iff $\min(u_i^*, v_i^*) > 0$ or $u_i^* v_i^* = 0$ holds for each $i \in \mathcal{I}_g(z^*) \cap \mathcal{I}_h(z^*)$;
- z^* is called *Strong* (S-) stationary to (1) iff $u_i^* \ge 0$ and $v_i^* \ge 0$ holds for each $i \in \mathcal{I}_g(z^*) \cap \mathcal{I}_h(z^*)$.

3 Sensitivity Analysis

We consider the following one stage SMPEC[4]:

$$\begin{array}{ll} \min & \mathbb{E}_P\big[f\big(z, \xi(\omega)\big)\big] \\ \text{s.t.} & z \in Z, \\ & 0 \le \mathbb{E}_P\big[G\big(z, \xi(\omega)\big)\big] \perp \mathbb{E}_P\big[H\big(z, \xi(\omega)\big)\big] \ge 0, \end{array} \tag{2}$$

where Z is a nonempty, closed and convex subset of $\mathbb{R}^n$, f, G and H are, respectively, continuously differentiable functions from $\mathbb{R}^n \times \mathbb{R}^q$ to $\mathbb{R}, \mathbb{R}^m, \mathbb{R}^m$, $\xi : \Omega \to \Xi$

is a vector of random variables defined on probability $(\Omega, \mathcal{F}, P)$ with support set $\Xi \subset \mathbb{R}^q$, and $\mathbb{E}_P[\cdot]$ denotes the expected value with respect to probability measure P, and '$\perp$' denotes the perpendicularity of two vectors.

Let $\mathscr{P}(\Omega)$ denote the set of all Borel probability measures. Assuming that $Q \in \mathscr{P}(\Omega)$ is close to P under some metric (to be defined shortly), we investigate in this section the following optimization problem:

$$\begin{array}{ll} \min & \mathbb{E}_Q\big[f\big(z, \xi(\omega)\big)\big] \\ \text{s.t.} & z \in Z, \\ & 0 \le \mathbb{E}_Q\big[G\big(z, \xi(\omega)\big)\big] \perp \mathbb{E}_Q\big[H\big(z, \xi(\omega)\big)\big] \ge 0, \end{array} \tag{3}$$

which is regarded as a perturbation of (2). Specifically, we study the relationship between the perturbed problem (3) and initial problem (2) in terms of optimal values, optimal solutions and stationary points when Q is close to P. To simplify notation and discussion, we assume throughout this paper that Z is bounded and the feasible set of (3) is nonempty. Our results can be easily extended to the case when Z is unbounded.

Let us start by introducing a distance function for the set $\mathscr{P}(\Omega)$, which is appropriate for our problem. Define the set of functions:

$$\begin{aligned} \mathscr{G} := & \big\{g(\cdot) = f(z, \cdot) : z \in Z, \big\} \cup \big\{g(\cdot) = G_i(z, \cdot) : z \in Z, i = 1, \dots, m\big\} \\ & \cup \big\{g(\cdot) = H_i(z, \cdot) : z \in Z, i = 1, \dots, m\big\}. \end{aligned}$$

The distance function for the elements in set $\mathscr{P}(\Omega)$ is defined by

$$\mathscr{D}(P, Q) := \sup_{g \in \mathscr{G}} \big|\mathbb{E}_P[g] - \mathbb{E}_Q[g]\big|.$$

This type of distance was introduced by Römisch [13, Sect. 2.2] for the stability analysis of stochastic programming and was called pseudo metric. It is well-known that $\mathscr{D}$ is non-negative, symmetric and satisfies the triangle inequality; see [13, Sect. 2.1]. Throughout this section, we use the following notation:

$$\begin{aligned} \mathcal{F}(Q) &:= \big\{z \in Z : 0 \le \mathbb{E}_Q\big[G\big(z, \xi(\omega)\big)\big] \perp \mathbb{E}_Q\big[H\big(z, \xi(\omega)\big)\big] \ge 0\big\}, \\ \vartheta(Q) &:= \inf\big\{\mathbb{E}_Q\big[f(z, \xi)\big] : z \in \mathcal{F}(Q)\big\}, \\ S_{opt}(Q) &:= \big\{z \in \mathcal{F}(Q) : \vartheta(Q) = \mathbb{E}_Q\big[f(z, \xi)\big]\big\}, \\ \mathscr{P}_{\mathscr{G}}(\Omega) &:= \Big\{Q \in \mathscr{P}(\Omega) : -\infty < \inf_{g(\xi) \in \mathscr{G}} \mathbb{E}_Q\big[g(\xi)\big] \text{ and } \inf_{g(\xi) \in \mathscr{G}} \mathbb{E}_Q\big[g(\xi)\big] < \infty\Big\}. \end{aligned}$$

It is easy to observe that, for $P, Q \in \mathscr{P}_{\mathscr{G}}(\Omega)$, $\mathscr{D}(P, Q) < \infty$.

In what follows, we use Klatte's stability results [17, 18] to derive the Lipschitz property of optimal value function and semi-continuity of the set-valued mapping of optimal solution. A sufficient condition for Klatte's result is the *pseudo-Lipschitz continuity* of the feasible set mapping $\mathcal{F}(\cdot)$ at every point of $z \in \mathcal{F}(P)$, that is, there exist a positive number β^*, a neighborhood U^* of P and a neighborhood Z^* of z such that

$$\mathrm{dist}_V\big(\mathcal{F}(Q_1) \cap Z^*, \mathcal{F}(Q_2)\big) \le \beta^* \mathscr{D}(Q_1, Q_2), \quad \forall Q_1, Q_2 \in U^*.$$

The property is also known as Aubin property; see [21]. To this end, we make the following assumptions.

Assumption 3.1 *There exists a neighborhood U_P of P and there exist positive constants β and δ such that for any $Q \in U_P$ and $z \in Z \cap \mathcal{B}(\mathcal{F}(Q), \delta)$*

$$d\big(z, \mathcal{F}(Q)\big) \le \beta \big\| \min\big\{\mathbb{E}_Q\big[G(z,\xi)\big], \mathbb{E}_Q\big[H(z,\xi)\big]\big\}\big\|, \tag{4}$$

where $\mathcal{B}(S, \delta)$ denotes the δ neighborhood of set S.

Assumption 3.2 *There exists a neighborhood U_P of P and there exist positive constants β and δ such that for any $Q \in U_P$ and $z \in Z \cap \mathcal{B}(\mathcal{F}(Q), \delta)$*

$$\begin{aligned} &d\big(z, \mathcal{F}(Q)\big) \\ &\quad \le \beta \big\| \big(-\mathbb{E}_Q\big[G(z,\xi)\big], -\mathbb{E}_Q\big[H(z,\xi)\big], \mathbb{E}_Q\big[G(z,\xi)\big] \circ \mathbb{E}_Q\big[H(z,\xi)\big]\big)_+ \big\|, \end{aligned} \tag{5}$$

where $(a)_+ := \max\{a, 0\}$ for a vector "a" and the maximum is taken componentwise and "$\circ$" denotes the Hadamard product.

In the literature [23, 24], the inequality (4) is known as *natural type error bound* whereas inequality (5) is known as *S-type error bound* of the complementarity constraint. In the case when $\mathbb{E}_Q[G(z,\xi)] := z$, $\mathbb{E}_Q[F(z,\xi)]$ is a Lipschiz continuous, uniform P-function with the Lipschiz modulus L_Q being upper bounded by a positive constant L, and the constant γ_Q (see Definition 2.1) being lower bounded by $\gamma > 0$ over U_P, Assumption 3.1 holds with $\beta = \frac{1+L}{\gamma}$, that is,

$$d\big(z, \mathcal{F}(Q)\big) \le \frac{1+L}{\gamma} \big\| \min\big\{\mathbb{E}_Q\big[G(z,\xi)\big], \mathbb{E}_Q\big[H(z,\xi)\big]\big\}\big\|, \quad Q \in U_P.$$

See [20, 25, 26] for a more detailed discussion of natural type error bound. S-type error bound is often related to monotone complementary problems. Let $\mathbb{E}[G(z,\xi)] := z$, and $H(z,\xi) := A(\xi)z$. If for all $Q \in U_P$, $\mathbb{E}_Q[A(\xi)]$ is a semi-definite matrix and $0 \le z \perp \mathbb{E}_Q[H(z,\xi)] \ge 0$ has a non-degenerate solution, then there exists a constant $\beta_Q > 0$ such that (5) hold. If β_Q is upper bounded by β, then Assumption 3.2 holds. Moreover, if (5) is replaced by the following S-type error bound:

$$\begin{aligned} &d(z, \mathcal{F}(Q)) \\ &\quad \le \beta\Big(\big\| \big(-\mathbb{E}_Q\big[G(z,\xi)\big], -\mathbb{E}_Q\big[H(z,\xi)\big], \mathbb{E}_Q\big[G(z,\xi)\big] \circ \mathbb{E}_Q\big[H(z,\xi)\big]\big)_+ \big\| \\ &\qquad + \sqrt{\big\| \big(-\mathbb{E}_Q\big[G(z,\xi)\big], -\mathbb{E}_Q\big[H(z,\xi)\big], \mathbb{E}_Q\big[G(z,\xi)\big] \circ \mathbb{E}_Q\big[H(z,\xi)\big]\big)_+ \big\|}\Big), \end{aligned} \tag{6}$$

then we can abandon the non-degenerate condition. For more information of the S-type error bound; see [24, 27–30]. We refer readers interested in the topic to monograph [31] and a survey paper by Pang [23] on error bound of variational inequalities and complementarity problems.

3.1 Stability of Optimal Value and Optimal Solution

With preparations in the preceding subsection, we are now ready to investigate the stability of SMPEC (2). The proposition below establishes the Lipschitz continuity of the feasible set mapping $\mathcal{F}(Q)$ under Assumption 3.1 or Assumption 3.2.

Proposition 3.1 *Let Assumption 3.1 or Assumption 3.2 hold. Suppose that there exist a neighborhood $\tilde{U}_P$ of P and a non-negative function $\kappa(\xi)$ such that* $\max(\|G(z,\xi)\|, \|H(z,\xi)\|) \le \kappa(\xi)$ *and $\mathbb{E}_Q[\kappa(\xi)] < \infty$ for $Q \in \tilde{U}_P$ and $z \in Z$. Then the following assertions hold*:

(i) *the solution set $S_{opt}(P)$ is nonempty and compact*;
(ii) *the graph of the feasible set mapping $\mathcal{F}(\cdot)$ is closed*;
(iii) *there exist a neighborhood U^* of P and a positive constant β^* such that the feasible set mapping $\mathcal{F}(Q)$ is Lipschitz continuous with modulus β^* on U^*, that is*,

$$\mathrm{dist_H}\big(\mathcal{F}(Q_1), \mathcal{F}(Q_2)\big) \le \beta^* \mathcal{D}(Q_1, Q_2), \quad \forall Q_1, Q_2 \in U^*.$$

Proof We first prove the theorem under Assumption 3.1.

Part (i) follows from the continuity of $\mathbb{E}_P[f(z,\xi)]$, $\mathbb{E}_P[G(z,\xi)]$ and $\mathbb{E}_P[H(z,\xi)]$ on Z and compactness of Z.

Part (ii). Note that the constraints of (3) can be reformulated as generalized equations:

$$0 \in \Gamma_Q(z) := -\Psi_Q(z) + \mathcal{N} \times Z,$$

where

$$\Psi_Q(z) := \begin{pmatrix} \mathbb{E}_Q[G(z,\xi)] \\ \mathbb{E}_Q[H(z,\xi)] \\ z \end{pmatrix}$$

and

$$\mathcal{N} := \big\{(x,y) : 0 \le x \perp y \ge 0,\ x \in \mathbb{R}^m,\ y \in \mathbb{R}^m\big\}.$$

By virtue of [32, Lemma 4.2], we can easily prove that the feasible set mapping $\mathcal{F}(\cdot)$, as the solution set mapping to the generalized equations, is upper semi-continuous and the compactness of Z restrict the set-valued mapping to be bounded. This implies the closeness of the graph of $\mathcal{F}(\cdot)$.

Part (iii). Let the neighborhood U_P and δ be given as in Assumption 3.1. By [32, Lemma 4.2], there exists a neighborhood U of P such that for $Q \in U$

$$\mathrm{dist_H}\big(\mathcal{F}(Q), \mathcal{F}(P)\big) \le \delta/2.$$

Let $U^* = U \cap U_P \cap \tilde{U}_P$ and $Q_1, Q_2 \in U^*$. Observe that for any $z \in \mathcal{F}(Q_1)$,

$$\big\|\min\big\{\mathbb{E}_{Q_1}\big[G(z,\xi)\big], \mathbb{E}_{Q_1}\big[H(z,\xi)\big]\big\}\big\| = 0.$$

By Assumption 3.1, there exists a positive constant β such that for any $z \in \mathcal{F}(Q_1)$

$$\begin{aligned}
&d\big(z, \mathcal{F}(Q_2)\big) \\
&\quad \le \beta \big\| \min\big\{\mathbb{E}_{Q_2}\big[G(z,\xi)\big], \mathbb{E}_{Q_2}\big[H(z,\xi)\big]\big\} \big\| \\
&\quad = \beta \big\| \min\big\{\mathbb{E}_{Q_2}\big[G(z,\xi)\big], \mathbb{E}_{Q_2}\big[H(z,\xi)\big]\big\} \big\| \\
&\qquad - \beta \big\| \min\big\{\mathbb{E}_{Q_1}\big[G(z,\xi)\big], \mathbb{E}_{Q_1}\big[H(z,\xi)\big]\big\} \big\| \\
&\quad \le \beta \big\| \min\big\{\mathbb{E}_{Q_2}\big[G(z,\xi)\big], \mathbb{E}_{Q_2}\big[H(z,\xi)\big]\big\} - \min\big\{\mathbb{E}_{Q_1}\big[G(z,\xi)\big], \mathbb{E}_{Q_1}\big[H(z,\xi)\big]\big\} \big\| \\
&\quad \le \beta \big\| \mathbb{E}_{Q_2}\big[G(z,\xi)\big] - \mathbb{E}_{Q_1}\big[G(z,\xi)\big] \big\| + \beta \big\| \mathbb{E}_{Q_2}\big[H(z,\xi)\big] - \mathbb{E}_{Q_1}\big[H(z,\xi)\big] \big\| \\
&\quad \le \beta \Big(\max_{z\in Z} \big\| \mathbb{E}_{Q_2}\big[G(z,\xi)\big] - \mathbb{E}_{Q_1}\big[G(z,\xi)\big] \big\| + \max_{z\in Z} \big\| \mathbb{E}_{Q_2}\big[H(z,\xi)\big] \\
&\qquad - \mathbb{E}_{Q_1}\big[H(z,\xi)\big] \big\| \Big) \\
&\quad \le 2m\beta \mathcal{D}(Q_1, Q_2),
\end{aligned}$$

where the third inequality follows from the fact that

$$\big|\min\{a_2, b_2\} - \min\{a_1, b_1\}\big| \le |a_2 - a_1| + |b_2 - b_1|, \quad \forall a_1, b_1, a_2, b_2 \in \mathbb{R},$$

and “m” is the dimension of $G(z,\xi)$. Then $\mathrm{dist}_{\mathrm{V}}(\mathcal{F}(Q_1), \mathcal{F}(Q_2)) \le 2m\beta\mathcal{D}(Q_1, Q_2)$. In the same manner, we can show that for any $z \in \mathcal{F}(Q_2)$,

$$d\big(z, \mathcal{F}(Q_1)\big) \le 2m\beta\mathcal{D}(Q_2, Q_1),$$

which yields $\mathrm{dist}_{\mathrm{V}}(\mathcal{F}(Q_2), \mathcal{F}(Q_1)) \le 2m\beta\mathcal{D}(Q_1, Q_2)$. Summarizing the discussions above, we have

$$\begin{aligned}
\mathrm{dist}_{\mathrm{H}}\big(\mathcal{F}(Q_1), \mathcal{F}(Q_2)\big) &= \max\big\{\mathrm{dist}_{\mathrm{V}}\big(\mathcal{F}(Q_1), \mathcal{F}(Q_2)\big), \mathrm{dist}_{\mathrm{V}}\big(\mathcal{F}(Q_2), \mathcal{F}(Q_1)\big)\big\} \\
&\le 2m\beta\mathcal{D}(Q_1, Q_2).
\end{aligned}$$

Part (iii) holds with $\beta^* := 2m\beta$.

Next, we prove the theorem under Assumption 3.2. Note that Part (i) and Part (ii) do not involve Assumption 3.1 or Assumption 3.2, we just need to show Part (iii).

Observe that for any $z \in \mathcal{F}(Q_1)$,

$$\big\| \big(-\mathbb{E}_{Q_1}\big[G(z,\xi)\big], -\mathbb{E}_{Q_1}\big[H(z,\xi)\big], \mathbb{E}_{Q_1}\big[G(z,\xi)\big]\mathbb{E}_{Q_1}\big[H(z,\xi)\big]\big)_+ \big\| = 0.$$

By Assumption 3.2, there exists a positive constant β such that for any $z \in \mathcal{F}(Q_1)$

$$\begin{aligned}
&d\big(z, \mathcal{F}(Q_2)\big) \\
&\quad \le \beta \big\| \big(-\mathbb{E}_{Q_2}\big[G(z,\xi)\big], -\mathbb{E}_{Q_2}\big[H(z,\xi)\big], \mathbb{E}_{Q_2}\big[G(z,\xi)\big] \circ \mathbb{E}_{Q_2}\big[H(z,\xi)\big]\big)_+ \big\| \\
&\quad = \beta \big\| \big(-\mathbb{E}_{Q_2}\big[G(z,\xi)\big], -\mathbb{E}_{Q_2}\big[H(z,\xi)\big], \mathbb{E}_{Q_2}\big[G(z,\xi)\big] \circ \mathbb{E}_{Q_2}\big[H(z,\xi)\big]\big)_+ \big\| \\
&\qquad - \beta \big\| \big(-\mathbb{E}_{Q_1}\big[G(z,\xi)\big], -\mathbb{E}_{Q_1}\big[H(z,\xi)\big], \mathbb{E}_{Q_1}\big[G(z,\xi)\big] \circ \mathbb{E}_{Q_1}\big[H(z,\xi)\big]\big)_+ \big\|
\end{aligned}$$

$$\le \beta \big\| \big(-\mathbb{E}_{Q_2}[G(z,\xi)], -\mathbb{E}_{Q_2}[H(z,\xi)], \mathbb{E}_{Q_2}[G(z,\xi)] \circ \mathbb{E}_{Q_2}[H(z,\xi)]\big)_+ - \big(-\mathbb{E}_{Q_1}[G(z,\xi)], -\mathbb{E}_{Q_1}[H(z,\xi)], \mathbb{E}_{Q_1}[G(z,\xi)] \circ \mathbb{E}_{Q_1}[H(z,\xi)]\big)_+ \big\|$$

$$\le 2\hat{\beta}\beta \Big(\max_{z\in Z} \big\| \mathbb{E}_{Q_2}[G(z,\xi)] - \mathbb{E}_{Q_1}[G(z,\xi)] \big\| + \max_{z\in Z} \big\| \mathbb{E}_{Q_2}[H(z,\xi)] - \mathbb{E}_{Q_1}[H(z,\xi)] \big\| \Big)$$

$$\le 4m\hat{\beta}\beta \mathcal{D}(Q_1, Q_2),$$

where

$$\hat{\beta} = \sup_{z\in Z, Q\in U_P} \max\big\{ \big\|\mathbb{E}_Q[G(z,\xi)]\big\|, \big\|\mathbb{E}_Q[G(z,\xi)]\big\| \big\} + 1$$

and "m" is the dimension of $G(z,\xi)$. Since $G(z,\xi)$ and $H(z,\xi)$ are integrable bounded uniformly over Z and $\bar{U}_P$, $\hat{\beta}$ is bounded. Then, there exists a positive constant $\beta^* := 4m\hat{\beta}\beta$ such that

$$\mathbb{D}\big(\mathcal{F}(Q_1), \mathcal{F}(Q_2)\big) \le \beta^* \mathcal{D}(Q_1, Q_2).$$

The rest are straightforward. The proof is complete. □

Proposition 3.1(iii) shows that the feasible set mapping of problem (3) is Lipschitz continuous with respect to probability measure over U^* under the distance $\mathcal{D}$. Using this property, we are ready to establish the first main stability result. Moreover, if S type error bound condition (5) is replaced by (6) it is not difficult to get a similar result of Proposition 3.1(iii), that is,

$$\mathrm{dist_H}\big(\mathcal{F}(Q_1), \mathcal{F}(Q_2)\big) \le \beta^* \widehat{\mathcal{D}}(Q_1, Q_2) := \beta^* \sqrt{\mathcal{D}(Q_1, Q_2)}, \quad \forall Q_1, Q_2 \in U^*.$$

Theorem 3.1 *Let the conditions of Proposition* 3.1 *hold. Suppose also that the Lipschitz modulus of $f(z,\xi)$ with respect to z is bounded by an integrable function $\kappa(\xi) > 0$. Then the following assertions hold*:

- (i) *there exists a neighborhood U_P^1 of P such that the optimal solution set of problem* (3), *denoted by $S_{opt}(Q)$, is not empty for $Q \in U_P^1$*;
- (ii) *the optimal solution set mapping $S_{opt}(\cdot)$ is upper semi-continuous at point P*;
- (iii) *there exist a neighborhood U_P^2 of P and a positive constant L^* such that the optimal value function of problem* (3) *is continuous at P and satisfies the following estimation* (*calmness at point P*):

$$\big|\vartheta(Q) - \vartheta(P)\big| \le L^* \mathcal{D}(Q, P), \quad \forall Q \in U_P^2.$$

Proof It follows from Proposition 3.1 that there exists a neighborhood U_P of P such that the feasible set mapping $\mathcal{F}(\cdot)$ is Lipschitz continuous on U_P (which implies the Aubin property). The rest follows straightforwardly from [18, Theorem 1] ([12, Theorem 2.3] or [13, Theorem 5] in stochastic programming). The proof is complete. □

Theorem 3.1 asserts that the optimal solution set mapping $S_{opt}(\cdot)$ is nonempty near P and upper semi-continuous at P. In order to quantify this upper semi-continuity property, we need a growth condition on the objective function in a neighborhood of the optimal solution $S_{opt}(P)$ to problem (2). Instead of imposing a specific growth condition, here we consider the following general growth function (see [12, 21]):

$$\Lambda(\nu) := \min\bigl\{\mathbb{E}_P\bigl[f(z,\xi)\bigr] - \vartheta^* : d\bigl(z, S_{opt}(P)\bigr) \geq \nu, z \in Z\bigr\} \tag{7}$$

of problem (2), where ϑ^* denotes the optimal value of problem (2), and the associated function,

$$\widetilde{\Lambda}(\varrho) := \varrho + \Lambda^{-1}(2\varrho), \quad \varrho \geq 0.$$

We have the following result.

Corollary 3.1 *Let the assumptions of Theorem 3.1 hold. Then there exist a neighborhood U_P of P and a positive constant L such that*

$$\emptyset \neq S_{opt}(Q) \subseteq S_{opt}(P) + \widetilde{\Lambda}\bigl(L\mathscr{D}(Q,P)\bigr)\mathcal{B},$$

for any $Q \in U_P$, where $\mathcal{B}$ denotes the closed unit ball.

Corollary 3.1 provides a quantitative upper semi-continuity of the set of optimal solutions; see [12, Theorem 2.4] for a detailed proof and [21, Theorem 7.64] for earlier discussions about functions $\Lambda(\cdot)$ and $\widetilde{\Lambda}(\cdot)$.

3.2 Stability of Stationary Points

It is well-known in the literature that MPEC problems are generically non-convex due to their combinatorial nature of the constraints, which means that one may obtain a stationary point in solving the perturbed SMPEC (3). This motivates us to undertake stability analysis of stationary points, in addition to that of optimal value and optimal solutions.

Following a recent work by Lin et al. [19], we can reformulate the first order optimality conditions which characterize the M-stationarity as a constrained generalized equation:

$$0 \in \Phi_P(z,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v) + \mathcal{N}_Z(z) \times 0_{7m+2}, \tag{8}$$

where 0_m denotes a m-dimensional zero vector, $(z,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v) \in \mathcal{W}$ and

$$\Phi_P(z,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v)$$

$$= \begin{pmatrix} \nabla\mathbb{E}_P[f(z,\xi)] - \nabla\mathbb{E}_P[G(z,\xi)]u - \nabla\mathbb{E}_P[H(z,\xi)]v \\ \alpha_1 - \mathbb{E}_P[G(z,\xi)] \\ \alpha_2 - \mathbb{E}_P[H(z,\xi)] \\ \alpha_1^T\alpha_2 \\ u \circ \alpha_1 \\ v \circ \alpha_2 \\ \beta_1 - u \circ v \\ \beta_3^T\beta_4 \\ \beta_2 - \beta_3 - u \\ \beta_2 - \beta_4 - v \end{pmatrix}, \tag{9}$$

$$\mathcal{W} = \big\{w \,|\, w = (z,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v), z \in Z, \alpha_1,\alpha_2 \ge 0; \beta_i \ge 0 \\ (i = 1,2,3,4)\big\}. \tag{10}$$

This means that $w \in \mathcal{W}$ is an M-stationary pair iff it is a solution of the stochastic generalized equation (8) and hence studying the stability of the stationary point amounts to that of the generalized equation.

For studying the stability of stationary points, we need to enlarge the set $\mathcal{G}$. Denote

$$\mathcal{G}^* := \mathcal{G} \cup \big\{g(\cdot) = \nabla f(z,\cdot), \big(\nabla G_i(z,\cdot)\big)_j, \big(\nabla H_i(z,\cdot)\big)_j : z \in Z, \\ 1 \le i \le m, 1 \le j \le n\big\}$$

and

$$\mathcal{D}^*(P,Q) := \sup_{g\in\mathcal{G}^*} \big|\mathbb{E}_P[g] - \mathbb{E}_Q[g]\big|.$$

We have the following stability result for stationary points.

Theorem 3.2 *Let Q^N be a sequence of probability measures which converge to P in distance $\mathcal{D}^*$ and w^N be the corresponding M-stationary pair. Let w^* be a limiting point of sequence $\{w^N\}$. Then w^* is an M-stationary pair of* (2).

Proof By virtue of [32, Lemma 4.2], it suffices to show that

$$\lim_{N\to\infty} \sup_{w\in\mathcal{B}(w^*)\cap\mathcal{W}} \big\|\Phi_{Q^N}(w) - \Phi_P(w)\big\| = 0, \tag{11}$$

where $\mathcal{W}$ is defined as in (10) and $\mathcal{B}(w^*)$ denotes the unit closed ball centered at w^*. Observe that

$$\begin{aligned} &\big\|\Phi_{Q^N}(w) - \Phi_P(w)\big\| \\ &\quad\le \big\|\big(\nabla\mathbb{E}_{Q^N}\big[f(z,\xi)\big] - \nabla\mathbb{E}_{Q^N}\big[G(z,\xi)\big]u - \nabla\mathbb{E}_{Q^N}\big[H(z,\xi)\big]v\big) \\ &\qquad - \big(\nabla\mathbb{E}_P\big[f(z,\xi)\big] - \nabla\mathbb{E}_P\big[G(z,\xi)\big]u - \nabla\mathbb{E}_P\big[H(z,\xi)\big]v\big)\big\| \\ &\qquad + \big\|\mathbb{E}_{Q^N}\big[G(z,\xi)\big] - \mathbb{E}_P\big[G(z,\xi)\big]\big\| + \big\|\mathbb{E}_{Q^N}\big[H(z,\xi)\big] - \mathbb{E}_P\big[H(z,\xi)\big]\big\| \\ &\quad\le \big\|\nabla\mathbb{E}_{Q^N}\big[f(z,\xi)\big] - \nabla\mathbb{E}_P\big[f(z,\xi)\big]\big\| + \gamma^*\big\|\nabla\mathbb{E}_{Q^N}\big[G(z,\xi)\big] - \nabla\mathbb{E}_P\big[G(z,\xi)\big]\big\| \end{aligned}$$

$$+\gamma^*\big\|\nabla\mathbb{E}_{Q^N}\big[H(z,\xi)\big]-\nabla\mathbb{E}_P\big[H(z,\xi)\big]\big\|$$
$$+\big\|\mathbb{E}_{Q^N}\big[G(z,\xi)\big]-\mathbb{E}_P\big[G(z,\xi)\big]\big\|+\big\|\mathbb{E}_{Q^N}\big[H(z,\xi)\big]-\mathbb{E}_P\big[H(z,\xi)\big]\big\|, \quad (12)$$

where

$$\gamma^*=\sup_{(u,v)\in\Pi_{(u,v)}(\mathcal{B}(w^*)\cap\mathcal{W})}\big\|(u,v)\big\|. \quad (13)$$

By the definition of $\mathcal{D}^*$, we have

$$\begin{aligned}\lim_{N\to\infty}\sup_{z\in Z}\big\|\mathbb{E}_{Q^N}\big[\psi(z,\xi)\big]-\mathbb{E}_P\big[\psi(z,\xi)\big]\big\|=0,\\ \lim_{N\to\infty}\sup_{z\in Z}\big\|\nabla\mathbb{E}_{Q^N}\big[\psi(z,\xi)\big]-\nabla\mathbb{E}_P\big[\psi(z,\xi)\big]\big\|=0,\end{aligned} \quad (14)$$

where $\psi := f, G$ or H. Combining (12)–(14), we obtain (11). The proof is complete. □

We make a few comments on Theorem 3.2. First, the theorem does not require error bound Assumption 3.1 or 3.2 in that the first order optimality condition is established through generalized equations (9) and [32, Lemma 4.2] instead of the pseudo-Lipschitz continuity of feasible solution set mapping as for the optimal values and optimal solutions. Second, it is possible to derive Lipschitz-like (calmness) property for the set of M-stationary points in terms of the perturbation of probability Q, as in Theorem 3.1, in the case when the generalized equations (9) satisfy metric regularity at M-stationary points of the true problem (2). Third, similar stability results can be derived for C- and S-stationary points by reformulating the first order optimality conditions characterizing the latter as a system of generalized equations. We omit the details.

4 Empirical Probability Measure

In this section, we discuss a popular special case when Q^N is an empirical probability measure. That is,

$$Q^N := \frac{1}{N}\sum_{k=1}^{N}\mathbb{I}_{\xi^k}(\omega),$$

where $\xi^1,\dots,\xi^N$ is an independent and identically distributed sampling of ξ and

$$\mathbb{I}_{\xi^k}(\omega) := \begin{cases}1, & \text{if } \xi(\omega)=\xi^k,\\ 0, & \text{if } \xi(\omega)\neq\xi^k.\end{cases}$$

In this section, we consider the case that P is approximated by empirical probability measure which is also known as sample average approximation:

$$\begin{array}{ll}\min & f^N(z)\\ \text{s.t.} & z\in Z,\\ & 0\le G^N(z)\perp H^N(z)\ge 0,\end{array} \quad (15)$$

where

$$f^N(z) := \frac{1}{N}\sum_{i=1}^{N} f\bigl(z,\xi^i\bigr), \qquad G^N(z) := \frac{1}{N}\sum_{i=1}^{N} G\bigl(z,\xi^i\bigr),$$
$$H^N(z) := \frac{1}{N}\sum_{i=1}^{N} H\bigl(z,\xi^i\bigr).$$

It is well-known that Q^N converges weakly to P w.p.1. In [7], Meng and Xu proved under some moderate conditions that weak stationary point of sample average approximated MPEC problems converges to its true counterpart with probability approaching one at exponential rate as the sample size tends to infinity. In this section, we derive the exponential rate of convergence for optimal value and M-stationary points which are of more interest in the literature of MPECs.

4.1 Optimal Value

If $f(z,\xi)$, $G(z,\xi)$, $H(z,\xi)$ are dominated by an integrable function on Z and Q^N be the empirical measure, it is easy to show by [33, Proposition 7, Chap. 6] that $\mathcal{D}(Q^N, P)$ tends to zero with probability one. Therefore Theorem 3.1 implies immediately almost sure convergence of optimal values and optimal solutions of the SAA problem to their true counterparts. Our focus here is to establish the exponential rate of convergence which provides some insight on the quantitative behavior of the SAA problem as sample size increases. We need the following assumption.

Assumption 4.1 *Let $\theta(z,\xi)$ denote any element in the collection of functions*

$$\bigl\{f(z,\xi), G_i(z,\xi), H_i(z,\xi), i=1,\dots,m\bigr\}.$$

Then $\theta(z,\xi)$ possess the following properties:

(a) *for every $z \in Z$ the moment generating function $\mathbb{E}[e^{(\theta(z,\xi)-\mathbb{E}_P[\theta(z,\xi)])t}]$ of the random variable $\theta(z,\xi)-\mathbb{E}_P[\theta(z,\xi)]$ is finite valued for t close to* 0;
(b) *there exist a* (*measurable*) *function $\kappa_1(\xi)$ and a constant $\gamma_1 > 0$, such that*

$$\bigl|\theta(z,\xi)-\theta(z',\xi)\bigr| \le \kappa_1(\xi)\|z-z'\|^{\gamma_1},$$

for all $\xi \in \Xi$ and $z', z \in Z$;
(c) *the moment generating function $M_{\kappa_1}(t)$ of $\kappa_1(\xi)$, is finite valued for all t in a neighborhood of zero.*

Assumption 4.1(a) means that the random variables $\theta(z,\xi)-\mathbb{E}_P[\theta(z,\xi)]$ does not have a heavy tail distribution. In particular, it holds if the random variable ξ has a bounded support set; see [9]. Assumption 4.1(b) requires global Hölder continuity of $\theta(z,\xi)$ in z independent of ξ. Assumption 4.1(c) requires $\mathbb{E}_P[\kappa_1(\xi)]$ to be finite.

Theorem 4.1 *Let the conditions of Theorem 3.1 and Assumptions 4.1 hold. Then, for any small positive number* ϵ, *there exist positive constants* $C(\epsilon)$ *and* $\beta(\epsilon)$ *independent of* N *such that*

$$\text{Prob}\{|\vartheta(Q^N) - \vartheta(P)| \geq \epsilon\} \leq C(\epsilon)e^{-N\beta(\epsilon)}, \tag{16}$$

for N *sufficiently large.*

Proof We first estimate $\text{Prob}\{\mathcal{D}(Q^N, P) \geq \epsilon\}$. By definition,

$$\begin{aligned}
&\text{Prob}\{\mathcal{D}(Q^N, P) \geq \epsilon\} \\
&\quad\leq \text{Prob}\Big\{\sup_{z\in Z}\big(|\mathbb{E}_{Q^N}[f(z,\xi)] - \mathbb{E}_P[f(z,\xi)]| + \|\mathbb{E}_{Q^N}[G(z,\xi)] - \mathbb{E}_P[G(z,\xi)]\| \\
&\qquad + \|\mathbb{E}_{Q^N}[H(z,\xi)] - \mathbb{E}_P[H(z,\xi)]\|\big) \geq \epsilon\Big\} \\
&\quad\leq \text{Prob}\Big\{\sup_{z\in Z}|\mathbb{E}_{Q^N}[f(z,\xi)] - \mathbb{E}_P[f(z,\xi)]| \geq \epsilon/3\Big\} + \text{Prob}\Big\{\sup_{z\in Z}\|\mathbb{E}_{Q^N}[G(z,\xi)] \\
&\qquad - \mathbb{E}_P[G(z,\xi)]\| \geq \epsilon/3\Big\} \\
&\qquad + \text{Prob}\Big\{\sup_{z\in Z}\|\mathbb{E}_{Q^N}[H(z,\xi)] - \mathbb{E}_P[H(z,\xi)]\| \geq \epsilon/3\Big\}.
\end{aligned}$$

By virtue of [9, Theorem 5.1] and Assumption 4.1, there exist $C_1(\epsilon/3)$, $\beta_1(\epsilon/3)$, $C_2(\epsilon/3)$, $\beta_2(\epsilon/3)$ and $C_3(\epsilon/3)$, $\beta_3(\epsilon/3)$ such that

$$\text{Prob}\Big\{\sup_{z\in Z}|\mathbb{E}_{Q^N}[f(z,\xi)] - \mathbb{E}_P[f(z,\xi)]| \geq \epsilon/3\Big\} \leq C_1(\epsilon/3)e^{-N\beta_1(\epsilon/3)},$$

$$\text{Prob}\Big\{\sup_{z\in Z}\|\mathbb{E}_{Q^N}[G(z,\xi)] - \mathbb{E}_P[G(z,\xi)]\| \geq \epsilon/3\Big\} \leq C_2(\epsilon/3)e^{-N\beta_2(\epsilon/3)}, \tag{17}$$

$$\text{Prob}\Big\{\sup_{z\in Z}\|\mathbb{E}_{Q^N}[H(z,\xi)] - \mathbb{E}_P[H(z,\xi)]\| \geq \epsilon/3\Big\} \leq C_3(\epsilon/3)e^{-N\beta_3(\epsilon/3)}. \tag{18}$$

Then

$$\text{Prob}\{\mathcal{D}(Q^N, P) \geq \epsilon\} \leq C^*(\epsilon)e^{-N\beta^*(\epsilon)} \tag{19}$$

holds with $C^*(\epsilon) = C_1(\epsilon/3) + C_2(\epsilon/3) + C_3(\epsilon/3)$ and $\beta^*(\epsilon) = \min\{\beta_1(\epsilon/3), \beta_2(\epsilon/3), \beta_3(\epsilon/3)\}$.

By Theorem 3.1(iii), there exist a neighborhood U_P of P and a positive L^* such that

$$|\vartheta(Q) - \vartheta(P)| \leq L^*\mathcal{D}(Q,P), \quad \forall Q \in U_P.$$

Subsequently, there exists a sufficiently large N^* such that $Q^N \in U_P$ and

$$\text{Prob}\{|\vartheta(Q^N) - \vartheta(P)| \geq \epsilon\} \leq \text{Prob}\{\mathcal{D}(Q^N, P) \geq \epsilon/L^*\}$$

for $N \geq N^*$. Together with (19), (16) holds with $C(\epsilon) = C^*(\epsilon/L^*)$ and $\beta(\epsilon) = \beta^*(\epsilon/L^*)$. The proof is complete. □

4.2 Stationary Points

We say $w^N = (z^N, \alpha_1^N, \alpha_2^N, \beta_1^N, \beta_2^N, \beta_3^N, \beta_4^N, u^N, v^N) \in \mathcal{W}$ is an M-stationary pair of (15) if

$$0 \in \Phi^N\big(z^N, \alpha_1^N, \alpha_2^N, \beta_1^N, \beta_2^N, \beta_3^N, \beta_4^N, u^N, v^N\big) + \mathcal{N}_Z\big(z^N\big) \times 0_{7m+2}, \qquad (20)$$

where

$$\Phi^N(z, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4, u, v) = \begin{pmatrix} \nabla f^N(z) - \nabla G^N(z)u - \nabla H^N(z)v \\ \alpha_1 - G^N(z) \\ \alpha_2 - H^N(z) \\ \alpha_1^T \alpha_2 \\ u \circ \alpha_1 \\ v \circ \alpha_2 \\ \beta_1 - u \circ v \\ \beta_3^T \beta_4 \\ \beta_2 - \beta_3 - u \\ \beta_2 - \beta_4 - v \end{pmatrix}.$$

Meng and Xu [7] studied the exponential rate convergence of weak stationary point. Here, we extend their result to M-stationary point. We need the following conditions on the moment of the underlying functions.

Assumption 4.2 *Let $\theta(z, \xi)$ denote any element in the collection of functions*

$$\big\{\big(\nabla f(z, \xi)\big)_j, \big(\nabla G_i(z, \xi)\big)_j, \big(\nabla H_i(z, \xi)\big)_j, i = 1, \ldots, m, j = 1, \ldots, n\big\}.$$

Then $\theta(z, \xi)$ possesses the following properties:

(a) *for every $z \in Z$ the moment generating function $\mathbb{E}[e^{(\theta(z,\xi)-\mathbb{E}_P[\theta(z,\xi)])t}]$ of the random variable $\theta(z, \xi) - \mathbb{E}_P[\theta(z, \xi)]$ is finite valued for t close to* 0;

(b) *there exist a* (*measurable*) *function $\kappa_2(\xi)$ and a constant $\gamma_2 > 0$, such that*

$$\big|\theta(z, \xi) - \theta(z', \xi)\big| \leq \kappa_2(\xi)\|z - z'\|^{\gamma_2},$$

for all $\xi \in \Xi$ and $z', z \in Z$;

(c) *the moment generating function $M_{\kappa_2}(t)$ of $\kappa_2(\xi)$, is finite valued for all t in a neighborhood of zero.*

Theorem 4.2 *Let Assumption 4.1 and Assumption 4.2 hold. Let w^N be a sequence of M-stationary pairs satisfying* (20) *and w^* a limiting point of sequence $\{w^N\}$ w.p.*1. *Suppose that*

$$\begin{aligned}&\Upsilon(z,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v)\\&\quad:=\Phi_P(z,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v)+\mathcal{N}_Z(z)\times 0_{7m+2}\end{aligned}$$

is metrically regular at point w^* for 0. *Then for any small positive number ϵ, there exist positive constants $C(\epsilon)>0$, $\beta(\epsilon)>0$ independent of N such that for N sufficiently large*

$$\text{Prob}\{d(w^N,S_P)\geq\lambda\epsilon\}\leq C(\epsilon)e^{-N\beta(\epsilon)},\tag{21}$$

where λ is the regularity modulus of Υ at w^* and S_P denotes the set of M-stationary pair of (2).

Proof By Theorem [34, Theorem 4.1] and the metric regularity of Υ,

$$d(w^N,S_P)\leq\lambda\|\Phi^N(w^N),\Phi_P(w^N)\|.$$

By virtue of the property of $\|\cdot\|$,

$$\begin{aligned}d(w^N,S_P)&\leq\lambda\|\Phi^N(w^N)-\Phi_P(w^N)\|\\&\leq\lambda\|(\nabla f^N(z^N)-\nabla G^N(z^N)u^N-\nabla H^N(z)v^N)\\&\quad-(\mathbb{E}_P[\nabla_z f(z^N,\xi)]-\mathbb{E}_P[\nabla_z G(z^N,\xi)]u^N-\mathbb{E}_P[\nabla_z H(z^N,\xi)]v^N)\|\\&\quad+\lambda\|(G^N(z^N)-\mathbb{E}_P[G(z^N,\xi)]\|+\lambda\|H^N(z^N)-\mathbb{E}_P[H(z^N,\xi)]\|.\end{aligned}$$

Then

$$\begin{aligned}&\text{Prob}\{d(w^N,S_P)\geq\lambda\epsilon\}\\&\quad\leq\text{Prob}\{\|(\nabla f^N(z^N)-\nabla G^N(z^N)u^N-\nabla H^N(z^N)v^N)\\&\qquad-(\mathbb{E}_P[\nabla_z f(z^N,\xi)]-\mathbb{E}_P[\nabla_z G(z^N,\xi)]u^N-\mathbb{E}_P[\nabla_z H(z^N,\xi)]v^N)\|\geq\epsilon/3\}\\&\qquad+\text{Prob}\{\|G^N(z^N)-\mathbb{E}_P[(Gz^N,\xi)]\|\geq\epsilon/3\}\\&\qquad+\text{Prob}\{\|H^N(z^N)-\mathbb{E}_P[H(z^N,\xi)]\|\geq\epsilon/3\}.\end{aligned}\tag{22}$$

As shown in the proof of Theorem 4.1, there exist positive constants $C_2(\epsilon/3)$, $\beta_2(\epsilon/3)$ and $C_3(\epsilon/3)$, $\beta_3(\epsilon/3)$ such that (17)–(18) hold. In the following, we study the exponential rate convergence of the first term on the right side of (22). Note that

$$\begin{aligned}&\text{Prob}\{\|(\nabla f^N(z^N)-\nabla G^N(z^N)u^N-\nabla H^N(z^N)v^N)\\&\qquad-(\mathbb{E}_P[\nabla_z f(z^N,\xi)]-\mathbb{E}_P[\nabla_z G(z^N,\xi)]u^N-\mathbb{E}_P[\nabla_z H(z^N,\xi)]v^N)\|\geq\epsilon/3\}\\&\quad\leq\text{Prob}\{\|\nabla f^N(z^N)-\mathbb{E}_P[\nabla_z f(z^N,\xi)]\|\geq\epsilon/9\}\\&\qquad+\text{Prob}\{\|\nabla G^N(z^N)-\mathbb{E}_P[\nabla_z G(z^N,\xi)]\|\geq\epsilon/9\gamma^*\}\\&\qquad+\text{Prob}\{\|\nabla H^N(z^N)-\mathbb{E}_P[\nabla_z H(z^N,\xi)]\|\geq\epsilon/9\gamma^*\}\end{aligned}$$

$$\le \text{Prob}\Big\{\sup_{z\in Z}\big\|\nabla f^N(z) - \mathbb{E}_P\big[\nabla_z f(z,\xi)\big]\big\| \ge \epsilon/9\Big\}$$

$$+ \text{Prob}\Big\{\sup_{z\in Z}\big\|\nabla G^N(z) - \mathbb{E}_P\big[\nabla_z G(z,\xi)\big]\big\| \ge \epsilon/9\gamma^*\Big\}$$

$$+ \text{Prob}\Big\{\sup_{z\in Z}\big\|\nabla H^N(z) - \mathbb{E}_P\big[\nabla_z H(z,\xi)\big]\big\| \ge \epsilon/9\gamma^*\Big\}, \tag{23}$$

where γ^* is as defined in (13). By virtue of [9, Theorem 5.1], there exist positive constants $C_4(\epsilon/9)$, $\beta_4(\epsilon/9)$, $C_5(\epsilon/9\gamma^*)$, $\beta_5(\epsilon/9\gamma^*)$ and $C_6(\epsilon/9\gamma^*)$, $\beta_6(\epsilon/9\gamma^*)$ such that

$$\text{Prob}\Big\{\sup_{z\in Z}\big\|\nabla f^N(z) - \mathbb{E}_P\big[\nabla_z f(z,\xi)\big]\big\| \ge \epsilon/9\Big\} \le C_4(\epsilon/9)e^{-N\beta_4(\epsilon/9)},$$

$$\text{Prob}\Big\{\sup_{z\in Z}\big\|\nabla G^N(z) - \mathbb{E}_P\big[\nabla_z G(z,\xi)\big]\big\| \ge \epsilon/9\gamma^*\Big\} \le C_5(\epsilon/9\gamma^*)e^{-N\beta_5(\epsilon/9\gamma^*)},$$

$$\text{Prob}\Big\{\sup_{z\in Z}\big\|\nabla H^N(z) - \mathbb{E}_P\big[\nabla_z H(z,\xi)\big]\big\| \ge \epsilon/9\gamma^*\Big\} \le C_6(\epsilon/9\gamma^*)e^{-N\beta_6(\epsilon/9\gamma^*)}.$$

Combining (22)–(23) and the estimate above, we obtain (21) with

$$C(\epsilon) = C_2(\epsilon/3) + C_3(\epsilon/3) + C_4(\epsilon/9\gamma^*) + C_5(\epsilon/9\gamma^*) + C_6(\epsilon/9\gamma^*)$$

and

$$\beta(\epsilon) = \min\big\{\beta_2(\epsilon/3), \beta_3(\epsilon/3), \beta_4(\epsilon/9\gamma^*), \beta_5(\epsilon/9\gamma^*), \beta_6(\epsilon/9\gamma^*)\big\}.$$

The proof is complete. □

Theorem 4.2 derives the exponential rate of convergence for M-stationary points obtained from solving the SAA problem. Similar results can be obtained for C- and S-stationary points. Similar to discussions in [9, 34], it is possible to obtain a precise estimate of constants $C(\epsilon)$ and $\beta(\epsilon)$ under some specific circumstance; we leave this to the interested readers.

5 Conclusions

This paper makes some contributions to the current research in a few aspects. First, it extends the stability analysis of classical one stage stochastic programs with equality and inequality constraints [12] to one stage SMPECs; second, it extends the current research on SMPECs, which is mostly focusing on Monte Carlo approximation (sample average approximation or sample path optimization) by providing an abstract approximation framework which may potentially cover numerical approximation schemes beyond the sample based approximations; third, it provides stability results of M-stationary points and exponential rate of convergence of the M-stationary points under the empirical probability measure approximation, which is usually difficult to obtain.

It might be possible to take this work further in a few directions: the stability analysis in Sect. 3 relies heavily on the error bounds in Assumption 3.1 or Assumption 3.2, which is apparently strong and it might be interesting to explore weaker conditions; the SMPEC model takes a complementarity form, it can be extended to a SMPEC with a general equilibrium constraint (e.g., characterized by a stochastic variational inequality); and finally, it might be interesting to carry out some numerical tests on the schemes covered by the stability analysis in this paper.

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