Math. Meth. Oper. Res. (2006)
DOI 10.1007/s00186-005-0061-1

ORIGINAL ARTICLE

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Modelling the effects of interconnection between electricity markets subject to uncertainty

Received: 19 December 2003 / Accepted: 2 August 2005

Abstract Interconnecting distinct electricity markets by adding a new transmission line affects the outcomes in these markets in a complicated way when there is uncertainty in demand or participant behaviour. We use market distribution functions to examine the effects of interconnection using a single transmission line under the assumption that this line has a differentiable loss function and agents in each of the interconnected markets do not change their behaviour in response to the interconnection. We also show how the case with capacity constraints on flows can be represented with appropriately formulated loss functions. We give analytical formulae for computing market outcomes when the uncertain events in the markets being connected are statistically independent, and show by example how to compute these outcomes when these events are correlated.

Keywords Electricity markets · Optimization · Network interconnection

1 Introduction

Over the last decade, there has been considerable interest in the design and operation of wholesale markets for electricity generation and distribution (see [16] and [15]

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for surveys of these developments). Recently attention has focussed on the transmission infrastructure needed to support these markets. This infrastructure can be provided by central planners, or through a merchant transmission investment model (see e.g. [12]). A question of interest to a provider of transmission investment is the effect on market outcomes (electricity dispatch and prices) of making changes to the transmission system. This question has been addressed by Borenstein et al. [6], who consider strategic generators operating in a deterministic setting.

This paper provides a methodology for investigating the effects of building a transmission line to connect two previously separate electricity markets when there is uncertainty in demand and participant behaviour. Cross-border interconnections between electricity markets are being planned or proposed in many regions around the world including North America, Australia, and Europe. It is not hard to see that connecting two separate networks by a single link will admit an increased trade in electricity. However quantifying the exact effect on dispatch and prices after the connection is not straightforward, and requires some careful modelling. A natural approach would be to construct a computer simulation of the new market. This paper shows how one might attack this problem analytically.

Our focus in this paper is on a model in which each market is a nodal electricity pool market, characterized by a central dispatch and pricing mechanism. In a pool market the price of electricity in each trading period is determined by solving an optimization problem that matches supply and demand so as to minimize the total revealed cost of power delivery. The cost to be minimized in pool markets is provided by the prices attached to offers made by generators in the form of supply curves that define how much a generator will supply at any given price. In a perfectly competitive market, the generation can be assumed to be offered at its marginal cost. However, in an oligopoly, generators will offer supply curves with the objective of maximizing their profit, and so the dispatch and price outcomes then depend on the strategic behaviour of generators.

One approach to modelling behaviour of electricity generators in nodal electricity markets is to look for a Nash equilibrium strategy for the market participants. Since the actions of the generators are represented by supply functions, we would like to obtain supply function equilibria. This is the approach taken by Green and Newbery [10] and Anderson and Philpott [3], building on the analysis of Klemperer and Meyer [13], but these papers consider the simplest case with just one node in the network. Unfortunately it is very difficult to find a supply function equilibrium in a network with transmission constraints, and in some cases an equilibrium may not exist. A number of authors have proposed simplifications. Often competition is restricted to Cournot bidding of quantities (see e.g. [7], [11], [14]). These formulations can lead to types of mixed linear complementarity problem, which can be solved for large systems. Alternatively one might work with supply functions but restrict the form of supply function offered, for example to affine functions [5] or piecewise affine functions [4]. A different approach adopted by [9] is to assume a conjectured supply function response of the other generators–this correctly supposes that changes in price will lead to changes in the quantity dispatched from other generators, but does so in a way that may not match the actual supply function response at equilibrium.

As mentioned earlier, this paper is concerned with the flow that occurs on a link joining two otherwise disconnected subnetworks. Our framework is stochastic–we provide formulae for the probability distributions for the flow on the link (and the line rental that it earns).

Our discussion illuminates the importance of correlation and independence between demand at different locations–this would not emerge from a deterministic analysis. It is important to realise that the probability distributions of nodal prices are not exogenous in our model–after interconnection they alter in response to the changes in transmission. However, a significant restriction is that we do not consider the strategic behaviour of generators in response to the addition of the interconnection. This would lead to equilibrium considerations, and as we have already mentioned, such an analysis is intractable.

The approach we follow is based on the concept of a *market distribution function* introduced in Anderson and Philpott [2]. Market distribution functions capture the effects on prices of changes in demand (or injections across a link), and so they provide a natural tool for investigating the effects of interconnecting two networks. The market distribution function $\psi(q, p)$ at any node is defined to be the probability that a generator at this node who makes a single offer to supply an amount q at price p is not fully dispatched by the market. In the case of a single node, $\psi(q, p)$ can be thought of in terms of different possible realisations for the supply functions $S_k(\cdot)$ used by the other generators ($k = 1, \dots, K$) and the (random) demand h. A single realisation ω gives supply functions $S_k^{(\omega)}(p)$ and a demand quantity $h^{(\omega)}$. In this framework $\psi(q, p)$ is the probability of those events ω where $h^{(\omega)}$ is less than $q + \sum_{k=1}^{K} S_k^{(\omega)}(p)$. This interpretation does not apply in a nodal market, in which generation is offered at nodes of a constrained transmission network. Here, the dispatch and prices are determined, not by the intersection of two curves, but by the primal and dual solutions of a convex optimization problem. In a nodal market, a separate market distribution function must be defined for each node.

The paper is laid out as follows. In the next section we give an overview of nodal electricity markets that are dispatched using convex optimization models. An understanding of these models is necessary to understand the effects of transmission losses on dispatch and prices. In Sect. 3 we study the connection of two markets in which the behaviour of participants and the demand in each component is assumed to be independent. We assume that the two systems are connected by a single lossy line without any capacity limit. However, we show with an example how this approach can be used to accommodate capacity constraints on the flow by using an appropriately formulated loss function. We prove three theorems that respectively give formulae for the probability distribution of line flows after connection, the probability distribution of line rentals after connection, and the market distribution function at the connecting nodes. In Sect.4 we consider the more realistic situation in which the market behaviours and demands are correlated, and illustrate by example some of the effects that different degrees of correlation have on market outcomes.

2 Nodal electricity markets

When the generators offer at nodes of a constrained transmission network, the dispatch and prices are determined by a convex optimal dispatch problem (called the *pricing problem* in [2]), which delivers an optimal primal solution defining the dispatch at each node and an optimal dual solution defining a set of nodal prices.

This problem can be formulated as follows. Suppose the network has N nodes with indices $i = 1, 2, \ldots, N$, and there are K market participants indexed $k = 1, 2, \ldots, K$, where each market participant k is assumed to be located at a single node. We denote by $K(i)$ the set of participants at node i.

Let u_k denote the net injection of energy of market participant k. The cost of injection u_k is represented by the nondecreasing convex function $C_k(u_k)$. For those participants k who buy power we treat their purchase as a negative injection, i.e. $u_k \leq 0$ with $C_k(u_k) \leq 0$ defined over this range. The cost function C_k should be thought of as a notional cost. For generators it is the integral of the inverse supply curve rather than the actual generation costs, and for demand-side participants it represents their bids into the market. The net injection u_k of each participant is typically constrained by capacity bounds, denoted collectively by $u \in U$, where u is the vector of K injections and U is a convex subset of R^K. We distinguish demand-side participants who bid for power (through a curve C_k) from inelastic demand, which is denoted by d_i, $i = 1, 2, \ldots, N$.

For each $j > i$, v_{ij} is defined to be the flow of energy sent from node i to node j. Here v_{ij} is unrestricted in sign, but must be chosen so that the vector $v \in V$, where V is constructed to account for the specific transmission and security constraints that apply to the market in which the pricing problem is solved. The form of our pricing model is motivated by the Australian and New Zealand pricing and dispatch models described by Alvey *et al.* [1]. In this context V includes the loop-flow constraints of a linearized DC load flow, along with linear constraints that are added to the model to ensure robustness of the solution in case of contingencies. In addition the losses on each link (i, j) are modelled by a strictly increasing concave differentiable function τ_{ij} whereby a flow of v_{ij} entering the line at i results in a flow out at j of $\tau_{ij}(v_{ij})$. (The standard approximations of active power losses in transmission networks based on DC power flow are based on quadratic loss functions, Wood and Wollenberg [17].) In practice this function will have $\tau_{ij}(0) = 0$, and losses in lines will be symmetric so that $\tau_{ij}(-v_{ij}) = -\tau_{ij}^{-1}(v_{ij})$ which gives $\tau'_{ij}(0) = 1$. We assume that these properties hold throughout the paper.

The framework we have given also allows the representation of lines with fixed capacities as a limiting case. As an example consider the function, defined for positive v_{ij}, by

$$\begin{aligned} \tau_{ij}(v_{ij}) &= v_{ij} - \tfrac{\delta^2 v_{ij}^2}{\Gamma(\Gamma+3\delta)(\Gamma+\delta-v_{ij})}, \quad \text{for } 0 \leq v_{ij} \leq \Gamma \\ &= \Gamma - \tfrac{\delta\Gamma^2}{(\Gamma+3\delta)v_{ij}}, \quad \text{for } v_{ij} > \Gamma, \end{aligned} \tag{1}$$

with values for negative v_{ij} defined by $\tau_{ij}(-v_{ij}) = -\tau_{ij}^{-1}(v_{ij})$. This is differentiable and concave and has value 0 and derivative 1 when $v_{ij} = 0$. We have plotted this function in Fig. 1 for $\delta = 0.1$ and $\Gamma = 4$. As δ approaches zero we approach the case where the line has no losses, but a fixed capacity of Γ.

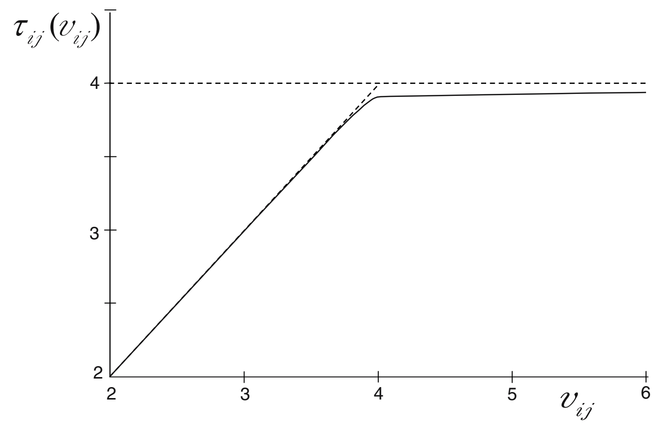

Fig. 1 Smooth loss function compared with lossless capacitated line

The pricing problem is now formulated as

$$\begin{aligned} \text{P: } & \underset{u,v}{\text{minimize}} \sum_k C_k(u_k) \\ & \text{subject to} \sum_{k \in K(i)} u_k + \sum_{j<i} \tau_{ji}(v_{ji}) - \sum_{j>i} v_{ij} \geq d_i,\ i = 1, 2, \ldots, N, \\ & u \in U,\ v \in V. \end{aligned}$$

Observe that to guarantee convexity, we use inequality flow balance constraints at each node. This allows the possibility of the free disposal of energy at each node, although in practice this will rarely occur. (A discussion of this issue can be found in Chao and Peck [8]).

We will make use of primal and dual solutions to P under different realisations ω of uncertain offers $C_k^{(\omega)}(\cdot)$, $k = 1, \ldots, K$, and demand $d_i^{(\omega)}$, $i = 1, 2, ..., N$. This will enable us to derive the market distribution function for a network that is constructed by linking two existing networks using a transmission line.

The nodal prices in a pool market can be found by solving a Lagrangian dual problem for P, which is to choose nonnegative prices p_i, $i = 1, 2, ..., N$, to maximize $E(p)$ where $E(p)$ is the optimal value of

$$\begin{aligned} & \underset{u,v}{\text{minimize}} \sum_k C_k(u_k) - \sum_i p_i \left(\sum_{k \in K(i)} u_k + \sum_{j<i} \tau_{ji}(v_{ji}) - \sum_{j>i} v_{ij} - d_i \right) \\ & \text{subject to } u \in U, \quad v \in V. \end{aligned}$$

Since P is a convex problem, there will be nonnegative prices p_i, injections u_{ki}, and link flows v_{ji}, that are feasible for P, and satisfy

$$E(p) = \sum_k C_k(u_k),$$

thereby solving P and its dual problem. It is useful to formalize the optimality conditions as follows:

$$p_i\left(\sum_{k\in K(i)} u_k+\sum_{j<i}\tau_{ji}(v_{ji})-\sum_{j>i} v_{ij}-d_i\right)=0,\quad i=1,2,\ldots,N, \tag{2}$$

$$\sum_{k\in K(i)} u_k+\sum_{j<i}\tau_{ji}(v_{ji})-\sum_{j>i} v_{ij}\geq d_i,\quad i=1,2,\ldots,N, \tag{3}$$

$$u_k\in A_k(p_i),\quad k\in K(i),\quad i=1,2,\ldots,N, \tag{4}$$

$$v\in B(p), \tag{5}$$

$$u\in U,\quad v\in V,\quad p_i\geq 0,\quad i=1,2,\ldots,N. \tag{6}$$

Here, $A_k(p_i)$ is the set of injections from participant k that minimizes $C_k(u_k)-p_i u_k$ over $u\in U$, and $B(p)$ denotes the set of v that minimizes the Lagrangian subject to $v\in V$. Thus, the optimality conditions (4) ensure that each generator (demand bid) is dispatched if the spot price exceeds (is below) their offer price. The optimality conditions (5) guarantee that the flow on each link is consistent with the prices at the endpoints. If for a particular link (i, j), V does not constrain v_{ij}, then (5) for this link is equivalent to

$$p_j\tau'_{ij}(v_{ij})-p_i=0. \tag{7}$$

When we use the formula for τ_{ij} given in (1) then this becomes

$$\begin{aligned} p_i &= p_j-\delta^2\tfrac{(2\Gamma+2\delta-v_{ij})v_{ij}}{\Gamma(\Gamma+3\delta)(\Gamma+\delta-v_{ij})^2}p_j, && \text{for } 0\leq v_{ij}\leq\Gamma \\ &= \tfrac{\delta\Gamma^2}{(\Gamma+3\delta)v_{ij}^2}p_j, && \text{for } v_{ij}>\Gamma. \end{aligned} \tag{8}$$

A standard analysis of a line with no losses but a fixed capacity has two regimes. Prices at the two ends of the line are equal when flow is below capacity. On the other hand if flow is at the capacity of the line then the only constraint on prices is that the direction of flow is from the node with lower price to that with higher price. For very small values of δ this is essentially the outcome from (8). When $v_{ij}<\Gamma-\eta$ with $\delta\ll\eta\ll\Gamma$ the second term is very small and we can take this as equalising the prices at the two ends of the line. This corresponds to a situation where the line is below capacity. However if v_{ij} is close to Γ then any outcome with $p_i<p_j$ is possible. Specifically p_i moves from more than $(3/4)p_j$ to less than $(\delta/\Gamma)p_j$ as v_{ij} moves from $\Gamma-\delta$ to Γ. Since δ is very small these different price outcomes reflect only very small changes in flow. In the limit as $\delta\to 0$ we exactly mimic the behaviour of the line with no losses and a capacity limit.

In the pricing problems we consider in this paper, we will often assume that both the primal and dual solutions are unique. Uniqueness of the primal solution occurs when there is only one possible dispatch that can solve problem P. A unique set of prices will occur when the value of the optimal solution is a smooth function of the right-hand sides. A range of possible prices signals a corner in the optimal value function. This may occur when either some loss function τ_{ij} or some C_k is not smooth.

One situation when this uniqueness will not hold occurs when offers at the same node are made at the same price (corresponding to affine segments of two C_k curves which have the same derivative). This leads to circumstances where the market dispatch mechanism must use some sort of tie breaking or sharing rule to decide which generator is dispatched. Note that our uniqueness assumption is stronger than specifying a tie-breaking rule as it requires the pricing problem to deliver the unique primal and dual solutions automatically.

We need to make a careful distinction between a situation in which an offer is made at a certain node (as is used to define the ψ function) and a situation in which an injection is made at the same node, under an assumption that all the other offers and demand remain fixed (denoted by realisation ω). In case of an injection there is change in the right-hand side of the flow balance constraint at the node in question, while in the case of the offer there is an additional u variable which appears in the objective function. However, the two situations are closely connected as the following lemma shows.

Lemma 1 *If for some realisation ω, P has a unique primal and dual optimal solution for every offer made at node n, then the following two statements are equivalent: (a) an offer of q at price p at node n is not fully dispatched; (b) with an injection q at node n, the price at n is strictly less than p.*

Proof Suppose that an offer of q at price p is made at node n for a fixed realisation ω of demand and player offers. Write q^* for the amount dispatched from the offer and let p_n^* be the price at node n, both under realisation ω. If (q, p) is not fully dispatched, then $q^* < q$ and $p_n^* \leq p$. This implies that when, instead of this offer being made at n, there is an injection of q^* at n, then the power flows in the network are still feasible. Moreover, the optimality conditions (2–6) are all still satisfied with the price at node n being p_n^*.

Now the optimal value function of the pricing problem is a convex function of demand for power at node n, with a (nondecreasing) slope equal to the clearing price at n. Thus increasing the injection (thereby decreasing demand) cannot increase the price, and so when the injection is q the price at node n is no greater than p_n^*, and hence no larger than p. If the injection of q leads to a price of p at node n, then the pricing problem with an offer of q at price p has an optimal solution under realisation ω with this offer fully dispatched (again by considering the optimality conditions). This gives a different solution to the pricing problem, which contradicts the uniqueness assumption, and so we obtain (b).

Now suppose that an offer of q at price p is fully dispatched. In other words $q^* = q$ and $p_n^* \geq p$. The flows in the network are feasible for the problem with an injection of q, and the prices satisfy (2–6) with this injection, and so p_n^* is the (unique) price at node n, which contradicts (b). □

3 Connecting two networks

We now consider two distinct transmission networks, called North and South. We are interested in connecting North and South by a transmission line that connects node s in South with node n in North. The new line connecting s to n is assumed to have unlimited capacity but will incur losses in transmission given by a strictly increasing concave differentiable loss function τ.

Since North and South are distinct, the link from s to n will not create any loop flows between North and South. We assume that there are no other constraints on the flow in this link. This means that the optimality condition (7) for the link sn is equivalent to

$$p_n \tau'_{sn}(v_{sn}) - p_s = 0.$$

In order to model the effects of interconnection, we need to consider the optimal dispatch problem in each network separately and its relationship to the optimal dispatch problem of the interconnected network. We write N and S for the North and South networks and we let S×N be the network consisting of N and S with interconnection from s to n. We denote by $\mathrm{P_S}$, $\mathrm{P_N}$, and $\mathrm{P_{S\times N}}$ the pricing problems in the respective networks.

We first consider a single realisation for the demands and participant behaviour in each network. Suppose under this realisation that the price at node s before interconnection is p_s and the price at node n is p_n. After interconnection the prices are $\tilde{p}_s$ and $\tilde{p}_n$ and the flow in the link from s to n is y.

Lemma 2 *Suppose $p_s < p_n$. Then after interconnection $y \geq 0$, and $p_s \leq \tilde{p}_s \leq \tilde{p}_n \leq p_n$. If $\mathrm{P_S}$ and $\mathrm{P_N}$ each have unique primal and dual solutions, then $y > 0$.*

Proof Suppose $p_s < p_n$ and $y < 0$ after interconnection. Then because $\mathrm{P_S}$ is convex, the optimal dual variable at s is increasing with y, and so $\tilde{p}_s \leq p_s$. Similarly, $\mathrm{P_N}$ is convex implies $\tilde{p}_n \geq p_n$. Thus $\tilde{p}_s < \tilde{p}_n$. However

$$\tilde{p}_n \tau'_{sn}(y) = \tilde{p}_s, \tag{9}$$

and $\tau'_{sn}(y) \geq 1$, as $y < 0$, which yields a contradiction.

Since $y \geq 0$, $p_s \leq \tilde{p}_s$ and $\tilde{p}_n \leq p_n$ both follow from convexity, and $\tau'_{sn}(y) \leq 1$ follows from the fact that τ_{sn} is concave with $\tau'_{sn}(0) = 1$. Thus (9) implies $\tilde{p}_s \leq \tilde{p}_n$, and so

$$p_s \leq \tilde{p}_s \leq \tilde{p}_n \leq p_n.$$

Now suppose $\mathrm{P_S}$ and $\mathrm{P_N}$ have unique primal and dual solutions for the realisation of demand and participant behaviour being considered. If $p_s < p_n$ and $y = 0$ after interconnection then $\tilde{p}_s = \tilde{p}_n \tau'_{sn}(0) = \tilde{p}_n$. Since $y = 0$, the flows and prices in S must satisfy the optimality conditions for $\mathrm{P_S}$ (otherwise the solution of $\mathrm{P_{S\times N}}$ could be improved). Since $\mathrm{P_S}$ has a unique dual solution, $\tilde{p}_s = p_s$. Similarly $\tilde{p}_n = p_n$, which is a contradiction. So $y > 0$. □

Lemma 2 demonstrates the unsurprising result that joining together separate networks at nodes with different prices will give flows in the direction of higher prices, and that these prices will tend to converge as the flow increases. Network owners or investors contemplating building an interconnecting link will of course be interested in the probability distributions of these effects. In order to compute the probability distribution of flows and prices after interconnection, we need to make use of the market distribution functions at s and n.

To do this suppose that we know $\psi^{(s)}$ and $\psi^{(n)}$, the market distribution functions at nodes s and n before the connection is made. We wish to use this information to compute the distribution of the flow in the link when the connection is made. More than this, we will develop formulae both to construct the market distribution function ψ at node A after the connection is made, and to produce the distribution of flow and prices at either end of the link. In our model we assume that all participants behave exactly as they did before the introduction of the link, and that this behaviour, and the demand distributions in the North and South networks are independent. We also make the assumption that $\psi^{(s)}$ and $\psi^{(n)}$ are continuous.

We now wish to study the dispatch of a generator who makes a single offer (q, p) at some node in $\mathrm{S} \times \mathrm{N}$. In this case the realisation $(\omega_\mathrm{S}, \omega_\mathrm{N})$ denotes all the *other* offers and loads in the network not including (q, p). We derive our construction by conditioning on the flow y in the link from s to n. By treating the flow y as an extra injection of flow at n we can use $\psi^{(n)}$ to model these outcomes. To do this rigorously requires a number of technical lemmas that allows us to compute the probability of dispatch and price outcomes at n when there is an injection of flow at this node. In these lemmas we write $\mathrm{P}_\mathrm{N}(\omega_\mathrm{N}, y)$ for the optimal dispatch problem in the network N with an injection of y at node n (y may be negative corresponding to an additional demand at n). This corresponds to decreasing the right-hand side of one of the constraints of the problem by y.

In the remainder of this section we shall require that $\mathrm{P}_\mathrm{N}(\omega_\mathrm{N}, 0)$ has a unique primal and dual solution, irrespective of any offers that might be made at n. This allows us to prove the following lemma.

Lemma 3 *Suppose the optimal solution for $P_{\mathrm{S}\times\mathrm{N}}(\omega_\mathrm{S}^1, \omega_\mathrm{N})$ gives p_s^1 as the price at node s, and a flow of y_1 entering the link at s, and the optimal solution for $P_{\mathrm{S}\times\mathrm{N}}(\omega_\mathrm{S}^2, \omega_\mathrm{N})$ gives p_s^2 and y_2, for these two quantities. If $y_1 > y_2$, then $p_s^1 < p_s^2$.*

Proof First observe that if the optimal solution for $\mathrm{P}_{\mathrm{S}\times\mathrm{N}}(\omega_\mathrm{S}, \omega_\mathrm{N})$ has a flow of y on sn then the dispatches in N under $(\omega_\mathrm{S}, \omega_\mathrm{N})$ are also optimal for $\mathrm{P}_\mathrm{N}(\omega_\mathrm{N}, \tau(y))$ (otherwise we could improve the solution to the $\mathrm{S} \times \mathrm{N}$ problem by substituting the P_N solution for the flows in N). Moreover, the prices that arise for $\mathrm{P}_{\mathrm{S}\times\mathrm{N}}(\omega_\mathrm{S}, \omega_\mathrm{N})$ in N are also the prices in $\mathrm{P}_\mathrm{N}(\omega_\mathrm{N}, \tau(y))$, since they are easily seen to satisfy the optimality conditions for this problem.

Now let $v(z)$ be the optimal value of $\mathrm{P}_\mathrm{N}(\omega_\mathrm{N}, z)$. Then the price p_n at node n for P_N is given by $p_n = -v'(z)$. Since $v(z)$ is convex, $v'(z)$ is a nondecreasing function, so p_n is nonincreasing with z. Thus, $p_n^1 \leq p_n^2$, since p_n^1 is the price at node n for $\mathrm{P}_\mathrm{N}(\omega_N, \tau(y_1))$, and p_n^2 is the price at node n for $\mathrm{P}_\mathrm{N}(\omega_\mathrm{N}, \tau(y_2))$, and $\tau(y_1) > \tau(y_2)$.

But if $p_n^1 = p_n^2$, then an offer of $(\tau(y_1), p_n^1)$ to $\mathrm{P}_N(\omega_\mathrm{N}, 0)$ could be dispatched either at $\tau(y_1)$ or $\tau(y_2)$ while remaining optimal. So $\mathrm{P}_\mathrm{N}(\omega_\mathrm{N}, 0)$ has alternative optima for this offer which contradicts our assumption. So the inequality is strict.

But since $y_1 > y_2$ and $p_n^1 < p_n^2$, we have $0 < \tau'(y_1) \leq \tau'(y_2)$, and so

$$p_s^1 = \tau'(y_1) p_n^1 < \tau'(y_2) p_n^2 = p_s^2.$$

Thus we have shown that $p_s^1 < p_s^2$ as we require. □

Lemma 4 *The value of y in an optimal solution to $P_{\mathrm{S}\times\mathrm{N}}(\omega_\mathrm{S}, \omega_\mathrm{N})$ is uniquely determined by ω_N and p_s.*

Proof Follows immediately from the previous lemma. □

Define $y(\omega_N, p)$ to be the flow along sn, if the price at s is p and ω_N occurs in N. From Lemma 4 we know that this is well defined and from Lemma 3 it is a nonincreasing function of p. The smoothness assumption on τ also enables us to write down the (unique) price at node n if the price at s is p and ω_N occurs in N. It is $p/\tau'(y(\omega_N, p))$.

Lemma 5 *For given price p at s the probability that ω_N falls in the set for which $y(\omega_N, p) < y$ is given by $\psi^{(n)}(\tau(y), p/\tau'(y(\omega_N, p)))$.*

Proof We let $\Gamma(y) = \{\omega_N : y(\omega_N, p) < y\}$. From Lemma 1 we know that $\psi^{(n)}(\tau(y), p/\tau'(y(\omega_N, p)))$ is given by the probability of ω_N being in the set $\Omega(\tau(y), p/\tau'(y(\omega_N, \text{p}))) = \{\omega_N :$ an injection of $\tau(y)$ at n leads to a price strictly less than $p/\tau'(y(\omega_N, p))\}$. Suppose that $\omega_N \notin \Gamma(y)$ so $y(\omega_N, p) \geq y$. Then from the optimality conditions an injection of $\tau(y(\omega_N, p))$ leads to a price of $p/\tau'(y(\omega_N, p))$. Since prices are decreasing in the injection, an injection of $\tau(y)$ leads to a price no lower than this and so $\omega_N \notin \Omega(\tau(y), p/\tau'(y(\omega_N, p)))$.

On the other hand if $\omega_N \in \Gamma(y)$, so $y(\omega_N, p) < y$, then an injection of $\tau(y(\omega_N, p))$ leads to a price of $p/\tau'(y(\omega_N, p))$ and an injection of $\tau(y)$ will result in a price the same or lower. Thus $\Gamma(y) \subset \Omega(\tau(y), z)$ for every $z > p/\tau'(y(\omega_N, p))$. Since $\psi^{(n)}$ is continuous, the result is established. □

Corollary 6 *For given price p,*

$$Pr(\{\omega_N : y(\omega_N, p) \leq y\}) = \psi^{(n)}\left(\tau(y), p/\tau'(y(\omega_N, p))\right).$$

Proof Follows immediately from Lemma 5 and the continuity of $\psi^{(n)}$. □

Our next result describes the distribution of the flow in a link joining two networks in terms of the market distribution functions of the two networks when separated. We have assumed up to now that P_N has a unique primal and dual solution, irrespective of any offers that might be made at n. To prove the next result we need in addition that P_S has a unique primal and dual solution, irrespective of any offers that might be made at s. For convenience we say that P_S and P_N satisfy the uniqueness assumption.

Theorem 7 *Suppose that events in N and S are independent, and that P_S and P_N satisfy the uniqueness assumption. Then the probability $H(y)$ that the flow from s to n is less than y is given by the Stieltjes integral*

$$H(y) = \int_{p=0}^{p=\infty} \psi^{(n)}\left(\tau(y), p/\tau'(y)\right) \mathrm{d}G \tag{10}$$

where G is the monotonic function $G(p) = \psi^{(s)}(-y, p)$.

Proof Define $k(\omega_S, y)$ to be the price at node s (considering S on its own) if there is a demand of y at node s, and ω_S is an instance of the set of possible demands and player offer curves in S. We first show that if $k(\omega_S, y) = p$ and $y(\omega_N, p) \le y$ then (ω_S, ω_N) results in a flow of no more than y. Suppose otherwise, and let $y_1 > y$ be the flow under (ω_S, ω_N). Then from Lemma 3 we must have $p_s < p$. We can divide up the network and observe that this means that with a demand of y_1 occurring at s, the price at s is p_s. But this contradicts the fact that the price at s is a nondecreasing function of y_1.

We also need the reverse: if $k(\omega_S, y) = p$ and (ω_S, ω_N) results in a flow of no more than y then $y(\omega_N, p) \le y$. Suppose that (ω_S, ω_N) results in a flow of $y_2 \le y$. Let $p_1 = k(\omega_S, y_2)$. Then $p_1 \le p$, since the price at s is a nondecreasing function of y. But we have $y(\omega_N, p_1) = y_2 \le y$. Since $y(\omega_N, p)$ is a nonincreasing function of p, this gives $y(\omega_N, p) \le y$ as required.

Now fix y. Let $K(p) = \{\omega_S \mid k(\omega_S, y) = p\}$. Let $W(p)$ be the events (ω_S, ω_N) that have $k(\omega_S, y) = p$ and also give rise to a flow of no more than y in the link. Then from our discussion above

$$\begin{aligned} W(p) &= \{(\omega_S, \omega_N) \mid y(\omega_N, p) \le y \text{ and } \omega_S \in K(p)\} \\ &= \{\omega_N \mid y(\omega_N, p) \le y\} \times K(p) \end{aligned}$$

using independence. Now, from Corollary 6, $\Pr\{\omega_N) \mid y(\omega_N, p) \le y\} = \psi^{(n)}(\tau(y), p/\tau'(y))$ and so we have established that

$$H(y) = \int_0^\infty \psi^{(n)}\left(\tau(y), p/\tau'(y)\right) \mathrm{d}\rho(p)$$

where $\mathrm{d}\rho(p)$ is a measure on p defined as the measure of the events in $K(p)$. If we let $M(p) = \{\omega_S \mid k(\omega_S, y) < p\}$ it only remains to show that $\Pr[M(p)] = \psi^{(s)}(-y, p)$. Now this is immediate since by Lemma 1 $k(\omega_S, y) < p$ if and only if an offer of $-y$ at price p is not fully dispatched under realisation ω_S. □

We now turn our attention to line rentals. In pool markets with location marginal prices, the rental earned by a transmission line is the difference in payment received by the system operator from purchases of transmitted power at the downstream end of the line as compared with payments made to generators for this power at the upstream end. Formally the line rental earned by a flow y on a line from s to n is defined by

$$r = p_n \tau(y) - p_s y,$$

where we denote the price at s by p_s, and the price at n by p_n. Our next result gives a formula for the distribution of line rentals on a line that links two separate markets.

To do this we require some definitions. We let

$$h(r, p) = \sup\left\{ y \,\middle|\, \frac{\tau(y)}{\tau'(y)} - y \le \frac{r}{p} \right\},$$

and

$$g(r,p) = \inf\left\{ y \,\middle|\, \frac{\tau(y)}{\tau'(y)} - y \le \frac{r}{p} \right\},$$

It is easy to show that $(\tau(y)/\tau'(y) - y)$ is a nondecreasing function of y for $y > 0$, and a nonincreasing function of y for $y < 0$, so for any r, $g(r,p) \le 0 \le h(r,p)$.

Theorem 8 *Suppose that at node s in S the market distribution function ψ is $\psi^{(s)}$ and at node n in N the market distribution function is $\psi^{(n)}$, that events in N and S are independent, and that P_S and P_N satisfy the uniqueness assumption. If the two networks are connected with a link from s to n with loss function τ, then the probability $L(r)$ that the line rental earned on the line from s to n is less than r is given by the Stieltjes integral*

$$L(r) = \int_{p=0}^{p=\infty} \psi^{(n+)}(r,p) dG^{(+)} - \int_{p=0}^{p=\infty} \psi^{(n-)}(r,p) \mathrm{d}G^{(-)},$$

where

$$\begin{aligned}
\psi^{(n+)}(r,p) &= \psi^{(n)}\left(\tau(h(r,p)), p/\tau'(h(r,p))\right),\\
\psi^{(n-)}(r,p) &= \psi^{(n)}\left(\tau(g(r,p)), p/\tau'(g(r,p))\right),\\
G^{(+)}(r,p) &= \psi^{(s)}(-h(r,p), p),\\
G^{(-)}(r,p) &= \psi^{(s)}(-g(r,p), p).
\end{aligned}$$

Proof The line rental on the line from s to n is defined by

$$r = p_n \tau(y) - py = p\left(\frac{\tau(y)}{\tau'(y)} - y\right),$$

where we denote the price at s by p, and the price at n by p_n. We seek $L(r)$ the probability of the set of events (ω_S, ω_N) that give rentals at most r. This is the set of events in (p, y) space giving line flows for which

$$p\left(\frac{\tau(y)}{\tau'(y)} - y\right) \le r.$$

Then since $g(r,p) \le 0 \le h(r,p)$, we have

$$\left\{ (\omega_S, \omega_N) \,\middle|\, p\left(\frac{\tau(y)}{\tau'(y)} - y\right) \le r \right\} = \{(\omega_S, \omega_N) \mid g(r,p) \le y \le h(r,p)\}.$$

Thus

$$\begin{aligned}
L(r) &= \Pr\{(\omega_S, \omega_N) \mid g(r,p) \le y \le h(r,p)\}\\
&= \Pr\{(\omega_S, \omega_N) \mid y \le h(r,p)\} - \Pr\{(\omega_S, \omega_N) \mid y \le g(r,p)\}.
\end{aligned}$$

It remains to show that

$$\Pr\{(\omega_S, \omega_N) \mid y \le h(r,p)\} = \int_0^\infty \psi^{(n+)}(r,p)\mathrm{d}G^{(+)}$$

and

$$\Pr\{(\omega_S, \omega_N) \mid y \le g(r,p)\} = \int_0^\infty \psi^{(n-)}(r,p)\mathrm{d}G^{(-)}.$$

Recall from the proof of Theorem 7 that under the independence assumption the set $W(r,p)$ of events (ω_S, ω_N) that have $k(\omega_S, h(r,p)) = p$ and also give rise to a flow $y \le h(r,p)$ in the link is

$$W(r,p) = \{\omega_N \mid y(\omega_N, p) \le h(r,p)\} \times \{\omega_S \mid k(\omega_S, h(r,p)) = p\}.$$

So

$$\Pr\{(\omega_S, \omega_N) \mid y \le h(r,p)\} = \int_{p=0}^{p=\infty} \psi^{(n)}\left(\tau(h(r,p)), p/\tau'(h(r,p))\right) \mathrm{d}\psi^{(s)}(-h(r,p),p).$$

The corresponding expression for $\Pr\{(\omega_S, \omega_N) \mid y \le g(r,p)\}$ is derived in the same way. This gives the result. □

We now prove a result that defines the market distribution function at node s after the networks N and S have been joined by a transmission line.

Theorem 9 *Suppose that at node s in S the market distribution function ψ is $\psi^{(s)}$ and at node n in N the market distribution function is $\psi^{(n)}$, events in N and S are independent, and P_N satisfies the uniqueness assumption. Then the market distribution function at s when the two networks are connected with a link with loss function τ is given by the Stieltjes integral*

$$\psi(q,p) = \int_{y=-\infty}^{y=\infty} \psi^{(s)}(q-y,p)\mathrm{d}F$$

where F is the monotonic function $F(y) = \psi^{(n)}(\tau(y), p/\tau'(y))$.

Proof Consider an offer at s (in S) of (q,p) for fixed q and p. Recall that $y(\omega_N, p)$ denotes the flow along the link starting at node s if the price at s is p and ω_N occurs in N. We first establish a preliminary result that uses $y(\omega_N, p)$ to decouple the independent events in S and N. (For convenience we shall use the notation $\omega \prec (q,p)_s$ to mean that (q,p) is not fully dispatched at node s under realisation ω.)

Let

$$\Omega_S(\omega_N) = \{\omega_S \mid \omega_S \prec (q - y(\omega_N,p), p)_s \text{ in S}\}.$$

We claim that $\Omega_S(\omega_N)$ is precisely the set of ω_S which leads to an offer of (q, p) at s not being fully dispatched in S $\times$ N when ω_N occurs in N. i.e. $\Omega_S(\omega_N) = X(\omega_N)$ where

$$X(\omega_N) = \{\omega_S \mid (\omega_S, \omega_N) \prec (q, p)_s \text{ in S} \times \text{N}\}.$$

We need to show the inclusion in two directions. First suppose that $\omega_S \in X(\omega_N)$. Suppose that the realisation (ω_S, ω_N) together with an offer (q, p) leads to a price p_s at node s and a flow y in the link. Since (q, p) is not fully dispatched $p_s \leq p$. By Lemma 3, $y \geq y(\omega_N, p)$. Now under ω_S an offer $(q - y, p)$ is not fully dispatched in S, hence an offer of $q - y(\omega_N, p)$ at price p is also not fully dispatched and so $\omega_S \in \Omega_S(\omega_N)$.

Now suppose $\omega_S \in \Omega_S(\omega_N)$. Suppose that the realisation (ω_S, ω_N) with an offer of (q, p) leads to a price p_s and flow y. If $p_s > p$ then $y < y(\omega_N, p)$ and so, from the definition of $\Omega_S(\omega_N)$, an offer of $q - y$ would not be fully dispatched in S. So, under the restriction that the flow in the link is y an offer of (q, p) at s in S$\times$N leads to a price no higher than p, which is a contradiction. If $p_s = p$ then $y = y(\omega_N, p)$ which gives from the definition of $\Omega_S(\omega_N)$ that the offer is not fully dispatched and so $\omega_S \in X(\omega_N)$. Finally when $p_s < p$ then clearly the offer is not fully dispatched and $\omega_S \in X(\omega_N)$.

Now to construct ψ we need to consider the set of events which lead to an offer (q, p) not being fully dispatched at s. We do this by conditioning on the value of $y(\omega_N, p)$ to give the set $Z(y)$ of such events that occur when the flow in the line is y. Let $B(y) = \{\omega_N \mid y(\omega_N, p) = y\}$ and define a measure on y by taking $d\mu(y)$ as the measure of the events in $B(y)$. Then

$$\begin{aligned} Z(y) &= \{(\omega_S, \omega_N) \mid \omega_N \in B(y) \text{ and } (\omega_S, \omega_N) \prec (q, p)_s \text{ in S} \times \text{N}\} \\ &= \{(\omega_S, \omega_N) \mid \omega_N \in B(y) \text{ and } \omega_S \in X(\omega_N)\} \\ &= \{(\omega_S, \omega_N) \mid \omega_N \in B(y) \text{ and } \omega_S \in \Omega_S(\omega_N)\} \\ &= \{\omega_S \mid \omega_S \prec (q - y(\omega_N, p), p)_s \text{ in S}\} \times B(y). \end{aligned}$$

The last equality follows from observing that $\Omega_S(\omega_N)$ is the same set for each $\omega_N \in B(y)$.

Now using independence, the probability measure for $Z(y)$ is given by the product of $\mu(y)$ and the probability of an offer at s of $q - y$ at price p not being fully dispatched in S. Hence we have shown that

$$\psi(q, p) = \int_{y=-\infty}^{y=\infty} \psi^{(s)}(q - y, p)\mathrm{d}\mu(y).$$

It only remains to show that $\mathrm{d}\mu$ is the same as $\mathrm{d}F$. But observe that if we let $\Gamma(y) = \{\omega_N \mid y(\omega_N, p) \leq y\}$ then by Lemma 1 $\mathrm{d}\mu$ is just the change in the probability of $\Gamma(y)$ as y varies. □

There are three significant restrictions on the results we have given. First observe that our results only apply to two networks linked by a single line. If the two networks are already connected and we are considering adding a new link then the situation is quite different. In fact if the network is a tree network then the methods we have described can be used to build up the market distribution function of the whole network by adding one node at a time and repeatedly using Theorem 9. The second restriction is that we have to assume that both demand and player bids are independent between the two networks. There will often be significant correlation between the demand at two ends of a link, and so this is a serious restriction. Finally, we have made various smoothness assumptions (notably on the losses in the link). Even though an accurate physical model might well have a smooth loss function, the actual dispatch is usually carried out by solving an approximation to P which assumes a piecewise linear loss function. (This is the approach followed in the New Zealand and Australian markets.)

Although the theory above assumes smooth strictly concave loss functions, we have indicated in the previous section how a limiting argument might be applied to a situation in which the losses were not smooth. Before finishing this section we revisit this issue, and give an example of how this approach might be applied to compute $H(y)$ for a capacitated line.

Consider a line joining S with N with no losses and capacity $\Gamma < 1$, at each end of which there is a generator offering $S(p) = p$ and an iid demand (denoted u in S, and v in N) with distribution

$$F(u) = \begin{cases} 0, & u \le 0, \\ 3u^2 - 2u^3, & 0 < u < 1, \\ 1, & u \ge 1. \end{cases}$$

This means in the absence of the line we have

$$\begin{aligned} \psi^{(n)}(q, p) &= \psi^{(s)}(q, p) \\ &= F(q + S(p)). \end{aligned}$$

We wish to use Theorem 7 to compute $H(y) = \Pr(x < y)$ where x is the flow from S to N. It is easy to see that for $y < \Gamma$ the flow in the line is at least y when $v - u \ge 2y$. Thus we have by direct computation

$$\begin{aligned} H(y) &= 1 - \int_{2y}^{1} F'(v) \left(\int_{0}^{v-2y} F'(u)\mathrm{d}u \right) \mathrm{d}v \\ &= \frac{1}{2} + \frac{12}{5}y - 16y^3 - \frac{64}{5}y^6 + 24y^4. \end{aligned}$$

The probability $H(y)$ is defined in Theorem 7 for smooth concave losses by

$$H(y) = \int_{p=0}^{p=\infty} \psi^{(n)}\left(\tau(y), p/\tau'(y)\right) \mathrm{d}G$$

where G is the monotonic function $G(p) = \psi^{(s)}(-y, p)$. Suppose we choose, as in our earlier discussion (1),

$$\begin{aligned} \tau(y) &= y - \frac{\delta^2 y^2}{\Gamma(\Gamma + 3\delta)(\Gamma + \delta - y)}, \quad \text{for } 0 \le y \le \Gamma \\ &= \Gamma - \frac{\delta \Gamma^2}{(\Gamma + 3\delta) y}, \quad \text{for } y > \Gamma, \end{aligned}$$

and let $\delta \to 0$. Observe that

$$G(p) = \psi^{(s)}(-y, p) = F(p - y),$$

so $G(p) = 0$ for $p \leq y$ and $G(p) = 1$ for $p \geq 1 + y$. Also

$$\psi^{(n)}\left(\tau(y), p/\tau'(y)\right) = F\left(\tau(y) + p/\tau'(y)\right)$$

which takes the value 1 when $\tau(y) + p/\tau'(y) \geq 1$. Hence

$$\begin{aligned} &\int_{p=0}^{p=\infty} \psi^{(n)}\left(\tau(y), p/\tau'(y)\right) \mathrm{d}G(p) \\ &= \int_{y}^{(1-\tau(y))\tau'(y)} \psi^{(n)}(\tau(y), p/\tau'(y)) \mathrm{d}G(p) \\ &+ \int_{(1-\tau(y))\tau'(y)}^{y+1} \mathrm{d}G(p). \end{aligned} \tag{11}$$

Now $(1 - \tau(y))\,\tau'(y)$ is a continuous function of δ at $\delta = 0$, as long as $y < \Gamma$. Moreover, the integrand $\psi^{(n)}(\tau(y), p/\tau'(y))\mathrm{d}G(p)$ is a continuous function of δ as long as $y < \Gamma$. Thus, if $y < \Gamma$, $H(y)$ can be found by setting $\delta = 0$ in (11) to give

$$\begin{aligned} H(y) &= \int_{y}^{1-y} \psi^{(n)}(y, p)\mathrm{d}G(p) + \int_{1-y}^{y+1} \mathrm{d}G(p) \\ &= \int_{y}^{1-y} \left(3(y+p)^2 - 2(y+p)^3\right)\left(6p - 6y - 6p^2 + 12py - 6y^2\right) \mathrm{d}p \\ &\quad +3(1)^2 - 2(1)^3 - (3(1-2y)^2 - 2(1-2y)^3) \\ &= \frac{1}{2} + \frac{12}{5}y + 24y^4 - \frac{64}{5}y^6 - 16y^3 \end{aligned}$$

as required.

It is interesting to observe what happens to $H(y)$ if $y \geq \Gamma$. Recall that $\mathrm{d}G(p) = 0$ unless $y < p < y + 1$, and $\psi^{(n)}(\tau(y), p/\tau'(y)) = 1$ if $p \geq (1 - \tau(y))\,\tau'(y)$. Now for $y > \Gamma$, as δ gets small $\tau(y) \to \Gamma$, and $\tau'(y) \to 0$. Thus for sufficiently small $\delta > 0$,

$$(1 - \tau(y))\,\tau'(y) < \Gamma \leq y.$$

Thus $\psi^{(n)}(\tau(y), p/\tau'(y)) = 1$, for every p with $y < p < y + 1$. It follows that

$$H_\delta(y) = \int_{p=0}^{p=\infty} \psi^{(n)}\left(\tau(y), p/\tau'(y)\right) \mathrm{d}G(p)$$
$$= \int_{y}^{y+1} 1 \cdot \mathrm{d}G(p)$$
$$= 1.$$

The interpretation here is that for sufficiently small δ below some threshold, the flow on the line y (in the approximating system) will lose so much at the margin that the nodal price at S will be extremely high. The probability of observing any higher flow is then zero, since this event would mean that the clearing price at S would dispatch more local generation at that node than is needed to meet the maximum possible residual demand.

In the next section we explore how, for a link connecting two single-node markets, we can investigate optimal bidding patterns when the second of our assumptions breaks down, and demand between two ends of a link is correlated. In these circumstances, the answers are provided by a direct calculation rather than the application of the theorems in this section.

4 Connecting two single-node markets

Up to now we have assumed that demands and generator offers in the two parts of the network are independent of each other. There are many practical situations in which this independence assumption is questionable. It is not possible to provide general formulae without an assumption of independence, but we can carry out some direct calculations in the case when demand at the two ends of a link are correlated. In this section we look at a simple two-node model in which the demands at the two nodes are drawn from a bivariate distribution.

It turns out to be simpler to work through an example where the loss function τ is not smooth. Our major theorems in the previous section are hard to establish with non-smooth losses as this requires us to work with left and right derivatives of τ. However, for the case of a two node network, it is possible to carry out the analysis directly.

Suppose that generator A is located at node s and the aggregate supply function for the other generators at node s is $S_1(p)$. The aggregate supply function at node n is $S_2(p)$. Suppose also that demand at node s is $D_1(p) + \epsilon_1$ and demand at node n is $D_2(p) + \epsilon_2$ where ϵ_1 and ϵ_2 are random shocks with (ϵ_1, ϵ_2) drawn from a bivariate distribution on the positive orthant.

We want to calculate the optimal offer for generator A and so we wish to derive the ψ function at node s. We suppose that generator A offers q at price p at node s. We will show that this offer fails to be completely dispatched exactly when (ϵ_1, ϵ_2) falls into a particular region in the positive orthant.

Because we will allow the loss in the link to be a non-smooth function, we need to adjust the optimality conditions. Specifically (7) becomes

$$p_j \tau'_{ij-}(v_{ij}) \geq p_i \geq p_j \tau'_{ij+}(v_{ij}),$$

where we write τ'_{ij-} and τ'_{ij+} for the left and right derivatives of τ_{ij}.

Let the function $y(\epsilon_2, p)$ be the flow along sn if the price at s is p and the demand shock at n is ϵ_2. We will show that this is well-defined, and decreasing in p, even though we may not have a smooth loss function in the link. First suppose that $y_1 < y_2$ are two possible flows, both having price at s being p, and demand shock ϵ_2, then $\tau(y_1) < \tau(y_2)$. From flow balance at node n, $D_2(p_{n1}) - S_2(p_{n1}) < D_2(p_{n2}) - S_2(p_{n2})$ where p_{ni} is the price at node n associated with flow y_i. Hence $p_{n1} > p_{n2}$. Thus

$$p \geq p_{n1}\tau'_{+}(y_1) > p_{n2}\tau'_{-}(y_2) \geq p$$

giving a contradiction.

Next, we consider the effect on $y(\epsilon_2, p)$ of a change in p. Let $p_1 > p_2$ and suppose that $y(\epsilon_2, p_1) > y(\epsilon_2, p_2)$. We write p_{n1} and p_{n2} for the price at node n associated with p_1 and p_2. Then, since $\tau(y(\epsilon_2, p_1)) > \tau(y(\epsilon_2, p_2))$ from the flow balance equations at node n, we can see that $p_{n1} < p_{n2}$. However

$$p_{n1} \geq \frac{p_1}{\tau'_{-}}(y(\epsilon_2, p_1)) > \frac{p_2}{\tau'_{+}}(y(\epsilon_2, p_2)) \geq p_{n2}$$

giving a contradiction.

We define

$$T(q, p, \epsilon_2) = q + S_1(p) - D_1(p) - y(\epsilon_2, p).$$

Observe that $y(\epsilon_2, p)$ is increasing in ϵ_2, so T is a decreasing function for fixed q and p. The equation $\epsilon_1 = T(q, p, \epsilon_2)$ defines a monotonic curve for fixed q and p, and as we show below an offer of (q, p) is not fully dispatched exactly when (ϵ_1, ϵ_2) lies to the left of this curve.

Lemma 10 $\psi(q, p) = \Pr\{(\epsilon_1, \epsilon_2) : \epsilon_1 < T(q, p, \epsilon_2)\}$.

Proof We want to show that an offer of q at price p is not fully dispatched if and only if $\epsilon_1 < T(q, p, \epsilon_2)$. Suppose that the offer is fully dispatched, so the price at s is at least p, and hence from our discussion above the flow entering the link is no more than $y(\epsilon_2, p)$. The supply from other generators at s is at least $S_1(p)$, and the demand is no more than $D_1(p)$. Hence the supply is at least $q + S_1(p)$, and the outflow is at most $\epsilon_1 + D_1(p) + y(\epsilon_2, p)$. Thus flow balance implies $\epsilon_1 \geq T(q, p, \epsilon_2)$.

On the other hand if the offer is not fully dispatched then the price is p or less and the total supply is strictly less than $q + S_1(p)$. The flow entering the link is at least $y(\epsilon_2, p)$ with total demand at node s of at least $\epsilon_1 + D_1(p)$. Thus flow balance implies that the total supply is at least $\epsilon_1 + D_1(p) + y(\epsilon_2, p)$. So $\epsilon_1 < T(q, p, \epsilon_2)$ as required. □

We consider a specific example. Consider a transmission line linking s and n which has no capacity limit and has linear losses so

$$\tau(y) = \begin{cases} \alpha y & y > 0 \\ (\frac{1}{\alpha})y & y \leq 0 \end{cases} \tag{12}$$

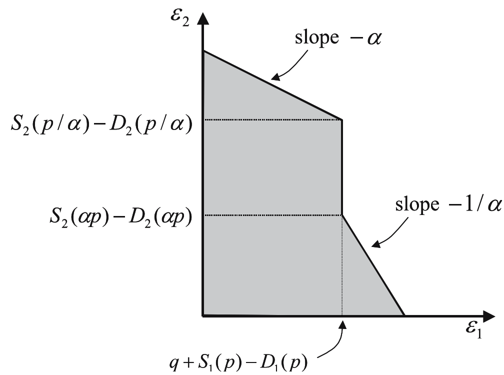

Fig. 2 The region $\epsilon_1 \le T(q, p, \epsilon_2)$

We need to start by calculating the function $y(\epsilon_2, p)$. We fix ϵ_2 and p and let p_n be the price at node n. Then y and p_n must satisfy

$$\tau(y) = D_2(p_n) + \epsilon_2 - S_2(p_n),$$
$$\tau'_-(y) \ge \frac{p}{p_n} \ge \tau'_+(y),$$

It is then straightforward to check that y defined as follows is the only solution to these conditions.

$$y(\epsilon_2, p) = \begin{cases} \alpha(D_2(\alpha p) + \epsilon_2 - S_2(\alpha p)) & \epsilon_2 < S_2(\alpha p) - D_2(\alpha p) \\ \left(\frac{1}{\alpha}\right)\left(D_2\left(\frac{p}{\alpha}\right) + \epsilon_2 - S_2\left(\frac{p}{\alpha}\right)\right) & \epsilon_2 > S_2\left(\frac{p}{\alpha}\right) - D_2\left(\frac{p}{\alpha}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Hence as a function of ϵ_2, T is piecewise linear and the region where the offer is not fully dispatched is simply the shaded region shown in Fig. 2.

4.1 Optimal offers with correlated and uncorrelated demand

Now we will specialise our example even further. Our aim is to investigate the form of an optimal offer when there is correlated demand at two ends of a link and compare this with the optimal offer when demand is uncorrelated. The optimal choice of offer depends on the profit function R. We shall suppose that there are no contracts and the generator costs are independent of quantity dispatched (as may occur with a hydro generator). Thus $R(q, p) = pq$.

Following [2] we will use ψ to calculate the optimal offer curve, which can be obtained using the function

$$Z(q, p) = \frac{\partial \psi}{\partial p}(q, p)\frac{\partial R}{\partial q}(q, p) - \frac{\partial \psi}{\partial q}(q, p)\frac{\partial R}{\partial p}(q, p). \tag{13}$$

If $Z(q, p) = 0$ defines an nondecreasing curve $q = S(p)$ with $Z(q, p) \ge 0$ when $q < S(p)$ and $Z(q, p) \le 0$ when $q > S(p)$, then $S(p)$ is a locally optimal offer curve.

We suppose that demand at the two ends of the link have the same form: both are givenly $1 - p + \epsilon$ where ϵ is a random shock with $0 \le \epsilon \le 1$. Generator A is the only generator at node s. At node n there is one other generator who offers an offer curve $S_2(p) = p$. The losses on the line are linear, as given by (12) with $\alpha = 4/5$.

As before we write ϵ_1 for the demand shock at node s and ϵ_2 for the demand shock at node n. In order to have continuous derivatives for ψ we need to be careful that the bivariate distribution of (ϵ_1, ϵ_2) has zero density at the boundaries of the unit square. We consider two cases: the strongly correlated case where ϵ_1 has a distribution given by

$$\Pr(\epsilon_1 \le x) = x^2(3 - 2x) \tag{14}$$

and $\epsilon_2 = \epsilon_1$; and the uncorrelated case where both ϵ_1 and ϵ_2 have a distribution given by (14) and corresponding joint density function on $[0, 1] \times [0, 1]$

$$\rho(u, v) = 36(u - u^2)(v - v^2). \tag{15}$$

We start by determining the $\epsilon_1 = T(q, p, \epsilon_2)$ line. Now in this example $S_1(p) = 0$, so

$$T(q, p, \epsilon_2) = q - 1 + p - y(\epsilon_2, p) \tag{16}$$

where

$$y(\epsilon_2, p) = \begin{cases} (4/5)\,(1 - (8/5)\,p + \epsilon_2) & \epsilon_2 < (8/5)\,p - 1 \\ (5/4)\,(1 - (5/2)\,p + \epsilon_2) & \epsilon_2 > (5/2)\,p - 1 \\ 0 & \text{otherwise.} \end{cases}$$

4.1.1 Strongly correlated case

With $\epsilon_2 = \epsilon_1$ we just need to calculate the probability that (ϵ_1, ϵ_2) lies on that part of the $\epsilon_1 = \epsilon_2$ line which is to the left of the $\epsilon_1 = T(q, p, \epsilon_2)$ line. We need to consider three cases depending on the segment of the T curve that the $\epsilon_1 = \epsilon_2$ line crosses. At the point where the lines cross we have $\epsilon_2 = q - 1 + p - y(\epsilon_2, p)$. The crossing occurs on the vertical section if $(8/5)p - 1 \le q - 1 + p \le (5/2)p - 1$, i.e. if $(3/5)p \le q \le (3/2)p$. In this case

$$\begin{aligned} \psi(q, p) &= \Pr(\epsilon_2 \le q - 1 + p) \\ &= \begin{cases} 0 & q + p < 1 \\ (q + p - 1)^2(5 - 2q - 2p) & 1 \le q + p \le 2 \\ 1 & q + p > 2 \end{cases} . \end{aligned}$$

Thus in this region

$$\begin{aligned} Z(q, p) &= \psi_p(q, p) R_q(q, p) - \psi_q(q, p) R_p(q, p) \\ &= (p - q)(-2(q + p - 1)^2 + 2(q + p - 1)(5 - 2q - 2p)) \\ &= 6(p - q)(2 - p - q)(p + q - 1). \end{aligned}$$

In this region, where $(3/5)p \le q \le (3/2)p$ and $0 \le \psi \le 1$, the only potential (locally optimal) offer curve has $S(p) = p$, where Z is zero (see Fig. 3). With this offer curve the generators at both ends of the link make the same offer and are dispatched equally (since $\epsilon_1 = \epsilon_2$).

The second case we need to consider, has the $\epsilon_1 = \epsilon_2$ line crossing the T curve on the section corresponding to flow from n to s (low ϵ_2): this occurs if $q < (3/5)p$. In this case, at the crossing we have

$$\begin{aligned}\epsilon_2 &= q - 1 + p - (4/5)(1 - (8/5)p + \epsilon_2)\\ \text{i.e. } \epsilon_2 &= \frac{19}{15}p + \frac{5}{9}q - 1.\end{aligned}$$

And so

$$\begin{aligned}\psi(q,p) &= \Pr\left(\epsilon_2 \le \frac{19}{15}p + \frac{5}{9}q - 1\right)\\ &= \begin{cases} 0 & \frac{19}{15}p + \frac{5}{9}q < 1\\ \left(\frac{19}{15}p + \frac{5}{9}q - 1\right)^2 (5 - \frac{38}{15}p - \frac{10}{9}q) & 1 \le \frac{19}{15}p + \frac{5}{9}q \le 2\,.\\ 1 & \frac{19}{15}p + \frac{5}{9}q > 2\end{cases}\end{aligned}$$

Hence in this region

$$\begin{aligned}Z(q,p) &= 2\left(\frac{19p}{15} - \frac{5q}{9}\right)\Bigg(-\left(\frac{19}{15}p + \frac{5}{9}q - 1\right)^2\\ &\quad + \left(\frac{19}{15}p + \frac{5}{9}q - 1\right)\left(5 - \frac{38}{15}p - \frac{10}{9}q\right)\Bigg)\\ &= 6\left(\frac{19p}{15} - \frac{5q}{9}\right)\left(2 - \frac{19p}{15} - \frac{5q}{9}\right)\left(\frac{19p}{15} + \frac{5q}{9} - 1\right).\end{aligned}$$

It is not hard to check that Z is positive throughout this region (where $q < (3/5)p$ and $0 \le \psi \le 1$) and so there are no potential optimal supply curves to consider.

The final case occurs when the crossing occurs on the section corresponding to flow from s to n, i.e. when $q > (3/2)p$. In this case

$$\begin{aligned}\epsilon_2 &= q - 1 + p - (5/4)\,(1 - (5/2)\,p + \epsilon_2)\\ \text{i.e. } \epsilon_2 &= \frac{11}{6}p + \frac{4}{9}q - 1,\end{aligned}$$

and so

$$\begin{aligned}\psi(q,p) &= \Pr(\epsilon_2 \le \frac{11}{6}p + \frac{4}{9}q - 1)\\ &= \begin{cases} 0 & \frac{11}{6}p + \frac{4}{9}q < 1\\ (\frac{11}{6}p + \frac{4}{9}q - 1)^2\,(5 - \frac{11}{3}p - \frac{8}{9}q) & 1 \le \frac{11}{6}p + \frac{4}{9}q \le 2\ .\\ 1 & \frac{11}{6}p + \frac{4}{9}q > 2\end{cases}\end{aligned}$$

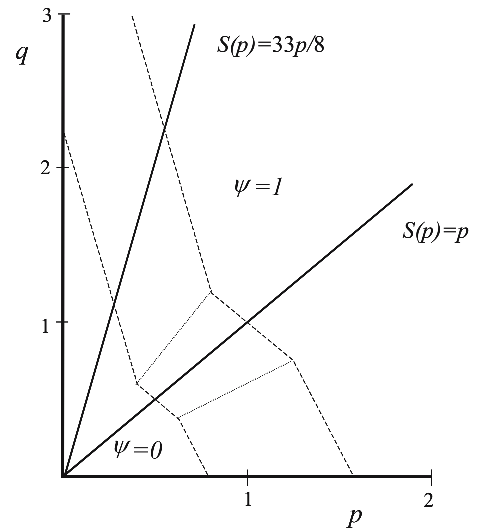

Fig. 3 Optimal offers for correlated and independent demand

Hence in this region

$$\begin{aligned}Z(q,p) &= 2\left(\frac{11}{6}p - \frac{4}{9}q\right)\left(-\left(\frac{11}{6}p + \frac{4}{9}q - 1\right)^2\right.\\&\quad\left.+\left(\frac{11}{6}p + \frac{4}{9}q - 1\right)\left(5 - \frac{11}{3}p - \frac{8}{9}q\right)\right)\\&= 6\left(\frac{11p}{6} - \frac{4q}{9}\right)\left(\frac{11p}{6} + \frac{4q}{9} - 1\right)\left(2 - \frac{11p}{6} - \frac{4q}{9}\right).\end{aligned}$$

We can check that the only potential locally optimal supply curve in this region (where $q > (3/2)p$ and $0 \le \psi \le 1$) is $S(p) = 33p/8$ (see Fig. 3).

Thus we have a second potential offer which is much more aggressive (offering more at any given price) with the aim of capturing larger market share. It remains to check which of these strategies produces the highest profit. For the first offer, $S(p) = p$ and on this curve

$$\begin{aligned}\psi(q,p) &= (2p-1)^2(5-2p)\\&= 28p^2 - 22p - 8p^3 + 5.\end{aligned}$$

The region over which we need to integrate starts at $p = 1/2$ and ends at $p = 1$. Hence the expected profit is

$$\int_S qp\,\mathrm{d}\psi = \int_{1/2}^{1} p^2\left(56p - 22 - 24p^2\right)\mathrm{d}p = \frac{247}{120}.$$

For the second offer curve

$$\psi(q,p) = \left(\frac{11}{6}p + \frac{4}{9}\left(\frac{33p}{8}\right) - 1\right)^2 \left(5 - \frac{11}{3}p - \frac{8}{9}(33p/8)\right)$$
$$= 121p^2 - 44p - \frac{2662}{27}p^3 + 5$$

and this is defined over the region $p \in (3/11, 6/11)$. Thus the expected profit is

$$\int_S qp\,\mathrm{d}\psi = \int_{3/11}^{6/11} \frac{33p^2}{8}\left(242p - 44 - \frac{2662}{9}p^2\right)\mathrm{d}p = \frac{621}{880}.$$

Consequently the first solution $S(p) = p$ is better.

4.1.2 Independent case

Now we consider the case where ϵ_1, ϵ_2 are independent. Rather than using the result of Theorem 9 we will calculate $\psi(q, p)$ directly from the regions determined by (16). Now ϵ_1, ϵ_2 have a joint continuous distribution with density function $\rho(u, v)$. Hence,

$$\psi(q,p) = \int_0^1 \int_0^{[T(q,p,v)]} \rho(u,v)\mathrm{d}u\mathrm{d}v$$
$$= \int_0^{[(8/5)p-1]} \int_0^{[q+(57/25)p-(9/5)-(4/5)v]} \rho(u,v)\mathrm{d}u\mathrm{d}v$$
$$+ \int_{[(8/5)p-1]}^{[(5/2)p-1]} \int_0^{[q+p-1]} \rho(u,v)\mathrm{d}u\mathrm{d}v$$
$$+ \int_{[(5/2)p-1]}^{1} \int_0^{[q+(33/8)p-(9/4)-(5/4)v]} \rho(u,v)\mathrm{d}u\mathrm{d}v.$$

Here, we have used the notation $[x]$ in the integral limits to indicate that where x lies outside the range $[0, 1]$ we correct back to this range (i.e. we take the projection of x onto $[0, 1]$).

Now we can calculate ψ_q and ψ_p.

$$\psi_q(q, p) = \int_0^{[(8/5)p-1]} \rho\left(q + \left(\frac{57}{25}\right)p - \left(\frac{9}{5}\right) - \left(\frac{4}{5}\right)v, v\right) \mathrm{d}v$$
$$+ \int_{[(8/5)p-1]}^{[(5/2)p-1]} \rho(q + p - 1, v)\mathrm{d}v$$
$$+ \int_{[(5/2)p-1]}^{1} \rho\left(q + \left(\frac{33}{8}\right)p - \left(\frac{9}{4}\right) - \left(\frac{5}{4}\right)v, v\right) \mathrm{d}v$$

where we take ρ to have value 0 outside the unit square $[0, 1] \times [0, 1]$. Also

$$\psi_p(q, p) = \int_0^{[(8/5)p-1]} \left(\frac{57}{25}\right)\rho\left(q + \left(\frac{57}{25}\right)p - \left(\frac{9}{5}\right) - \left(\frac{4}{5}\right)v, v\right) \mathrm{d}v$$
$$+ \int_{[(8/5)p-1]}^{[(5/2)p-1]} \rho(q + p - 1, v)\mathrm{d}v$$
$$+ \int_{[(5/2)p-1]}^{1} (33/8)\,\rho\left(q + \left(\frac{33}{8}\right)p - \left(\frac{9}{4}\right) - \left(\frac{5}{4}\right)v, v\right) \mathrm{d}v.$$

The next step is to calculate the Z function.

$$Z(q, p) = \psi_p(q, p)R_q(q, p) - \psi_q(q, p)R_p(q, p)$$
$$= \int_0^{[(8/5)p-1]} \left(\left(\frac{57p}{25}\right) - q\right)\rho\left(q + \left(\frac{57}{25}\right)p - \left(\frac{9}{5}\right) - \left(\frac{4}{5}\right)v, v\right) \mathrm{d}v$$
$$+ \int_{[(8/5)p-1]}^{[(5/2)p-1]} (p - q)\rho(q + p - 1, v)\mathrm{d}v$$
$$+ \int_{[(5/2)p-1]}^{1} \left(\left(\frac{33p}{8}\right) - q\right)\rho\left(q + \left(\frac{33}{8}\right)p - \left(\frac{9}{4}\right) - \left(\frac{5}{4}\right)v, v\right) \mathrm{d}v.$$

The only $Z = 0$ line, and the optimal solution, is given by the supply function which occurred as a local, but not global, optimum for the strongly correlated case, i.e. $S(p) = 33p/8$. This is shown in Fig. 3 with the solution from the strongly correlated case. The offers are only significant in the band where ψ is between 0 and 1 (which is shown by the dashed lines in the figure).

It is interesting that with uncorrelated demand it is no longer possible to keep flows in the link at zero by offering the same as the other generator, and so the previous optimal offer is no longer effective. We can expect this type of behaviour to occur in other examples. The losses in the link will give an incentive to offering in ways that balance offers at the two ends of the link, but this incentive will be weakened as correlation between demand at the two ends of the link decreases.

5 Conclusions

The market distribution function introduced by [2] is a powerful tool for representing the probabilistic behaviour of an electricity market. Previous work has focussed on its use in optimizing the offer of a generator. In this paper we have shown how to use the market distribution function to analyse the probabilistic behaviour of flow on an interconnector. This enables the calculation of the distribution of line rentals for the interconnector. We have also shown how to derive the market distribution function for the interconnected market, allowing the optimization of offers in this environment.

In practice the analysis of the effects of interconnection under uncertainty would typically be studied using a simulation model. In this paper we have explored the extent to which an analytical treatment is possible. This leads to an understanding of the limits of an analytical treatment in contrast to one based on simulation models.

A significant restriction on our analysis is that it takes no account of changes in participant behaviour that arise from the interconnection. As shown by Borenstein et al. [6] an interconnection between two markets is likely to induce more competitive behaviour. We conjecture that when the generators adjust their offers after interconnection the expected flow in the interconnector will be less than computed by our analysis. Similarly we expect the price difference to be reduced. This will imply that our estimate of line rentals will be higher than the value that is observed when the generators change their behaviour.

The analysis of interconnection in the first half of the paper is very general. In this framework it is very difficult to establish any results unless the random effects in each market are independent. In practice the variations in electricity demand are dependent on climate and time of day, so there will be significant correlations between demand at the two ends of an interconnector.

In general correlated demand will tend to give an increased correlation in prices at the endpoints of the interconnector. Thus compared with the uncorrelated case we expect a reduction in the flow in the interconnector and the associated rentals. The effect of ignoring correlated demand and strategic generator behaviour is to overestimate the flows and rentals for the interconnector.

6 Acknowledgements

The authors would like to thank the anonymous referees for their comments.

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