Math Meth Oper Res
DOI 10.1007/s00186-009-0283-8

ORIGINAL ARTICLE

A two stage stochastic equilibrium model for electricity markets with two way contracts

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Received: 9 June 2008 / Accepted: 15 December 2008

Abstract This paper investigates generators' strategic behaviors in contract signing in the forward market and power transaction in the electricity spot market. A stochastic equilibrium program with equilibrium constraints (SEPEC) model is proposed to characterize the interaction of generators' competition in the two markets. The model is an extension of a similar model proposed by Gans et al. (Aust J Manage 23:83–96, 1998) for a duopoly market to an oligopoly market. The main results of the paper concern the structure of a Nash–Cournot equilibrium in the forward-spot market: first, we develop a result on the existence and uniqueness of the equilibrium in the spot market for every demand scenario. Then, we show the monotonicity and convexity of each generator's dispatch quantity in the spot equilibrium by taking it as a function of the forward contracts. Finally, we establish some sufficient conditions for the existence of a local and global Nash equilibrium in the forward-spot markets. Numerical experiments are carried out to illustrate how the proposed SEPEC model can be used to analyze interactions of the markets.

Keywords Electricity market · Nash equilibrium · Stochastic equilibrium programs with equilibrium constraints

1 Introduction

Over the past two decades, the electricity industry in many countries has been deregulated. One of the main consequences of deregulation is that the governments under-

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Published online: 18 February 2009

take their efforts to develop fully competitive electricity spot markets. In most of the wholesale spot markets (pool-type systems), generators make daily (or hourly) bids of generation at different prices, and then an independent system operator (ISO) decides how actual demand is to be met by dispatching cheaper power first. In these pool-type electricity markets (found in Australia, New Zealand, Norway, at one time in UK, and some parts of US), a single market clearing price is determined by a sealed-bid auction and paid to each generator for all the power they dispatch.

Along with the spot market emerges the forward market where generators and retailers may enter into hedge contracts before bidding in the spot market. For example, in the early 1990s, during the restructuring of the electricity market in UK, some long term, "take-or-pay" contracts (or agreements) are stipulated by three main Scottish electricity generators, see Onofri (2005). Moreover, various contract markets have also been established in Europe, Australia, New Zealand and North America. By participating in the forward markets, generators and retailers may share their risks associated with a fluctuating pool price for the real power dispatching. The most common type of contract is known as a (two-way) contract-for-difference (or hedge contract), which operates between a retailer and a generator for a given amount of power at a given strike price. The signing of this type of contracts is separate from the market dispatching mechanism and can be taken as financial instruments without an actual transfer of power.

In this paper, we formulate generators' competition in the forward-spot market mathematically as a two stage stochastic equilibrium problem where each generator first aims at maximizing its expected profit by signing a certain mount of long term contracts and then bids for dispatches in the spot market on a daily or hourly basis. Differing from the two stage competition model, a volume of previous research has been performed to study the effect on the competition in the spot market from the contract quantities, in which the competition of signing contracts in the forward market is not considered. von der Fehr and Harbord (1992) investigate the spot market by modeling it as a multi-unit auction and demonstrate that contracts give generators a strategic advantage in the spot market by allowing them to commit to dispatch greater quantities during peak demand periods. Powell (1993) explores the interaction between the forward market and the spot market by characterizing the competition in the spot market within a framework of Nash–Cournot equilibrium, and shows that risk-neutral generators can raise their profits by selling contracts for more than the expected spot price. Moreover, Green and Newbery (1992) appropriately look at the endogenous formation of both pool and contract prices in a supply function model, and apply their analysis to the British electricity market.

By modeling the mechanism of the competition in the forward market as a Nash–Cournot game, previous contributions, such as (Allaz and Vila 1993; Willems 2005; Gans et al. 1998), focus on the impact of the forward market on the spot price and show that generators have incentives to trade in the forward market whereas forward contracting reduces spot prices and increases consumption levels. The exploration of the bilevel deterministic Nash–Cournot model for a duopoly forward-spot market is first carried out by Allaz and Vila (1993), which identifies two critical assumptions: One is the so-called Cournot behavior where producers (generators) act as though the quantity offered by the other competitors is fixed; the other is the connection to the prisoner's dilemma where each producer (generator) will sell forward so as to make them worse off and make consumers better off than would be in the case if the forward market did not exist. Applying this type of Nash–Cournot models of electricity pools, Gans et al. (1998) demonstrate that the contract market can make the duopolistic spot market more competitive, and hence the existence of the contract market lowers prices in pool markets. By replacing two way contracts with call options, Willems (2005) extends the results in Allaz and Vila (1993) to the Cournot type market with options, and compares it with the market efficiency effects of the Cournot game with two way contracts. Instead of duopoly markets in Allaz and Vila (1993), Bushnell (2007) presents some estimation of the impact of forward contracts and load obligations on spot market prices for a Cournot type environment with multiple generators.

Differing from much of previous work concerning on the influence on spot market efficiency from contracts, our work provides a new model for the entire forward-spot market by formulating it as a two stage stochastic equilibrium problem with equilibrium constraints (EPEC), which refers to generators' competition in the forward market as an equilibrium problem subject to the equilibrium in the spot market described by a complementarity problem. Over the past few years, EPEC models have been applied to some hierarchical decision-making problems in a wide domain in engineering design, management, and economics. Recently, a number of EPEC models have been performed for electricity markets. In modeling the forward-spot market, Su (2007) and Shanbhag (2005, Chapter 5) study the Nash–Cournot equilibrium by modeling the bilevel markets as an EPEC. Su (2007) investigates the existence results for the deterministic forward-spot market equilibrium introduced by Allaz and Vila (1993). Shanbhag (2005, Chapter 5) introduces a 2-node forward-spot model and considers it as an expected profit maximization problem subject to the complementarity constraints for every scenario in the spot market. He also investigates existence of the *simultaneous stochastic Nash equilibrium* (SSNE) in the context of the forward-spot electricity market. Moreover, besides the application in the forward-spot market, the EPEC models are also used by Yao et al. (2007) to investigate the equilibrium in the spatial electricity market, where they capture the congestion effects and bilevel competitions by formulating each generator's objective as a maximization problem in the forward market subject to the Karush–Kuhn–Tucker (KKT) optimal conditions in the spot market and the network constraints. More recently, Hu and Ralph (2007) use EPEC to model a bilevel electricity market, where generators and customers bid cost and utility functions in a nodal market and the ISO determines the dispatch quantities by minimizing the overall social cost in an upper optimization level.

Apart from Cournot-type models, another well established approach is the *supply function equilibrium* (SFE) model, which clearly encapsulates the underlying structure of bidders' strategy on the quantity–price relationship. SFE is originally proposed by Klemperer and Meyer (1989) to model competition in a general oligopolistic market where the market demand is uncertain and each firm aims to develop a supply function to maximize its profit in any demand scenario. By applying the SFE to predict the performance of the pioneer England and Wales market, Green and Newbery (1992) analyze the behavior of the duopoly and characterize the England and Wales electricity market during its first years of operation under the SFE approach. Anderson and Philpott (2002) first propose an optimal supply function model with discontinuous supply functions to address the fact that supply functions in practice are not continuous as assumed in SFE model and they use this model to investigate generators' optimal strategies of bidding a stack of price–quantity offers into electricity markets in circumstances where demand is unknown in advance. Anderson and Xu (2005) extend the optimal supply function approach to consider both second order necessary conditions and sufficient conditions of the optimality for each generator's price-quantity offers given its rivals' offers are fixed. Besides the analysis on the optimality conditions for the spot market, the SFE framework has also been applied to investigate the interactions between the forward market and the spot market. Green (1999) and Newbery (1998) are among the first researchers who study the impact of two-way contracts in conjunction with the SFE model and observe that contracts provide incentives for generators to supply more in a spot market. Anderson and Xu (2006) make further investigations in this direction by considering the optimal supply functions in electricity markets with option contracts and nonsmooth costs. However, calculating an SFE requires solving a set of differential equations instead of the typical set of algebraic equations as in Cournot models, which presents considerable limitations on the equilibrium conditions and the numerical tractability. Indeed, the existence of the SFE has been proved only for linear supply function models (Rudkevich 2005) and for symmetric models without capacity limit (Klemperer and Meyer 1989), with capacity constraints (Anderson and Xu 2005; Holmberg 2008), and there is no discussion about an SFE model for a two stage forward-spot market.

Along the direction of the research on EPEC and Cournot models, this paper makes a number of contributions. First, we present mathematical models for generator's optimal decisions and Nash–Cournot equilibrium problems in the forward-spot market. Second, we discuss the existence and uniqueness of Nash–Cournot equilibrium in the spot market and investigate properties of such equilibrium. Third, we show the existence of Nash–Cournot equilibrium in the forward market.

The rest of the paper is laid out as follows: in the next section, we give a detailed description of an SEPEC model for the forward-spot market competition, and show that the equilibrium in the spot market depends on the contract quantities rather than the strike price. In Sect. 3, we use a complementary program model to solve the equilibrium problem in the spot market, and obtain the existence and uniqueness results and the monotonicity of the supply functions with respect to the contract quantities. In Sect. 4, we show the existence of Nash–Cournot equilibrium of the forward-spot market interaction, and the continuity of each generator's profit in the forward market. In Sect. 5, we present some numerical tests to illustrate the theoretical results in this paper. Finally, in Sect. 6, we point out the restrictions of the paper and further work.

2 Mathematical description of the problem

In this section, we present mathematical details on modeling competition in the forward market and the spot market, and show that the optimization problem in the forward-spot market can be structured as a two stage stochastic equilibrium model. This model can be viewed as an extension of a similar model by Gans et al. (1998) in a duopoly to an oligopoly.

We suppose that there are M generators competing in a non-collaborative manner for dispatch in the spot market on daily basis and these generators are economically rational and risk neutral. In the spot market, *market demand* is characterized by an inverse demand function $p(Q, \xi(\omega))$, where $p(Q, \xi)$ is the spot price, Q is the aggregate dispatch quantity and $\xi(\omega)$ is a random shock. Here, $\xi : \Omega \to \mathbb{R}$ is a continuous random variable defined on probability space $(\Omega, \mathcal{F}, P)$ with known distribution. To ease the notation, we will write $\xi(\omega)$ as ξ and the context will make it clear when ξ should be interpreted as a deterministic vector. We denote by $\rho(\xi)$ the density function of the random shock and assume that ρ is well defined and has a support set Ξ.

Since the outcome of the clearing price $p(Q, \xi)$ is fluctuating in the spot market, both generators and retailers wishing to ensure a fixed or a stable electricity price to hedge the risks rising from the variation of the spot price can do so by signing forward electricity contracts. This kind of contracts can be taken as a financial instrument and does not involve actual transaction of power. There are essentially two types of contracts: a one-way contract such as a put option or a call option where only one side of the contract commits to pay the difference between the strike price and the spot price for the contracted quantity, and a two-way contract where both sides of the contracts commit to pay the prices difference as opposed to the one way contract. In this paper, we simplify the discussion by focusing on two-way contract, that is, each generator signs a two-way contract with retailers.

2.1 Generator's optimal decision problem in the spot market

We begin the model of the spot market by formulating a generator's profit function which involves three terms: a revenue from selling electricity in the spot market, the cost of generating the electricity and the difference due to the commitment to a contract.

First, we look into the term of each generator's commitment to its contract by giving details on the contract signing and the mechanism of generators' fulfillment in the spot market. We assume that, in the forward market, generator i, $i = 1, \ldots, M$, enters into a two-way contract at a fixed price $z_i(x_i, x_{-i})$ for an amount x_i, where $x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M)^T$ denotes the vector of contract quantities signed by its rivals and the superscript T denotes transpose. Here z_i is a function of x_i and x_{-i}. For the simplicity of notation, we write $z_i(x_i, x_{-i})$ as $z_i(x)$, where $x := (x_1, \ldots, x_M)^T$. We will come back to investigate the property of function z_i later on. Taking all forward contracts as financial instruments, we may regard the fulfillment of these contracts equally as generators' commitment to daily power supply over a certain time period. Under such contracts, generator i gets paid $x_i(p(Q, \xi) - z_i(x))$ from the other party of the contract when the market clearing price $p(Q, \xi)$ is greater than $z_i(x)$ and pays the other party by $x_i(z_i(x) - p(Q, \xi))$ otherwise.

Consider a spot market in which generators set their dispatch quantities before the realization of the market demand uncertainties. If generator i's dispatch quantity is q_i and the aggregate dispatch from its rivals is Q_{-i}, then at a demand scenario $p(\cdot, \xi)$, the market is cleared at the price $p(q_i + Q_{-i}, \xi)$ and each generator is paid at the price for their dispatch. Hence, we can formulate generator i's revenue from selling electricity q_i by $q_i p(q_i + Q_{-i}, \xi)$. Note that in this model, a generator can influence the market clearing price and hence its revenue by choosing a proper q_i. In reality, some markets allow generators to bid in a stack of quantities at an increasing order of prices for dispatch and the ISO forms a schedule of aggregate quantities at each price by putting them together. After the realization of the demand shock, the market clearing price is determined and all bids below the price get dispatched which are paid at the same price, see for instance Anderson and Philpott (2002) and references therein. Our work simplifies the bidding and clearing mechanism in the real market by looking at a generator's total dispatch/supply and aiming to capture some insights on how a generator plays its strategy to influence the spot market by adjusting its total supply of power, which is a Cournot model.

Finally, we assume that generation of an amount q_i by generator i incurs a total cost of $c_i(q_i)$, which is twice continuously differentiable for any $q_i \geq 0, i = 1, 2, \ldots, M$. Accordingly, generator i's profit in the spot market is

$$R_i(q_i, x, Q_{-i}, \xi) := q_i p(q_i + Q_{-i}, \xi) - c_i(q_i) - x_i \left(p(q_i + Q_{-i}, \xi) - z_i(x)\right).$$

Therefore, generator i's decision problem is to choose q_i to maximize $R_i(q_i, x, Q_{-i}, \xi)$, where x, ξ, Q_{-i} and $z_i(x)$ treated as fixed parameters, that is,

$$\max_{q_i \geq 0} R_i(q_i, x, Q_{-i}, \xi) := q_i p(q_i + Q_{-i}, \xi) - c_i(q_i) - x_i p(q_i + Q_{-i}, \xi) + x_i z_i(x). \tag{2.1}$$

In the following, we state two assumptions on each generator's implicit capacity limit, the differentiability of $p(\cdot, \xi)$ for $\xi \in \Xi$ and $c_i(\cdot)$ for $i = 1, 2 \ldots, M$. We first make the following assumption on generators' capacity limits.

Assumption 2.1 For each generator $i, i = 1, 2, \ldots, M$, there is a capacity limit q_i^u, such that

$$c_i'(q_i) \geq p(q_i, \xi), \quad \text{for} \quad q_i \geq q_i^u, \quad \xi \in \Xi.$$

Observe that, Assumption 2.1 is an implicit way of ensuring that each generator's dispatch quantity is upper bounded. This type of assumptions has been used by Sherali et al. (1983); DeWolf and Smeers (1997) in a deterministic version, and by DeMiguel and Xu (2008) in a stochastic version, for the same purpose. The assumption implies that even generator i was a monopoly, its marginal cost at output level q_i^u or above would exceed any possible market price. Therefore, none of the firms would wish to supply more than q_i^u. Moreover, we proceed to make some fairly standard assumptions on the inverse demand function and generators' cost functions.

Assumption 2.2 For $Q \geq 0$ and $q_i \geq 0, i = 1, 2, \ldots, M$, the inverse demand function $p(Q, \xi)$ and the cost function $c_i(q_i)$ satisfy the following:

(a) $p(Q, \xi)$ is twice continuously differentiable w.r.t. Q, and $p(Q, \xi)$ is a strictly decreasing and convex function of Q for every fixed $\xi \in \Xi$.

(b) $p'_Q(Q,\xi) + Qp''_Q(Q,\xi) \le 0$, for every $Q \ge 0$ and $\xi \in \Xi$.
(c) The cost function $c_i(q_i)$, $i = 1, 2, \ldots, M$, is twice continuously differentiable and $c'_i(q_i) \ge 0$ and $c''_i(q_i) \ge 0$ for any $q_i \ge 0$.

The assumption is fairly standard and used in Sherali et al. (1983), DeWolf and Smeers (1997) and Xu (2005) except the convexity of the inverse demand function. The convexity is required to establish some technical results in Lemma 3.1 and it covers a variety of demand functions such as linear multiplicative function, isoelastic function and logarithmic function. From the above assumptions and generators' profit functions, we give the following proposition to show that each generator's optimal dispatch quantity in the spot market does not depend on the strike price.

Proposition 2.3 *Generator i's optimal solution to* (2.1) *depends on the vector of contract quantities x, the spot market scenario ξ and the spot dispatches $\{q_1, \ldots, q_M\}$ but not the strike prices $\{z_1(x), \ldots, z_i(x), \ldots, z_M(x)\}$. Moreover, if generator i's contract quantity x_i is less than q_i^u, then its marginal profit is negative for $q_i > q_i^u$ under Assumptions* 2.1 *and* 2.2.

Proof Consider the derivative of generator i's profit maximization problem (2.1). Since ξ, x_i, Q_{-i} and $z_i(x)$ are fixed, differentiating R_i w.r.t. q_i, we have,

$$\frac{\partial R_i(q_i, x, Q_{-i}, \xi)}{\partial q_i} = p(q_i + Q_{-i}, \xi) + (q_i - x_i)p'_{q_i}(q_i + Q_{-i}, \xi) - c'_i(q_i). \quad (2.2)$$

Since the optimal solution is determined by the above derivative which is independent of $z_i(x)$, the first part of the conclusion follows.

To show the second part of the proposition, note that $p(q_i + Q_{-i}, \xi) - c'_i(q_i) < 0$ for $q_i \ge q_i^u$ under Assumption 2.1 and $(q_i - x_i)p'_{q_i}(q_i + Q_{-i}, \xi) < 0$ when $q_i \ge q_i^u$ as $q_i^u \ge x_i$ and $p'_{q_i}(q_i + Q_{-i}, \xi) < 0$. The conclusion follows. □

By Proposition 2.3, we can add the capacity constraint explicitly to the profit maximization problem (2.1):

$$\begin{aligned} \max_{q_i \in [0, q_i^u]} \quad & R_i(q_i, x, Q_{-i}, \xi) = q_i p(q_i + Q_{-i}, \xi) \\ & -c_i(q_i) - x_i p(q_i + Q_{-i}, \xi) + x_i z_i(x). \end{aligned} \quad (2.3)$$

A referee raised a question of whether we can replace the explicit capacity limit by assuming that $c'_i(q)$ increases steeply as q_i approaches q_i^u but not mentioning q_i^u explicitly. The potential benefit of doing this is that we don't need to consider the upper bound in the first order optimality conditions to be discussed in Sect. 3. The answer is yes. However, following Proposition 2.3, we can ignore the upper bound in the derivation of first order optimality conditions anyway because generator i's optimum will not be achieved beyond q_i^u. The additional benefit of giving an explicit q_i^u makes our profit maximization problem (2.3) well defined without specifying the properties of the underlying objective function.

2.2 Nash–Cournot equilibrium in the spot market

In the spot market, when market demand is realized, that is, every generator knows the inverse demand function $p(\cdot, \xi)$ giving the relationship between the clearing price and the aggregate dispatch quantity, and each generator sets its optimal dispatch quantity to the pool market by solving profit maximization problem (2.1), which means that generators play a Nash–Cournot game in the spot market, a situation that no generator can improve its profit in the spot market by changing its dispatch unilaterally while the other players keep their bids fixed. Following Proposition 2.3, if there exists a Nash–Cournot equilibrium in the spot market, it must be independent of strike price $z_i(x)$, for $i = 1, 2, \dots, M$. A formal definition of such an equilibrium can be given as follows.

Definition 2.4 A *Nash–Cournot equilibrium* in the spot market at demand scenario $p(\cdot, \xi)$ is an M-tuple $(q_1(x, \xi), \dots, q_M(x, \xi))$ where $q_i(x, \xi)$ solves (2.3) for $i = 1, \dots, M$.

Remark 2.5 The dependence of $q_i(x, \xi)$ on x_i is intuitive and follows from Proposition 2.3. However, the dependence of $q_i(x, \xi)$ on x_j needs some clarification. Let us look at (2.2). If we change x_j but q_j is not changed accordingly (e.g., $q_j \equiv 0$) for $j = 1, 2, \dots, M$ and $j \neq i$, then Q_{-i} does not change. In this case, $q_i(x, \xi)$ is not affected by the change of x_j. This implies that only when the change of x_j has an impact on Q_{-i}, it has an impact on $\frac{\partial R_i(q_i, x, Q_{-i}, \xi)}{\partial q_i}$, hence the optimal solution $q_i(x, \xi)$. Practically, it means that a generator can influence a market equilibrium in the spot market only by changing its dispatch quantity to the spot market. We will use this observation in Proposition 3.7.

From theoretical point of view, there may exist multiple equilibria although in practice only one of them is reached. We denote the set of these equilibria by $\mathbf{q}(x, \xi)$. We also use $q(x, \xi) = (q_1(x, \xi), \dots, q_M(x, \xi))^T$ to denote an equilibrium in the set $\mathbf{q}(x, \xi)$. Note also that the market clearing price $p(Q(x, \xi), \xi)$ is determined by the market equilibrium at the end of competition because the aggregate dispatch is $Q(x, \xi) = \sum_{i=1}^{M} q_i(x, \xi)$.

2.3 Generator's optimal decision problem in the forward market

In the forward market, when generators compete to sign contracts, they do not know what market clearing price will be in the spot market. We assume here that each generator knows: (a) generators play a Nash–Cournot game in the spot market; (b) there is an equilibrium in every scenario; (c) the inverse demand function $p(\cdot, \xi)$ and the distribution of ξ.

Under these assumptions, generator i's expected profit can be written as

$$\pi_i(x_i, x_{-i}) := \mathbb{E}\left[R_i(q_i(x, \xi), x, Q_{-i}(x, \xi), \xi)\right], \tag{2.4}$$

where $q_i(x, \xi)$ and $Q_{-i}(x, \xi)$ correspond to some equilibrium $q(x, \xi)$ in the spot market, and generator i aims to maximize its expected profit by choosing an optimal contract quantity x_i. It is important to note that this is a statistical average that generator i may expect before the competition in the spot market is realized.

Observe that if the spot market has multiple equilibria, then each generator may have its own prediction on an equilibrium $q(x,\xi) \in \mathbf{q}(x,\xi)$, and consequently $q(x,\xi)$ in the term $R_i(q_i(x,\xi), x, Q_{-i}(x,\xi), \xi)$ in (2.4) may depend on i, that is, it takes a value depending on generator i's view about the market equilibria. For instance, if generator i is optimistic, then it may expect the best equilibrium situation, that is, to choose $q(x,\xi) \in \mathbf{q}(x,\xi)$ such that $R_i(q_i(x,\xi), x, Q_{-i}(x,\xi), \xi)$ is maximized. See a similar discussion by Pang and Fukushima (2005) in a deterministic Nash equilibrium model and Shapiro and Xu (2005) in a stochastic mathematical program with equilibrium constraints (SMPEC) model. Therefore, the expected profit of generator i at the forward market can be formulated as:

$$\hat{\pi}_i(x_i, x_{-i}) := \mathbb{E}\Bigg[\max_{q(x,\xi)\in\mathbf{q}(x,\xi)} q_i(x,\xi)p\left(Q(x,\xi),\xi\right) \\ -c_i\left(q_i(x,\xi)\right) - x_i p(Q(x,\xi),\xi) + x_i z_i(x)\Bigg].$$

On the other hand, for a pessimistic generator i, it may expect the worst equilibrium situation, that is, to choose $q(x,\xi) \in \mathbf{q}(x,\xi)$ such that $R_i(q_i(x,\xi), x, Q_{-i}(x,\xi), \xi)$ is minimized, and the expected profit of generator i at the forward market can be formulated as:

$$\check{\pi}_i(x_i, x_{-i}) := \mathbb{E}\Bigg[\min_{q(x,\xi)\in\mathbf{q}(x,\xi)} q_i(x,\xi)p\left(Q(x,\xi),\xi\right) \\ -c_i\left(q_i(x,\xi)\right) - x_i p(Q(x,\xi),\xi) + x_i z_i(x)\Bigg].$$

Let us now focus on the strike price in the forward market. In practice, most generators are risk neutral. That means, with the perfect knowledge of the distribution of the demand scenario ξ, no generator will sign a contract at a strike price lower than the expected spot price, and similarly retailers will find no advantage to sign a contract at a strike price higher than the expected spot price. For the simplicity of discussion, we assume that every generator and retailer are risk neutral and they have the same view on a market equilibrium. This leads to the following assumption.

Assumption 2.6 The strike price in the forward market equals the expected spot market price, that is,

$$z_i(x) \in \left\{\mathbb{E}[p(Q(x,\xi),\xi)] : \ Q(x,\xi) = q^T(x,\xi)e, \ q(x,\xi) \in \mathbf{q}(x,\xi)\right\}, \quad (2.5)$$

where e is an M-dimensional vector with unit components.

This kind of assumption is not new and has been made by Gans et al. (1998), Su (2007) and Shanbhag (2005, Chapter 5). Under the risk neutrality assumption, if the spot market has a unique equilibrium in every demand scenario, then we have an identical strike price, that is, $z_1(x) = \cdots = z_M(x)$. Of course, if the spot market has multiple equilibria, and each generator has different view on a market equilibrium, then $z_i(x)$, $i = 1, \ldots, M$ may take different values and a contract can be agreed only when both parties of the contract have the same view on spot market equilibrium.

2.4 Nash–Cournot equilibrium in the forward market

For the simplification of discussion, we assume that $z_1(x) = \cdots = z_M(x)$ either because generators have the same views on spot market equilibrium or there is a unique equilibrium in every scenario. From a practical perspective, it means that, to each generator, every unit of contract defines the same obligation of energy dispatching in the spot market. Therefore, the expected profits of generators at the forward market can be rewritten as

$$\pi_i(x_i, x_{-i}) = \mathbb{E}\left[q_i(x,\xi)p\left(Q(x,\xi),\xi\right) - c_i\left(q_i(x,\xi)\right)\right],$$

for $i = 1, \ldots, M$ and its decision problem in the forward market is

$$\max_{x_i \geq 0} \pi_i(x_i, x_{-i}) = \mathbb{E}\left[q_i(x,\xi)p\left(Q(x,\xi),\xi\right) - c_i\left(q_i(x,\xi)\right)\right], \quad i = 1, \ldots, M, \tag{2.6}$$

that is, generators play a Nash–Cournot game when they compete to sign contracts in the forward market. We are interested in the outcome of competition by looking into an equilibrium of the Nash–Cournot game.

Definition 2.7 A stochastic equilibrium in the forward-spot market is a $2M$ tuple $(x_1^*, \ldots, x_M^*, q_1^*(x^*, \xi), \ldots, q_M^*(x^*, \xi))$ such that

$$\pi_i(x_i^*, x_{-i}^*) = \max_{x_i \geq 0} \pi_i(x_i, x_{-i}^*), \quad i = 1, \ldots, M, \tag{2.7}$$

$$q_i(x^*, \xi) \in \arg\max_{q_i \geq 0} R_i\left(q_i(x^*, \xi), x^*, Q_{-i}(x^*, \xi), \xi\right), \quad i = 1, \ldots, M, \quad \forall \xi \in \Xi, \tag{2.8}$$

and $(q_1(x^*, \xi), \ldots, q_M(x^*, \xi))$ is a Nash–Cournot equilibrium in demand scenario $p(\cdot, \xi)$.

The problem is essentially an SEPEC. Recently DeMiguel and Xu (2008) propose a stochastic multiple leader Stackelberg (SMS) model for a general oligopoly market where a group of firms compete to supply homogeneous goods to a future market and they model the problem as an SEPEC. The model extends Sherali's *deterministic* multiple-leader model (Sherali 1984) and De Wolf and Smeers' stochastic *single-leader* model (DeWolf and Smeers 1997). However, there are some fundamental differences between this model and the SMS model: (a) In the SMS model, only a few strategic firms (leaders) play a Nash–Cournot game at the first stage and the non-strategic firms (followers) do not participate in the competition. In our model, all generators compete in the forward market. (b) In the SMS model, leaders do not compete at the second stage after market demand is realized, and their commitments (supply) at the first stage are treated as given and consequently followers only compete for a residual demand. In our model, every generator must compete for dispatch in the spot market and their optimal strategy is affected by their commitments to forward contracts.

3 Equilibrium in the spot market

In this section, we investigate in detail Nash–Cournot equilibrium in the spot market at demand scenario $p(\cdot, \xi)$. We are particularly concerned with existence, uniqueness of equilibrium and properties of equilibrium as a function of forward contracts.

3.1 Existence and uniqueness of the equilibrium

First, before presenting further analysis on the existence and uniqueness of the equilibrium, we give some results on the strict concavity of each generator's profit function.

Lemma 3.1 *Under Assumption 2.2, for every $Q \geq 0$ and $\xi \in \Xi$*

(i) *$Qp(Q + K, \xi)$ is a concave function for any fixed $K \geq 0$.*
(ii) *For any fixed $K \geq 0$ and $X \geq 0$, $(Q - X)p(Q + K, \xi)$ is a strictly concave function of Q for $Q \geq 0$.*

The proof to Lemma 3.1 is given in the appendix. From the strict concavity of the function $(Q - X)p(Q, \xi)$, we can verify that each generator's objective function, $R_i(q_i, x, Q_{-i}, \xi)$, $i = 1, 2, \ldots, M$, is strictly concave w.r.t. q_i for fixed $Q_{-i} \geq 0$, $x \geq 0$ and $\xi \in \Xi$.

Proposition 3.2 *Let $R_i(q_i, x, Q_{-i}, \xi)$ be defined as in (2.1). Under Assumptions 2.6 and 2.2, $R_i(q_i, x, Q_{-i}, \xi)$ is strictly concave w.r.t. q_i.*

The conclusion follows straightforwardly from the convexity of $c_i(q_i)$ and the concavity of $(q_i - x_i)p(q_i + Q_{-i}, \xi)$ that is proved in Lemma 3.1 (ii).

Proposition 3.3 *Under Assumptions 2.1, 2.6 and 2.2, for every fixed $x_i \in [0, +\infty)$, $i = 1, 2, \ldots, M$ and $\xi \in \Xi$, there exists a unique Nash–Cournot equilibrium in the spot market, $q(x, \xi) = (q_1(x, \xi), \ldots, q_M(x, \xi))^T$, which solves the following problem*

$$q_i(x, \xi) \in \arg\max_{q_i \geq 0} \{R_i(q_i, x, Q_{-i}, \xi) = (q_i - x_i)p(q_i + Q_{-i}, \xi) - c_i(q_i) + x_i z_i(x)\}.$$

Moreover, $q_i(x, \xi) \in [0, \max\{q_i^u, x_i\}]$, for any fixed x and ξ with $i = 1, \ldots, M$.

Proof Since generator i's objective function $R_i(q_i, x, Q_{-i}, \xi)$, is strictly concave in q_i (here x, ξ are parameters), the existence of equilibrium follows from Rosen (1965, Theorem 1) while the uniqueness follows from Rosen (1965, Theorem 2) because the strict concavity implies the diagonally strict concavity of a weighted non-negative sum of the objective functions. Let us now look into the boundedness of the equilibrium. Because, for any fixed $\xi \in \Xi$ and $x_i \geq 0$, $R_i(q_i, x, Q_{-i}, \xi)$ is strictly concave, we have

$$\begin{aligned}\frac{dR_i(q_i, x, Q_{-i}, \xi)}{dq_i} &= p(q_i + Q_{-i}, \xi) + q_i p'_Q(q_i + Q_{-i}, \xi) \\ &\quad -c'_i(q_i) - x_i p'_Q(q_i + Q_{-i}, \xi) \\ &\leq p(q_i, \xi) + q_i p'_Q(q_i + Q_{-i}, \xi) - c'_i(q_i) - x_i p'_Q(q_i + Q_{-i}, \xi) \\ &\leq (q_i - x_i) p'_Q(q_i + q_{-i}, \xi) \leq 0,\end{aligned}$$

for any $q_i \geq \max\{q_i^u, x_i\}$. Hence, R_i achieves maximum in $[0, \max\{q_i^u, x_i\}]$. □

3.2 Properties of the equilibrium in the spot market

We now investigate properties of Nash–Cournot equilibrium $q(x, \xi)$ in the spot market by taking it as a function of x and ξ. We will also investigate the monotonicity of aggregate dispatch function $Q(x, \xi)$ w.r.t.x_i for $i = 1, 2, \ldots, M$. We do so by reformulating the Nash–Cournot equilibrium problem in the spot market as a nonlinear complementarity problem. The KKT conditions of the Nash–Cournot equilibrium problem can be written as

$$\begin{array}{c} p(Q, \xi) + (q_i - x_i) p'_Q(Q, \xi) - c'_i(q_i) + \mu_i = 0, \\ 0 \leq \mu_i \perp q_i \geq 0, \end{array} \tag{3.9}$$

for $i = 1, 2, \ldots, M$, where $0 \leq \mu_i \perp q_i \geq 0$ denotes that $q_i \geq 0$, $\mu_i \geq 0$ and at least one of them is equal to zero.

Denote generators' cost functions in a vector-valued form as $\mathbf{c}(q) = (c_1(q_1), \ldots, c_M(q_M))^T$ and $e = (1, \ldots, 1)^T$ with an appropriate dimension. Define a vector-valued function

$$G(q, x, \xi) := -p(q^T e, \xi)e - (q - x) p'_Q(q^T e, \xi) + \nabla \mathbf{c}(q),$$

where $\nabla \mathbf{c}(q) := (c'_1(q_1), \ldots, c'_M(q_M))^T$. The complementarity problem (3.9) can be rewritten as

$$0 \leq q \perp G(q, x, \xi) \geq 0. \tag{3.10}$$

Consequently, each generator's decision problem can be reformulated as a stochastic mathematical program with complementary constraints (SMPCC), where, for every $i = 1, \ldots, M$, generator i's decision problem is

$$\begin{aligned} &\max_{x_i \geq 0} \mathbb{E}\left[q_i(x, \xi) p(Q(x, \xi), \xi) - c_i(q_i(x, \xi))\right] \\ &\text{ s.t. } \; q(x, \xi) \text{ solves } 0 \leq q \perp G(q, x, \xi) \geq 0, \; \xi \in \Xi. \end{aligned}$$

It is well known that (3.10) can be reformulated as a system of nonsmooth equations as

$$F(q, x, \xi) := \min(G(q, x, \xi), q) = 0, \tag{3.11}$$

where 'min' is taken componentwise.

In what follows, we use Eq. (3.11) to investigate the dependence of q on x and ξ. Observe that F is only piecewise smooth, therefore we need to use the Clarke generalized implicit function theorem rather than the classical implicit function theorem to derive the implicit function $q(x, \xi)$ defined by (3.11).

Definition 3.4 (Clarke generalized Jacobian/subdifferential) Let $H : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz continuous function. The Clarke generalized Jacobian (Clarke 1983) of H at $w \in \mathbb{R}^n$ is defined as

$$\partial H(w) \equiv \text{conv}\left\{ \lim_{y \in D_H,\ y \to w} \nabla H(w) \right\},$$

where 'conv' denotes the convex hull of a set and D_H denotes the set of points in a neighborhood of x at which H is Frechét differentiable.

When $m = 1$ or $n = 1$, ∂H is also called Clarke subdifferential. When $n = m$, the Clarke generalized Jacobian $\partial H(x)$ is said to be *non-singular* if every matrix in $\partial H(x)$ is non-singular. From Definition 3.4, we can observe that the Clarke subdifferential coincides with the usual gradient $\nabla H(x)$ at the point x where $H(\cdot)$ is strictly differentiable. Note that a number of functions in this paper are piecewise continuously differentiable, which means that at "most" points, the Clarke subgradient coincides with the classical gradient. The additional benefit of the Clarke notion provides us a derivative tool to deal with a "few" points where the classical derivatives do not exist and traditional right/left derivative approach make discussions complicated and indeed not working when dealing with vector valued functions. By using the Clarke notion, we have a unified derivative tool for both "differentiable points" and "nondifferentiable points".

Theorem 3.5 *Let $F(q, x, \xi)$ be defined as in* (3.11)*. Under Assumptions* 2.1*,* 2.2 *and* 2.6*, the following results hold.*

(i) *$\partial_q F(q, x, \xi)$ is non-singular for $q \geq 0$ and $x \geq 0$.*
(ii) *For every $x \geq 0$ and $\xi \in \Xi$, there exists a unique q such that $F(q, x, \xi) = 0$.*
(iii) *There exists a unique Lipschitz continuous and piecewise smooth function $q(x, \xi)$ defined on $[0, +\infty) \times \Xi$ such that*

$$F(q(x, \xi), x, \xi) = 0.$$

The theorem above shows that under Assumptions 2.1, 2.2 and 2.6, there exists a unique Nash–Cournot equilibrium in the spot market for every x and ξ, and the equilibrium is a vector valued function of x and ξ which is Lipschitz continuous and piecewise smooth. In what follows, we investigate the subdifferentials of the dispatch function $q(x, \xi)$ in the spot equilibrium and the aggregate dispatch $Q(x, \xi)$ w.r.t. x_i and ξ. This is to examine the impact of the changes of individual generator's contract level and random shock ξ on the market equilibrium and the aggregate dispatch in the spot market. We need the following assumption to guarantee that, for every demand scenario, there is at least one generator whose dispatch quantity to the spot market is strictly positive. Obviously, this is always satisfied in the real electricity market.

Assumption 3.6 Suppose that, for every $\xi \in \Xi$ and x signed in the forward market, the inverse demand function $p(\cdot, \xi)$ and the cost functions $c_i(\cdot)$ satisfy

$$\min_{i=1,\ldots,M} c_i'(0) < p(Q(x, \xi), \xi). \tag{3.12}$$

The assumption implies that at any demand scenario, and for any contract quantities x signed in the forward market, there is at least one generator whose marginal cost of producing a very small amount of electricity is strictly lower than the market clearing price, which means that there exists at least one generator which is profitable by supplying a small amount of electricity in the spot market. This assumption excludes the case that no generator is willing to sell electricity in a particular scenario.

Proposition 3.7 *Let $F(q, x, \xi)$ be defined as in (3.11). Under Assumptions 2.1, 2.2, 2.6 and 3.6, we have the following.*

(i) *The Clarke generalized Jacobian of $q(x, \xi)$ w.r.t. x can be estimated as follows:*

$$\partial_x q(x, \xi) \subset \text{conv}\Big\{-W^{-1}U : (W, U, V) \in \partial F(q(x, \xi), x, \xi), \\ W \in \mathbb{R}^{M\times M},\ U \in \mathbb{R}^M,\ V \in \mathbb{R}\Big\}. \tag{3.13}$$

(ii) *The Clarke subdifferential of the aggregate dispatch function, $Q(x, \xi)$, w.r.t. x_i, for $i = 1, \ldots, M$, can be estimated as*

$$\partial_{x_i} Q(x, \xi) \subset [0, 1).$$

The lower bound is reached only when $q_i(x, \xi) = 0$.

(iii) *The Clarke subdifferential of generator i's dispatch function, $q_i(x, \xi)$, w.r.t. x_i, can be estimated as*

$$\partial_{x_i} q_i(x, \xi) \subset [0, 1).$$

The lower bound is reached only when $q_i(x, \xi) = 0$.

(iv) *The Clarke subdifferential of $q_i(x, \xi)$, w.r.t. x_j can be estimated as*

$$\partial_{x_j} q_i(x, \xi) \subset (-1, 0].$$

The upper bound is reached only when $q_j(x, \xi) = 0$.

(v) *If* $p''_{Q,\xi}(Q,\xi) = 0$, *then* $q_i(x,\xi)$ *is an increasing function of* ξ; *moreover, if there exists a constant* $C \geq 0$ *such that*

$$p'_Q(Q,\xi) + p''_Q(Q,\xi)(q-x)^T e < -C, \quad \text{for } Q \geq 0, x \geq 0 \text{ and } \xi \in \Xi,$$

then the Clarke subdifferential of $Q(x,\xi)$ *w.r.t.* ξ *can be estimated as follows:*

$$\partial_\xi Q(x,\xi) \subset \left(0, \frac{1}{C} p'_\xi(Q(x,\xi),\xi)\right].$$

We provide a proof on these technical results in the appendix. Moreover, some economic interpretations for these results can be given as following: Part (ii) indicates that every unit increase of contract quantity by a generator in the forward market will result in an increase of the aggregate dispatch of all generators in the spot market by less than one unit. Part (iii) has a similar interpretation for an individual generator. Part (iv) means that generator i's dispatch will be reduced by less than one unit if one of its rivals increases one unit in its contract quantity.

To give an intuitive interpretation of the results in this section, we present a simple example of a duopoly market.

Example 3.8 Consider an electricity market with two generators, A and B. The generators' cost functions are

$$c_A(q_A) = 0.8q_A, \qquad c_B(q_B) = q_B,$$

where q_A and q_B denote A and B's quantities for dispatches in the spot market, respectively. We assume that the inverse demand function is

$$p(q_A + q_B, \xi) = \alpha(\xi) - \beta(q_A + q_B),$$

where $\alpha(\xi) = 7 + \xi$, $\beta = 2$, and the random shock ξ follows a uniform distribution on the set [0, 1]. Denote the contract positions of A and B in the forward market by x_A and x_B. The inverse demand function after the realization of the random shock ξ is

$$p(q_A + q_B, \xi) = 7 + \xi - 2(q_A + q_B).$$

Let $q_A^u = 3.6$ and $q_B^u = 3.5$ be the capacity limits of A and B. In the spot market, generator A and B's profit maximization problems can be respectively written as

$$\begin{aligned}
&\max_{q_A \in [0, q_A^u]} R_A(q_A, q_B, x, \xi) = -2q_A^2 + q_A(6.2 - 2q_B + \xi + 2x_A) - x_A(7 - 2q_B + \xi),\\
&\max_{q_B \in [0, q_B^u]} R_B(q_B, q_A, x, \xi) = -2q_B^2 + q_B(6 - 2q_A + \xi + 2x_B) - x_B(7 - 2q_A + \xi),
\end{aligned} \tag{3.14}$$

It is easily verify that, for any $\xi \in [0, 1]$, $\forall q_A \geq q_A^u$ and $\forall q_B \geq q_B^u$, we have the following inequalities,

$$\begin{aligned} p(q_A, \xi) &\leq 7 + \xi - 2q_A^u \leq 0.8 = c'_A(q_A), \\ p(q_B, \xi) &\leq 7 + \xi - 2q_B^u \leq 1 = c'_B(q_B), \end{aligned} \tag{3.15}$$

which imply that Assumption 2.1 holds in this example. According to our discussion following Assumption 2.1, (3.15) implicitly ensures that the optimal solution $q_i^*(x, \xi)$ satisfy $q_i^*(x, \xi) \leq q_i^u$ for $i = A, B$ and will never go beyond q_A^u and q_B^u in every scenario $\xi \in \Xi$. Therefore, the constraints $q_i \leq q_i^u$ for $i = A, B$ in (3.14) are not active, and generator A and B's profit maximization problems can be respectively reformulated as

$$\begin{aligned} &\max_{q_A \geq 0} R_A(q_A, q_B, x, \xi) = -2q_A^2 + q_A(6.2 - 2q_B + \xi + 2x_A) - x_A(7 - 2q_B + \xi), \\ &\max_{q_B \geq 0} R_B(q_B, q_A, x, \xi) = -2q_B^2 + q_B(6 - 2q_A + \xi + 2x_B) - x_B(7 - 2q_A + \xi), \end{aligned}$$

where R_A and R_B are quadratic functions. Therefore, the optimal dispatches in the spot market satisfy the following first-order conditions:

$$\begin{aligned} 0 \leq q_A(x, \xi) \perp 4q_A(x, \xi) - (6.2 - 2q_B(x, \xi) + \xi + 2x_A) \geq 0, \\ 0 \leq q_B(x, \xi) \perp 4q_B(x, \xi) - (6 - 2q_A(x, \xi) + \xi + 2x_B) \geq 0. \end{aligned} \tag{3.16}$$

Note that, the case $q_A = q_B = 0$ is excluded by Assumption 3.6 for (3.17). From (3.16), we have

$$\begin{aligned} &(q_A(x, \xi), q_B(x, \xi)) \\ &= \begin{cases} \left(0,\ \frac{1}{4}(2x_B + 6 + \xi)\right), & \text{if } q_A = 0; \\ \left(\frac{1}{4}(2x_A + 6.2 + \xi),\ 0\right), & \text{if } q_B = 0; \\ \left(\frac{1}{6}(4x_A - 2x_B + \xi + 6.4),\ \frac{1}{6}(4x_B - 2x_A + \xi + 5.8)\right), & \text{otherwise.} \end{cases} \end{aligned} \tag{3.17}$$

Equation (3.17) implies that $\partial_{x_i} q_i$ is a subset of $[0, 1/2]$ or $[1/2, 2/3]$ for $i = A, B$, and $\partial_{x_j} q_i \subset [-1/3, 0]$ for $i, j = A, B$ and $i \neq j$, which verifies the results (iii) and (iv) in Proposition 3.7. Moreover, the aggregated dispatch quantity can be written as

$$Q(x, \xi) = \begin{cases} \frac{1}{4}(6 + \xi + 2x_B), & \text{if } q_A = 0; \\ \frac{1}{4}(6.2 + \xi + 2x_A), & \text{if } q_B = 0; \\ \frac{1}{3}(6.1 + \xi + x_A + x_B), & \text{otherwise,} \end{cases} \tag{3.18}$$

which implies $\partial_{x_i} Q$ is a subset of $[0, 1/2]$ or $[1/3, 1/2]$, and hence the result (ii) in Proposition 3.7. Observe that (3.17) and (3.18) provide us with a further properties, that is, at the demand scenario ξ, if $q_i(x, \xi) \equiv 0$ for every x, then $\partial_{x_j} q_i(x, \xi) \equiv \{0\}$ for $i, j = A, B$ and $\partial_{x_i} Q(x, \xi) \equiv \{0\}$. This fact verifies the lower bounds in the results (ii) and (iii), and the upper bound in the result (iv) in Proposition 3.7.

4 Equilibrium in the forward market

In this section, we investigate the competition in the forward market. We do so by looking into the existence of a Nash–Cournot equilibrium in the forward market as defined in Definition 2.7. For the simplification of discussion, we assume that the spot market has a unique Nash–Cournot equilibrium, $q(x,\xi) = (q_1(x,\xi), q_2(x,\xi), \ldots, q_M(x,\xi))^T$ for every x and ξ. First, from Proposition 3.7, we can establish a relationship between the strike price and the contract quantities in the following proposition.

Proposition 4.1 *Under Assumptions 2.1, 2.2 and 2.6, the strike price z is a function of the contract quantities x, that is, $z(x) = \mathbb{E}[p(Q(x,\xi),\xi)]$. Moreover, the elements in the set $\partial_{x_i} z(x)$ are all non-positive.*

Proof Under Assumption 2.6 and the uniqueness of the supply functions $q_i(x,\xi)$ in the spot equilibrium, we have, $z(x) = \mathbb{E}[p(Q(x,\xi),\xi)]$. The Clarke subdifferential of $z(x)$ is

$$\partial_{x_i} z(x) = \partial_{x_i} \mathbb{E}[p(Q(x,\xi),\xi)].$$

Since the inverse demand function $p(Q,\xi)$ is a continuously differentiable function of Q(see Assumption 2.2), and $Q(x,\xi)$ is a Lipschitz continuous function of each x_i proved in Proposition 3.5(iii), we have, $p(Q(x,\xi),\xi)$ is also a Lipschitz continuous function of x_i. Therefore, from Clarke (1983, Theorem 2.7.5),

$$\partial_{x_i} \mathbb{E}[p(Q(x,\xi),\xi)] \subset \mathbb{E}[\partial_{x_i} p(Q(x,\xi),\xi)] \subset \mathbb{E}[p'_Q(x,\xi)\partial_{x_i} Q(x,\xi)].$$

Moreover, by Part (ii) of Theorem 3.5,

$$\mathbb{E}[p'_Q(Q(x,\xi),\xi)\partial_{x_i} Q(x,\xi)] \subset (p'_Q(Q(x,\xi),\xi), 0] \subset (-\infty, 0].$$

This completes the proof. □

Proposition 4.1 establishes a relationship between the strike price and a generator's contract quantity in the forward market, in which the negativity of the elements in $\partial_{x_i} z(x)$ implies that any unilateral increase of the contract quantity by a generator never results in an increase of the strike price.

4.1 Differentiability of the expected profit

We now discuss the continuity and differentiability of a generator's objective function in the forward market and investigate the change of the expected profit of an individual generator against the change of its contract quantity. To avoid too much mathematical details and make our analysis more readable, we move all the detailed proofs of the lemmas and theorem in this subsection to the appendix. We start by considering the first order derivative. Recall that

$$\pi_i(x_i, x_{-i}) := \int\limits_{\xi \in \Xi} [q_i(x,\xi) p(Q(x,\xi),\xi) - c_i(q_i(x,\xi))]\,\rho(\xi) d\xi,$$
$$\text{for} \quad i = 1, 2, \ldots, M.$$

Obviously, the only component in the integrand which may cause nondifferentiability of the integrand and hence $\pi_i(x_i, x_{-i})$ is $q_j(x,\xi)$, $j = 1, \ldots, M$ and $j \neq i$. In what follows, we demonstrate that under some moderate condition, the piecewise smoothness of $q(x,\xi)$ may not cause nondifferentiability of $\pi_i(x_i, x_{-i})$.

Assumption 4.2 The inverse demand function and the cost functions satisfy the following.

(i) For any fixed $\xi \in \Xi$, there exists an $L_1(\xi) \geq 0$ such that

$$\max\left(-p'_Q(Q(x,\xi),\xi), p''_Q(Q(x,\xi),\xi), c_i(q_i(x,\xi))\right) \leq L_1(\xi),$$
$$\forall x_i \geq 0, \quad i = 1, 2, \ldots, M,$$

and $\sup_{\xi \in \Xi} L_1(\xi) < \infty$.

(ii) There exists a constant $\sigma \geq 0$ such that $-p'_Q(Q(x,\xi),\xi) + c''_i(q_i(x,\xi)) > \sigma$, for all $\xi \in \Xi$ and $x_i \geq 0$ for $i = 1, 2, \ldots, M$.

Under Assumption 4.2, we need a couple of intermediate results, Lemmas 4.3 and 4.4, to obtain the main result on the twice continuous differentiability of $q_i(x,\xi)$ w.r.t. x_i in Theorem 4.5. For the clarity of notation, we write $q_i(x,\xi)$ as $q_i(x_i, x_{-i}, \xi)$ to distinguish x_i and x_{-i} because x_{-i} will be treated as parameters when we analyze the sensitivity of the quantities w.r.t. x_i.

Lemma 4.3 *Under Assumptions 2.1, 2.6, 2.2, 3.6 and 4.2, the following results hold.*

(i) *For each $i = 1, \ldots, M$, $q_i(x_i, x_{-i}, \xi)$ is a piecewise continuously differentiable and increasing function of x_i.*

(ii) *For $x_j \geq 0$, $j = 1, 2, \ldots, M$, $j \neq i$ and $\xi \in \Xi$, $q_i(x_i, x_{-i}, \xi)$ is globally Lipschitz continuous w.r.t. x_i; that is, there exists a function $L_2^i(\xi)$, $i = 1, \ldots, M$, such that*

$$|q_i(x_i^{(1)}, x_{-i}, \xi) - q_i(x_i^{(2)}, x_{-i}, \xi)| \leq L_2^i(\xi)|x_i^{(1)} - x_i^{(2)}|, \ \forall x_i^{(1)}, x_i^{(2)} \geq 0,$$

where $\int_{\xi \in \Xi} L_2^i(\xi)\rho(\xi)d\xi < \infty$.

From the part (i) of Lemma 4.3, we know that $q_i(x_i, x_{-i}, \xi)$ is a nondecreasing function in x_i, and thus there exists a unique point at which $q_i(x_i, x_{-i}, \xi)$ turns from zero to positive as x_i increases, and we denote this point by $x_i(\xi)$. In economic terms, given the contract position x_{-i} signed by generator i's rivals, for a realized demand shock $\xi \in \Xi$, $x_i(\xi)$ is the contract position at which generator i's marginal profit in the spot market becomes from zero to positive, and its dispatch quantity also becomes from zero to positive. Mathematically, $x_i(\xi)$ can be regarded as a degenerate point of the complementarity problem (3.10) because at this point, both $G_i(q(x,\xi), x_i, \xi)$ and $q_i(x_i, x_{-i}, \xi)$ are equal to zero. Note that $q_i(x_i, x_{-i}, \xi)$ is not differentiable w.r.t. x_i at the point $x_i(\xi)$. From a practical perspective, part (i) of Lemma 4.3 implies that, the more contracts (in the sense of quantities) a generator signs in the forward market, the more dispatch the generator will commit in the spot market.

In what follows, we investigate the set of degenerate points $x_i(\xi)$ for a given x. This is because these degenerate points may result in non-differentiability of the integrand of $\pi_i(x_i, x_{-i})$ and potentially further result in the non-differentiability of $\pi_i(x_i, x_{-i})$ if there are too many such points (in the sense that the Lebesgue measure of the set of such points is non-zero). The following lemma states that the number of degenerate points are actually finite which implies that they will not cause non-differentiability of $\pi_i(x_i, x_{-i})$.

Lemma 4.4 *Let* $\Xi_i(x) := \{\xi \in \Xi | x_i = x_i(\xi)\}$ *and* $\Xi(x) := \bigcup_{i=1}^{M} \Xi_i(x)$. *Under Assumptions 2.1, 2.6, 2.2, 3.6 and 4.2,* $\Xi(x)$ *is a finite set for any* x.

Note that, from the definition of $\Xi(x)$, given x, $\Xi_i(x) \neq \phi$ means that there is a ξ such that generator i's dispatch quantity $q_i(x_i, x_{-i}, \xi)$ turns from zero to strictly positive, that is, the ith element of x is $x_i(\xi)$. Therefore, $\Xi_i(x)$ is the set of points $\xi \in \Xi$ at which generator i's dispatch quantity turns from zero to positive, and $\Xi(x)$ is the set of points $\xi \in \Xi$ at which the dispatch quantity of at least one of generators turns from zero to positive.

As observed in the proof of Lemma 4.4 in the appendix, $x_i(\xi)$ is a decreasing function of ξ to maintain the property that $q_i(x_i(\xi), x_{-i}, \xi) \equiv 0$ for $x_i \leq x_i(\xi)$ and $q_i(x_i(\xi), x_{-i}, \xi) > 0$ for $x_i > x_i(\xi)$. For given x_{-i} and x_i, let us define

$$v_i(x_i, \xi) := (q_i(x_i, x_{-i}, \xi) - x_i) p(Q(x, \xi), \xi) - c_i(q_i(x_i, x_{-i}, \xi)). \quad (4.19)$$

The only values of ξ at which $v_i(\cdot, \xi)$ might not be differentiable w.r.t. x_i are points ξ at which the dispatch of one of generator turns from positive to zero. These are only points at which $Q(x, \xi)$ might not be differentiable w.r.t. x_i and thus $v_i(\cdot, \xi)$ might not be differentiable w.r.t. x_i. By Lemma 3.4 in DeMiguel and Xu (2008), $\Xi_i(x_i)$ is a finite set, which implies that $Q(x, \xi)$ is differentiable w.r.t. x_i for almost every $\xi \in \Xi$ and thus $v_i(x_i, \xi)$ is differentiable w.r.t. x_i for almost every $\xi \in \Xi$. We are now able to address the main results of this section.

Theorem 4.5 *Suppose that there exists* $L_3(\xi) \geq 0$ *such that* $\int_\Xi L_3(\xi)\rho(\xi)d\xi < \infty$ *and*

$$\max \left(p(Q, \xi), |p'''_Q(Q, \xi)|, |Q''_{x_i}(x, \xi)| \right) \leq L_3(\xi),$$

for all $Q \geq 0, \xi \in \Xi$ *and* x_i *with* $i = 1, \ldots, M$, *at which* $Q(x, \xi)$ *is twice continuously differentiable w.r.t.* x_i. *Then, under Assumptions 2.1, 2.6, 2.2, 3.6 and 4.2,* $\pi_i(x_i, x_{-i})$ *is twice continuously differentiable.*

In what follows, we explain Theorem 4.5 through a simple example based on Example 3.8.

Example 4.6 (Continued from Example 3.8) Consider a duopoly market as described in Example 3.8. From the definition, for any fixed x_B, the degenerate point $x_A(\xi)$ (at which $q_A(x_A, x_B, \xi)$ turns from zero to positive as x_A increases) can be identified by solving the following equations

$$q_A(x_A(\xi), x_B, \xi) = 0; \quad \text{and}$$
$$4q_A(x, \xi) - (6.2 - 2q_B(x_A(\xi), x_B, \xi) + \xi + 2x_A(\xi)) = 0,$$

where $x_A(\xi) \geq 0$. By solving (3.16), we obtain

$$x_A(\xi) = \frac{1}{2}(x_B - 0.5\xi - 3.2), \quad \text{for fixed } x_B. \tag{4.20}$$

Combining the condition that $x_A(\xi) \geq 0$, we can see that for fixed x_A there exists at most one ξ such that $q_A(x_A, x_B, \xi)$ is possibly *not* differentiable. This implies that the cardinality of the set $\Xi_A(x)$ is at most 1, and hence verifies Lemma 4.4.

In what follows, we look into Theorem 4.5. For the sake of simplicity, we only verify the differentiability of $\pi_A(x_A, x_B)$ in x_A [1] and $x_B \leq 3.2$[2]. We can obtain q_A and q_B by solving the following complementarity problem:

$$\begin{gathered} 4q_A(x_A, x_B, \xi) - (6.2 - 2q_B(x_A, x_B, \xi) + \xi + 2x_A) = 0, \\ 0 \leq q_B(x_A, x_B, \xi) \perp 4q_B(x_A, x_B, \xi) - (6 - 2q_A(x_A, x_B, \xi) + \xi + 2x_B) \geq 0. \end{gathered} \tag{4.21}$$

From (3.15) in Example 3.8, q_A and q_B can be expressed as:

$$\begin{cases} (q_A^I, q_B^I) = \left(\frac{1}{4}(6.2 + \xi + 2x_A), 0\right), & \text{if } \xi \in [0, 2x_A - 4x_B - 5.8], \\ (q_A^{II}, q_B^{II}) = \left(\frac{1}{6}(6.4 + \xi + 4x_A - 2x_B)\right., & \\ \qquad\qquad \left.\frac{1}{6}(5.8 + \xi + 4x_B - 2x_A)\right), & \text{if } \xi \in [2x_A - 4x_B - 5.8, 1], \end{cases}$$

where the two smooth pieces $\{(q_A^I, q_B^I)\}$ and $\{(q_A^{II}, q_B^{II})\}$ intersect at the point $x_A = 2x_B + 2.9 + 0.5\xi$, where $(q_A^I, q_B^I) = (q_A^{II}, q_B^{II}) = \left(\frac{6.2+\xi+2x_A}{4}, 0\right)$. In other words, at any fixed point x_A and x_B, there is at most one ξ such that q_A and q_B are not differentiable w.r.t. variable x_A. Consequently, generator A's expected profit in the forward market can be calculated as follows:

[1] As we can do for x_B in the same way.

[2] There will be two nondifferentiable points when $x_B > 3.2$.

$$\pi_A(x_A, x_B) = \begin{cases} \int_0^1 \left[q_A^I p(Q^I, \xi) - c_A(q_A^I)\right] \rho(\xi) d\xi, & \text{if } \bar{\xi} \geq 1; \\ \int_0^{\bar{\xi}} \left[q_A^I p(Q^I, \xi) - c_A(q_A^I)\right] \rho(\xi) d\xi + \int_{\bar{\xi}}^1 \left[q_A^{II} p(Q^{II}, \xi) - c_A(q_A^{II})\right] \rho(\xi) d\xi, & \text{if } 0 < \bar{\xi} < 1; \\ \int_0^1 \left[q_A^{II} p(Q^{II}, \xi) - c_A(q_A^{II})\right] \rho(\xi) d\xi, & \text{if } \bar{\xi} \leq 0, \end{cases}$$

where $Q^I = q_A^I + q_B^I$, $Q^{II} = q_A^{II} + q_B^{II}$ and $\bar{\xi}$ denotes $\bar{\xi}(x_A) := 2(x_A - 2x_B - 2.9)$.

Calculating the derivative $\frac{\partial \pi_A(x_A, x_B)}{\partial x_A}$ for the case $0 < \bar{\xi} < 1$, we have

$$\begin{aligned} \frac{\partial \pi_A(x_A, x_B)}{\partial x_A} = & \int_0^{\bar{\xi}(x_A)} \frac{\partial \left[q_A^I p(Q^I, \xi) - c_A(q_A^I)\right]}{\partial x_A} \rho(\xi) d\xi \\ & + \frac{\partial \bar{\xi}(x_A)}{\partial x_A} \Big[q_A^I(x_A, x_B, \bar{\xi}(x_A)) p(Q^I(x, \bar{\xi}(x_A)), \bar{\xi}(x_A)) \\ & \qquad - c_A(q_A^I(x_A, x_B, \bar{\xi}(x_A))) \Big] \rho(\bar{\xi}(x_A)) \\ & + \int_{\bar{\xi}(x_A)}^1 \frac{\partial \left[q_A^{II} p(Q^{II}, \xi) - c_A(q_A^{II})\right]}{\partial x_A} \rho(\xi) d\xi \\ & - \frac{\partial \bar{\xi}(x_A)}{\partial x_A} \Big[q_A^{II}(x_A, x_B, \bar{\xi}(x_A)) p(Q^{II}(x, \bar{\xi}(x_A)), \bar{\xi}(x_A)) \\ & \qquad - c_A(q_A^{II}(x_A, x_B, \bar{\xi}(x_A))) \Big] \rho(\bar{\xi}(x_A)). \end{aligned}$$

Since at the point $x_A = 2x_B + 2.9 + 0.5\bar{\xi}(x_A)$,

$$\begin{aligned} & \left(q_A^I(x_A, x_B, \bar{\xi}(x_A)), q_B^I(x_A, x_B, \bar{\xi}(x_A)) \right) \\ & = \left(q_A^{II}(x_A, x_B, \bar{\xi}(x_A)), q_B^{II}(x_A, x_B, \bar{\xi}(x_A)) \right). \end{aligned}$$

and then $\partial \pi_A(x_A, x_B)/\partial x_A$ above can be simplified as

$$\begin{aligned} \frac{\partial \pi_A(x_A, x_B)}{\partial x_A} = & \int_0^{\bar{\xi}(x_A)} \frac{\partial \left[q_A^I p(Q^I, \xi) - c_A(q_A^I)\right]}{\partial x_A} \rho(\xi) d\xi \\ & + \int_{\bar{\xi}(x_A)}^1 \frac{\partial \left[q_A^{II} p(Q^{II}, \xi) - c_A(q_A^{II})\right]}{\partial x_A} \rho(\xi) d\xi. \end{aligned} \tag{4.22}$$

Moreover, for $\bar{\xi} \geq 1$ and $\bar{\xi} \leq 0$ we have

$$\frac{\partial \pi_A(x_A, x_B)}{\partial x_A} = \begin{cases} \int_0^1 \frac{\partial [q_A^I p(Q^I,\xi) - c_A(q_A^I)]}{\partial x_A} \rho(\xi) d\xi, & \text{if } \bar{\xi} \geq 1; \\ \int_0^1 \frac{\partial [q_A^{II} p(Q^{II},\xi) - c_A(q_A^{II})]}{\partial x_A} \rho(\xi) d\xi, & \text{if } \bar{\xi} \leq 0, \end{cases} \tag{4.23}$$

Combining both (4.22) and (4.23), we can see that $\frac{\partial \pi_A(x_A,x_B)}{\partial x_A}$ is a continuous function of x_A and hence $\pi_A(x_A, x_B)$ is continuously differentiable w.r.t. x_A. Repeating the process above on derivative $\frac{\partial \pi_A(x_A,x_B)}{\partial x_A}$, we can show that $\pi_A(x_A, x_B)$ is twice continuously differentiable. This verifies the result in Theorem 4.5.

4.2 Existence of the forward-spot equilibrium

We now move on to discuss the existence of Nash–Cournot equilibrium in the forward-spot market. A well known sufficient condition for the existence is the concavity or quasi-concavity of each generator's objective function on its strategy space. See for instance Rosen (1965, Theorem 1) and Yuan and Tarafdar (1996, Theorem 1). It turns out, however, very difficult to show this kind of 'global' concavity here. For this reason, we look into the local concavity and consequently investigate the existence of 'local Nash equilibrium'. The notion is used by Hu and Ralph for modeling a bilevel games in an electricity market with locational prices, see Hu and Ralph (2007) for details. As noted in Hu and Ralph (2007), the concept of local Nash equilibrium is proposed as a weaker alternative to Nash equilibrium for the electricity market. From a viewpoint of the real market, given that the global optima of nonconcave maximization problems are difficult to identify, the limitation of knowledge of generators may lead to meaningful local Nash equilibria, in which the local optimality is sufficient for the satisfaction of generators. Moreover, given the condition that the spot market is always profitable for every generator at every scenario ξ, we establish our main results on the existence of the global Nash equilibrium in the forward-spot market. We start by giving a definition on local Nash equilibrium.

Definition 4.7 (*Local Nash equilibrium*) x^* is a *local Nash–Cournot equilibrium* of the forward market if for each i, x_i^* is a local optimal solution to the problem

$$\max_{x_i \geq 0} \pi_i(x_i, x_{-i}^*) = \mathbb{E}\left[q_i(x_i, x_{-i}^*, \xi) p(Q(x_i, x_{-i}^*, \xi), \xi) \right.$$
$$\left. - c_i(q_i(x_i, x_{-i}^*, \xi))\right], \quad i = 1, 2, \ldots, M.$$

Comparing to their global counterparts, local Nash equilibria seem deficient. However, for some decision-making problems, given that global optima are difficult to identify because of the nonconcave objective functions, local optimality may be sufficient for the satisfaction of players. For instance, generators may only optimize their contract positions locally due to limited information on the forward market or general conservativeness. To illustrate the existence of the local Nash–Cournot equilibrium in the forward-spot market, we present the following example based on the duopoly model in Example 3.8.

Example 4.8 (Continued from Example 4.6) Consider a duopoly market described in Example 4.6, in which the capacity limits of generator A and B are $q_A^u = 3.6$ and $q_B^u = 3.5$, respectively.

Define

$$X = \{(x_A, x_B) \,|\, x_A > 0.4,\ x_B > 0.6\},$$

and $x := (x_A, x_B) \in X$. Let

$$X^+ = \{x = (x_A, x_B) \,|\, q_A(x, \xi) > 0,\ q_B(x, \xi) > 0,\ \forall \xi \in [0, 1]\}.$$

That is, if contract position $x = (x_A, x_B)$ is in X^+, then for all $\xi \in \Xi$, the dispatch of each generator in the spot market is always strictly positive. It is easy to verify that X^+ is an open convex set.

Let $x \in X \cap X^+$ (the set $X \cap X^+$ is nonempty, open and convex). It is easy to derive that the optimal dispatches in the spot market satisfy the following:

$$\begin{aligned} q_A(x, \xi) &= \tfrac{1}{4}(6.2 - 2q_B(x, \xi) + \xi + 2x_A), \\ q_B(x, \xi) &= \tfrac{1}{4}(6 - 2q_A(x, \xi) + \xi + 2x_B), \\ q_A(x, \xi) &> 0, \quad \text{for all} \quad \xi \in \Xi, \\ q_B(x, \xi) &> 0, \quad \text{for all} \quad \xi \in \Xi, \end{aligned} \tag{4.24}$$

and the spot price is

$$\begin{aligned} p(q_A + q_B, \xi) &= 7 + \xi - 2(q_A + q_B) \\ &= \frac{1}{3}(8.8 + \xi - 2x_A - 2x_B). \end{aligned}$$

Consequently, we have the generators' profit functions in the forward market

$$\begin{aligned} \pi_A(x_A, x_B) &= \int_{\xi=0}^{1} [q_A p - c_A(q_A)]\rho(\xi) d\xi \\ &= \frac{1}{18}\left[\frac{1}{3} + \frac{1}{2}(12.8 + 2x_A - 4x_B)\right. \\ &\qquad \left. + (6.4 + 4x_A - 2x_B)(6.4 - 2x_A - 2x_B)\right], \end{aligned}$$

$$\begin{aligned} \pi_B(x_B, x_A) &= \int_{\xi=0}^{1} [q_B p - c_B(q_B)]\rho(\xi) d\xi \\ &= \frac{1}{18}\left[\frac{1}{3} + \frac{1}{2}(11.6 + 2x_B - 4x_A)\right. \\ &\qquad \left. + (5.8 + 4x_B - 2x_A)(5.8 - 2x_A - 2x_B)\right]. \end{aligned}$$

Accordingly, the first order derivative of π_i w.r.t. x_i, $i = A, B$, are

$$\frac{\partial \pi_A}{\partial x_A} = \frac{1}{18}(13.8 - 16x_A - 4x_B),$$
$$\frac{\partial \pi_B}{\partial x_B} = \frac{1}{18}(11.6 - 16x_A - 4x_B).$$

By solving the system of equations

$$\frac{1}{18}(13.8 - 16x_A - 4x_B) = 0,$$
$$\frac{1}{18}(11.6 - 16x_A - 4x_B) = 0,$$

we obtain $x^* = (x_A^*, x_B^*) = (0.71, 0.61)$. It is easy to verify that $x^* \in X^+ \cap X$. Moreover, since

$$\frac{\partial^2 \pi_A(x^*)}{\partial x_A^2} = -\frac{8}{9},$$
$$\frac{\partial^2 \pi_B(x^*)}{\partial x_B^2} = -\frac{8}{9},$$

then the expected profit functions are concave. Therefore x^* is a local Nash–Cournot equilibrium.

Before presenting further analysis on the existence of local Nash equilibrium, we need the following result on the concavity of each generator's dispatch function.

Proposition 4.9 *Let $p(Q, \xi) = \alpha(\xi) - \beta(\xi)Q$, where $\xi : \Omega \to \Xi \subset \mathbb{R}$ is a random variable defined on probability space $(\Omega, \mathcal{F}, P)$, $\alpha(\xi) : \Xi \to \mathbb{R}_+$ and $\beta(\xi) : \Xi \to \mathbb{R}_+$ are continuous functions for all $\xi \in \Xi$. Assume that the marginal cost functions $c_i(q_i)$, $i = 1, \dots, M$, satisfy one of the following conditions:*

(i) *$c_j(q_j)$ is linear on $q_j \in [0, q_j^u]$ for $j = 1, \dots, M$;*
(ii) *all generators' marginal cost functions are identical and nondecreasing, that is, for any $\bar{q} > 0$*

$$c_1'(\bar{q}) = c_2'(\bar{q}) = \cdots = c_M'(\bar{q}).$$

Under Assumptions 2.1, 2.2, 2.6, and 3.6, the aggregate dispatch quantity $Q(x, \xi)$ is convex w.r.t. x_i for $x_i \geq 0$.

The proof of this proposition is attached in the appendix of this paper. The proof not only shows the convexity of the aggregate dispatch quantity, but also gives the formulation of $Q'_{x_i}(x, \xi)$ in (6.47), which implies that the more contract is signed, the higher rate of increase in the aggregate dispatch is. It also shows that the rate of increase is a piecewise smooth function of x for any ξ.

Lemma 4.10 *Under Assumptions 2.1, 2.2, 2.6, and 3.6, for fixed x_{-i} and $\xi \in \Xi$, the function $p(Q(x,\xi),\xi) - c_i'(q_i(x_i, x_{-i}, \xi))$ is a decreasing function w.r.t. x_i.*

Proof Let

$$h(x_i, x_{-i}, \xi) := p(Q(x,\xi),\xi) - c_i'(q_i(x_i, x_{-i}, \xi)).$$

Under the assumption of Proposition 4.9, we have

$$h(x_i, x_{-i}, \xi) = \alpha(\xi) - \beta(\xi)Q(x,\xi) - c_i'(q_i(x_i, x_{-i}, \xi)).$$

For each fixed $\xi \in \Xi$,

$$\partial_{x_i} h(x_i, x_{-i}, \xi) \subset -\beta(\xi)\partial_{x_i} Q(x,\xi) - c_i''(q_i(x_i, x_{-i}, \xi))\partial_{x_i} q_i(x_i, x_{-i}, \xi).$$

By Proposition 3.7 and Assumption 2.2, $\partial_{x_i} Q(x,\xi) \subset [0,1)$, $\partial_{x_i} q_i(x_i, x_{-i}, \xi) \subset [0,1)$ and $c_i''(q_i) \geq 0$. Thus

$$\partial_{x_i} h(x_i, x_{-i}, \xi) \subset -\beta(\xi)\partial_{x_i} Q(x,\xi) - c''\partial_{x_i} q_i(x_i, x_{-i}, \xi) \subset (-\infty, 0],$$

which implies that $p(Q(x,\xi),\xi) - c'(q_i(x_i, x_{-i}, \xi))$ is a decreasing function of x_i for every ξ. □

We are now ready to state a couple of existence results on equilibrium in the forward-spot market. Before that, we define the index set $\mathcal{I}(x,\xi) = \{j | q_j(x,\xi) > 0\}$ which is slightly different from the definition of $\mathcal{I}(x_i,\xi) = \{j | q_j(x_i, x_{-i}, \xi) > 0,\ j \neq i\}$ for fixed x_{-i} in the proof of Proposition 4.9.

Theorem 4.11 (Existence of local equilibrium) *Let assumptions in Proposition 4.9 hold. There exists at least one local Nash–Cournot equilibrium in the forward market, if the following conditions are satisfied*:

(1) *There exist open and convex sets X_i, $i = 1, \ldots, M$, such that for any $\xi \in \Xi$, $\mathcal{I}(x,\xi)$ is constant on $X := X_1 \times X_2 \times \cdots \times X_M$.*
(2) *For $i = 1, 2, \ldots, M$, there exist a non-empty compact convex subset X_i^0 of X_i and a non-empty compact subset K_i of X_i such that, for each $x \in X \backslash K$, there exists $y \in \mathrm{conv}(X^0 \bigcup \{x\})$ satisfying*

$$\sum_{i=1}^{M} \pi_i(x_i, x_{-i}) < \sum_{i=1}^{M} \pi_i(y_i, x_{-i}),$$

where $X^0 := \prod_{i=1}^{M} X_i^0$ and $K := \prod_{i=1}^{M} K_i$.

Proof We first consider a local forward-spot equilibrium problem formulated as

$$\begin{cases} \pi_i(x_i^*, x_{-i}^*) = \max\limits_{x_i \in X_i} \pi_i(x_i, x_{-i}), \\ q_i^*(x^*,\xi) \in \arg\max\limits_{q_i \geq 0} R_i(q_i(x^*,\xi), x^*, Q_{-i}(x^*,\xi), \xi), \qquad \forall \xi \in \Xi, \end{cases} \tag{4.25}$$

for $i = 1, 2, \ldots, M$, where $(q_1(x^*, \xi), \ldots, q_M(x^*, \xi))$ is the global Nash–Cournot equilibrium in the spot market for fixed x^*. Note that, in this local equilibrium problem, the decision variables x_i for $i = 1, 2, \ldots, M$ take their values in a noncompact and convex subset X_i of the feasible strategy set $[0, +\infty)$ in the global problem (2.7). Let

$$f_i(x_i, x_{-i}, \xi) := q_i^*(x_i, x_{-i}, \xi) p(Q^*(x, \xi), \xi) - c_i(q_i^*(x_i, x_{-i}, \xi)). \quad (4.26)$$

We reformulate (4.25) as

$$\max_{x_i \in X_i} \pi_i(x_i, x_{-i}) = \mathbb{E}[f_i(x_i, x_{-i}, \xi)]. \quad (4.27)$$

We prove the existence of a local Nash–Cournot equilibrium satisfying (4.27) by virtue of Yuan and Tarafdar (1996, Theorem 1) which addresses the existence of Nash equilibrium problem with noncompact feasible sets of strategies.

To apply this theorem, we need to verify Conditions (1) to (4) in Yuan and Tarafdar (1996, Theorem 1) Conditions (1) and (2) in Yuan and Tarafdar (1996, Theorem 1) can be easily verified by the twice continuously differentiability of $\pi_i(x_i, x_{-i})$ proved in Theorem 4.5. Condition (4) in Yuan and Tarafdar (1996, Theorem 1) is equivalent to Condition (2) of Theorem 4.11.

To verify Condition (3), we need to show that f_i is concave w.r.t. x_i on the noncompact feasible set X_i for every fixed ξ. For this purpose, we need to prove that the right-hand derivative of $f_i(x_i, x_{-i}, \xi)$ is a non-increasing function of x_i on X_i. Denote the right-hand derivative of $f_i(x_i, x_{-i}, \xi)$ by $f_i^+(x_i, x_{-i}, \xi)$. Then

$$\begin{aligned} f_i^+(x_i, x_{-i}, \xi) = {} & q_i^+(x_i, x_{-i}, \xi)[p(Q(x, \xi), \xi) - c_i'(q_i(x_i, x_{-i}, \xi))] \\ & - q_i(x_i, x_{-i}, \xi)\beta(\xi) Q_{x_i}^+(x, \xi). \end{aligned}$$

Similar to the proof of Proposition 4.9 and Lemma 4.10, we divide the right-hand derivative of $f_i(x_i, x_{-i}, \xi)$ w.r.t. x_i into two cases depending on whether $i \in \mathcal{I}(x, \xi)$ or not. Case 1, $i \in \mathcal{I}(x, \xi)$. We have

$$\begin{aligned} f_i^+(x_i, x_{-i}, \xi) = {} & \frac{1 + |\mathcal{I}(x, \xi)|}{2 + |\mathcal{I}(x, \xi)|}[p(Q(x, \xi), \xi) - c_i'(q_i(x_i, x_{-i}, \xi))] \\ & - \frac{1}{2 + |\mathcal{I}(x, \xi)|} q_i(x_i, x_{-i}, \xi)\beta(\xi). \end{aligned}$$

Case 2, $i \notin \mathcal{I}(x, \xi)$. We have

$$f_i^+(x_i, x_{-i}, \xi) = -\frac{1}{2 + |\mathcal{I}(x, \xi)|} q_i(x_i, x_{-i}, \xi)\beta(\xi).$$

Because $|\mathcal{I}(x, \xi)|$ is constant on X_i, $q_i(x, \xi)$ is a monotonically increasing function of x_i and $p(Q(x, \xi), \xi) - c_i'(q_i(x_i, x_{-i}^*, \xi))$ is decreasing by Lemma 4.10, we can easily see that $f_i^+(x_i, x_{-i}, \xi)$ is a decreasing function of x_i in either case. This shows the concavity of $f_i(x_i, x_{-i}, \xi)$ and hence of $\pi_i(x_i, x_{-i}) = \mathbb{E}[f_i(x_i, x_{-i}, \xi)]$ on the set $X = \prod_{i=1}^{M} X_i$, because concavity is preserved under the integration w.r.t. ξ, which verifies Condition (3) in Yuan and Tarafdar (1996, Theorem 1).

Therefore, Yuan and Tarafdar (1996, Theorem 1), there exists at least one Nash–Cournot equilibrium for $\{(4.27)\}_{i=1}^{M}$. Let us denote the equilibrium by $x^* = (x_1^*, x_2^*, \ldots, x_M^*)^T$. Since, for every $i = 1, 2, \ldots, M$, X_i is an open subset of $[0, +\infty)$, then x_i^* is a local maximizer of $\pi_i(x_i, x_{-i}^*)$ for $x_i \geq 0$. Hence, x^* is also a local Nash–Cournot equilibrium for the global equilibrium problem

$$\max_{x_i \geq 0} \pi_i(x_i, x_{-i}) = \mathbb{E}[f_i(x_i, x_{-i}, \xi)], \qquad i = 1, 2, \ldots, M.$$

By Proposition 3.3, there exists a unique equilibrium $q(x^*, \xi) = (q_1(x^*, \xi), \ldots, q_M(x^*, \xi))$ for the game problem in the spot market given that generators reach the local equilibrium x^* in the forward market. Therefore, 2M tuple $(x_1^*, x_2^*, \ldots, x_M^*, q_1(x^*, \xi), \ldots, q_M(x^*, \xi))$ is a local Nash–Cournot equilibrium for the forward-spot competition problem,

$$\begin{cases} \pi_i(x_i^*, x_{-i}^*) = \max\limits_{x_i \geq 0} \pi_i(x_i, x_{-i}), \\ q_i^*(x^*, \xi) \in \arg\max\limits_{q_i \geq 0} R_i(q_i(x^*, \xi), x^*, Q_{-i}(x^*, \xi), \xi), \quad i = 1, 2, \ldots, M, \quad \forall \xi \in \Xi, \end{cases} \tag{4.28}$$

This completes the proof. □

From a practical perspective, Theorem 4.11, giving a result on the existence of local Nash equilibrium, implies that, if every generator would like to accept a local optimal solution subject to its limited knowledge of the nonconcave profit function, then all generators will reach an equilibrium in the forward-spot market. On the other hand, the restrictions of Theorem 4.11 are straightforward. First, the theorem only gives a result on the existence of local Nash equilibrium which is not necessarily an optimal choice for each generator. Second, Condition (2) in Theorem 4.11 on the structure of the feasible sets may not be easily verified in the real system because it is given purely for a mathematical purpose. In order to get a result with more practical implication, we need to consider a particular type of markets in which every generator is profitable for every demand scenario. In the following theorem, we will show the existence of the global Nash–Cournot equilibrium for a class of forward-spot markets.

Theorem 4.12 (Existence of global equilibrium) *Let conditions in Proposition 4.9 hold. If for any contracts $x := (x_1, \ldots, x_M)$ signed in the forward market, the spot equilibrium $(q_1(x, \xi), \ldots, q_M(x, \xi))$ satisfies the condition that for any scenario $\xi \in \Xi$,*

$$p(Q(x, \xi), \xi) - c_i'(q_i(x, \xi)) + \beta(\xi) q_i(x, \xi) > 0, \tag{4.29}$$

for $i = 1, 2, \ldots, M$, then there exists a global Nash–Cournot equilibrium in the forward-spot market.

Remark 4.13 We make a few comments on the condition (4.29) before providing a proof.

(i) The condition implies that every generator makes a positive dispatch in the spot equilibrium for any $x \in X$ and $\xi \in \Xi$. To see this, let us assume for a contradiction that there exists i such that $q_i(x,\xi) = 0$ in the spot equilibrium. From the condition (4.29), we have $p(Q_{-i}(x,\xi),\xi) - c_i'(0) > 0$. Therefore, from the continuity of functions $p(\cdot,\xi)$ and $c_i'(\cdot)$, there exists a small positive value ϵ satisfying that $p(\epsilon + Q_{-i}(x,\xi),\xi) - c_i'(\epsilon) > 0$ and hence generator i's profit function $R_i(\epsilon, x, Q_{-i}, \xi)$ in the spot market can be calculated as

$$\begin{aligned} R_i(\epsilon, x, Q_{-i}, \xi) &= (\epsilon - x_i)p(\epsilon + Q_{-i}(x,\xi),\xi) - c_i(\epsilon) \\ &= (\epsilon - x_i)p(\epsilon + Q_{-i}(x,\xi),\xi) - \left(\int_0^{\epsilon} c_i'(q)dq + c_i(0)\right) \\ &> (\epsilon - x_i)p(\epsilon + Q_{-i}(x,\xi),\xi) - (\epsilon c_i'(\epsilon) + c_i(0)) \\ &> -x_i p(\epsilon + Q_{-i}(x,\xi),\xi) - c_i(0), \end{aligned} \tag{4.30}$$

where the first inequality is from the convexity of $c_i(\cdot)$ assumed in (iii) of Proposition 2.2. Consequently, we have

$$\begin{aligned} R_i(\epsilon, x, Q_{-i}, \xi) &> -x_i p(\epsilon + Q_{-i}(x,\xi),\xi) - c_i(0) \\ &> (0 - x_i)p(Q_{-i}(x,\xi),\xi) - c_i(0) \\ &= R_i(0, x, Q_{-i}, \xi), \end{aligned} \tag{4.31}$$

which implies that $q_i = 0$ is not the optimal decision of generator i given its rivals' decision $q_{-i}(x,\xi)$ in the spot equilibrium, hence $(0, q_{-i}(x,\xi))$ is not an equilibrium, a contradiction! Therefore, for any fixed x_{-i}, $i \in \mathcal{I}(x,\xi)$.

(ii) Theorem 4.12 may be viewed as a special case of Theorem 4.11 on local Nash equilibrium. Since $q_i(x,\xi) > 0$ is for generator i in the spot equilibrium, we have that every generator dispatches a positive quantity, and hence $\mathcal{I}(x,\xi) = \{1, 2, \ldots, M\}$ is constant in the whole strategy space, which satisfies condition (i) of Theorem 4.11. From the proof of Theorem 4.12, we can identify the concavity of generator i's profit function in the whole strategy space X, and hence condition (2) is also satisfied. Therefore, the condition in Theorem 4.12 implies both conditions in Theorem 4.11.

Proof of Theorem 4.12 Under the assumption $p(Q,\xi) = \alpha(\xi) - \beta(\xi)Q$ in Proposition 4.9, we have

$$p(Q_{-i}(x,\xi),\xi) - c_i'(q_i(x,\xi)) = \alpha(\xi) - \beta(\xi)Q_{-i}(x,\xi) - c_i'(q_i(x,\xi)) \geq 0, \tag{4.32}$$

for all $\xi \in \Xi$.

Now, we look into the right-hand derivative of $f_i(x_i, x_{-i}, \xi)$, which can be written as following,

$$\begin{aligned} f_i^+(x_i, x_{-i}, \xi) = q_i^+(x_i, x_{-i}, \xi)[p(Q(x_i, x_{-i}, \xi), \xi) - c_i'(q_i(x_i, x_{-i}, \xi))] \\ -q_i(x_i, x_{-i}, \xi)\beta(\xi)Q_{x_i}^+(x, \xi). \end{aligned} \tag{4.33}$$

By (4.30), we have that every generator will dispatch a positive quantity and hence $i \in \mathcal{I}(x, \xi)$ for every generator i and any fixed contract quantity x. From the proof of Proposition 4.9, we can reformulate (4.33) as

$$\begin{aligned} f_i^+(x_i, x_{-i}, \xi) = \frac{1 + |\mathcal{I}(x_i, x_{-i}, \xi)|}{2 + |\mathcal{I}(x_i, x_{-i}, \xi)|}[p(Q(x, \xi), \xi) - c_i'(q_i(x_i, x_{-i}, \xi))] \\ -\frac{1}{2 + |\mathcal{I}(x_i, x_{-i}, \xi)|} q_i(x_i, x_{-i}, \xi)\beta(\xi), \end{aligned}$$

where we reformulate $\mathcal{I}(x, \xi)$ by $\mathcal{I}(x_i, x_{-i}, \xi)$ to emphasize generator i's decision. As proved in Sect. 3, $q_i(x_i, x_{-i}, \xi)$ and hence $f_i^+(x_i, x_{-i}, \xi)$ are piecewise smooth functions of x_j for any $i, j = 1, 2, \ldots, M$. For any fixed x_{-i}, we proceed the proof by dividing our discussion on the monotonicity of the right-hand derivative of $f_i(x_i, x_{-i}, \xi)$ in two cases depending on the smoothness of f_i:

Case 1, we consider the monotonicity of $f_i^+(x_i, x_{-i}, \xi)$ at the point x_i where the set $\mathcal{I}(x_i, x_{-i}, \xi)$ is constant and hence $f_i^+(x_i, x_{-i}, \xi)$ is continuous. From Lemma 4.10, $p(Q(x, \xi), \xi) - c_i'(q_i(x_i, x_{-i}, \xi))$ is a decreasing function of x_i for any fixed x_{-i}, and $q_i(x_i, x_{-i}, \xi)$ is an increasing function of x_i. Therefore, we have $f_i^+(x_i, x_{-i}, \xi)$ is a decreasing function of x_i for any fixed x_{-i} and any scenario ξ in every smooth piece of x_i.

Case 2, we consider the monotonicity of $f_i^+(x_i, x_{-i}, \xi)$ at the point x_i where $\mathcal{I}(x_i, \xi)$ is not constant and hence $f_i^+(x_i, x_{-i}, \xi)$ is not continuous. Let

$$x_i^- = \lim_{\delta \to 0} x_i - \delta, \quad \text{and} \quad x_i^+ = \lim_{\delta \to 0} x_i + \delta, \quad \text{for a } \delta > 0,$$

which are on the left and right sides of x_i, respectively. Since $|\mathcal{I}(x, \xi)|$ is a decreasing function of x_i for any fixed x_{-i} which has been shown in the proof of Proposition 4.9, we have $I^+ := |\mathcal{I}(x_i^+, x_{-i}, \xi)|$ is less than or equal to $I^- := |\mathcal{I}(x_i^-, x_{-i}, \xi)|$ for every fixed x_{-i} and ξ. Moreover, because of the Lipschitz continuity of $q_i(x_i, x_{-i}, \xi)$ w.r.t x_j for any $i, j = 1, 2, \ldots, M$, we have $q_i(x_i^+, x_{-i}, \xi) = q_i(x_i^-, x_{-i}, \xi) = q_i(x_i, x_{-i}, \xi)$ for any x_{-i}, and hence

$$\begin{aligned} & f_i^+(x_i^+, x_{-i}, \xi) - f_i^+(x_i^-, x_{-i}, \xi) \\ & = \left\{ \frac{1 + I^+}{2 + I^+}[p(Q, \xi) - c_i'(q_i)] - \frac{1}{2 + I^+} q_i \beta(\xi) \right\} \end{aligned}$$

$$
\begin{aligned}
&- \left\{ \frac{1+I^-}{2+I^-}[p(Q,\xi) - c_i'(q_i)] - \frac{1}{2+I^-} q_i \beta(\xi) \right\} \\
&= \frac{1}{(2+I^+)(2+I^-)} \left[\left(I^+ - I^-\right) \left(p(Q,\xi) - c_i'(q_i) + q_i \beta(\xi)\right) \right],
\end{aligned}
$$

where the second equality is from $q_i(x_i^+, x_{-i}, \xi) = q_i(x_i^-, x_{-i}, \xi)$. Due to (4.29) in this theorem, we have

$$
\begin{aligned}
& f_i^+(x_i^+, x_{-i}, \xi) - f_i^+(x_i^-, x_{-i}, \xi) \\
&= \frac{1}{(2+I^+)(2+I^-)} \left[\left(I^+ - I^-\right) \left(p(Q,\xi) - c_i'(q_i) + q_i \beta(\xi)\right) \right] < 0,
\end{aligned}
$$

which means that, at the point x_i, $f_i^+(x_i, x_{-i}, \xi)$ is also a decreasing function of x_i for fixed x_{-i}.

By combining the results in both cases, we can show that the right-hand derivative of $f_i(x_i, x_{-i}, \xi)$ is a decreasing function of x_i for $i = 1, 2, \ldots, M$, which indicates that the function $f_i(x_i, x_{-i}, \xi)$ and hence $\pi_i(x_i, x_{-i})$ are concave functions of x_i. From the proof of Rosen (1965, Theorem 1), we know that there exists a global Nash–Cournot equilibrium in the forward-spot market. □

From Theorem 4.12 and Remark 4.13, we can make the following qualitative statement.

Corollary 4.14 *If for all possible demand shock, every generator makes a positive dispatch in the spot equilibrium, then there exists a global Nash–Cournot equilibrium in the forward-spot electricity market.*

5 Numerical examples

In this section, we present a simple example to illustrate how the forward-spot market equilibrium can be obtained numerically and how the SEPEC model can be used to analyze the interaction of the markets. We carry out some computer simulations for the SEPEC model with two players. We investigate how the dispatches, expected profits and strike prices vary on the change of a generator's contract position.

Note that it is very difficult to obtain a closed form of the expected value of the objective functions. Consequently, we use a well known sample average approximation (SAA) approach to approximate the expected values. SAA is a popular method in stochastic programming; see Gürkan et al. (1999), Robinson (1996), DeMiguel and Xu (2008) and the references therein. The basic idea behind the SAA method is to approximate the expected value function by a sample average. Here we use the SAA approach as in DeMiguel and Xu (2008) to solve our SEPEC problem. We skip the theoretical analysis of convergence of this method because we believe similar conclusion can be drawn as in DeMiguel and Xu (2008) and it is not the focus of this paper.

We now move on to computer simulations for the SEPEC model to look into specifically dependence of dispatches, expected profits and strike prices on forward contracts.

Let $\xi^1, \ldots, \xi^N$ be an independent identically distributed (i.i.d) sample of $\xi(\omega)$, where N is the sample size. The sample average approximation problem for generator i is,

$$\max_{x_i \geq 0} \frac{1}{N} \sum_{k=1}^{N} q_i^N(x, \xi^k) p(Q^N(x, \xi^k), \xi^k) - c_i(q_i^N(x, \xi^k)), \tag{5.34}$$

where for $i = 1, 2, \ldots, M$, $q_i^N(x, \xi^k)$ is defined implicitly as the equilibrium in the spot market at demand scenario ξ^k, and $Q^N(x, \xi^k) = \sum_{i=1}^M q_i^N(x, \xi^k)$. Note that as discussed in Sect. 3.2, $q^N(x, \xi^k) = (q_1^N(x, \xi^k), \ldots, q_M^N(x, \xi^k))^T$ is a solution to the nonlinear complementarity problem $0 \leq q^N(x, \xi^k) \perp G(x, q^N, \xi^k) \geq 0$, where

$$G(x, q^N, \xi^k) = -p(Q^N(x, \xi^k), \xi^k)e - (q^N - x)p'_Q(Q^N(x, \xi^k), \xi^k) + \nabla \mathbf{c}(q^N).$$

Consequently the problem can be reformulated as the following standard nonlinear programming problem:

$$\begin{array}{ll} \max_{x_i \geq 0} \frac{1}{N} \sum_{k=1}^{N} q_i^N(x, \xi^k) p(Q^N(x, \xi^k), \xi^k) - c_i(q_i^N(x, \xi^k)) & \\ \text{s.t.} \qquad\qquad q^N \geq 0 \quad \forall k, & \\ \qquad\qquad G(x, q^N, \xi^k) \geq 0 \quad \forall k, & \\ \qquad\qquad -q^N \circ G(x, q^N, \xi^k) \geq 0 \quad \forall k, & \end{array} \tag{5.35}$$

where $\circ$ represents the componentwise scalar product.

Example 5.1 Consider two generators, A and B, competing in a forward market. Assume that the inverse demand function takes the following form

$$p(Q, \xi) = \alpha(\xi) - \beta(\xi)Q,$$

where ξ is a random variable following a truncated normal distribution with zero mean, standard deviation of 1, and truncated at two deviations above and below the mean. Let $\alpha(\xi) = 2 + \xi$, $\beta(\xi) = 7 + 0.5\xi$, and each generator's cost function be as follows:

$$\begin{aligned} \text{Generator} A: \quad & c_A(q_A) = 0.1q_A^2 + 1q_A; \\ \text{Generator} B: \quad & c_B(q_B) = 0.1q_B^2 + 0.5q_B. \end{aligned}$$

By fixing the Generator A's contract level x_A, we carry out some static analysis on generator's dispatch q_i, for $i = A$ and B, expected profit $\pi_B(x_A, x_B)$ and market clearing price $p(Q, \xi)$ in the spot market, w.r.t. the different values of x_B.

In Figs. 1 and 2, we let $x_A = 0$, that is, generator A has no contract. We examine how the optimal dispatch of A varies as x_B increases. The results show that generator A's average dispatch decreases as x_B increases from 0 to 0.1, and it becomes zero when $x_B \in [0.1, 0.3]$. This demonstrates that generator A's dispatch is a decreasing function of x_B. The results also show that generator B's average dispatch q_B increases as x_B increases and the curve of q_B is concave.

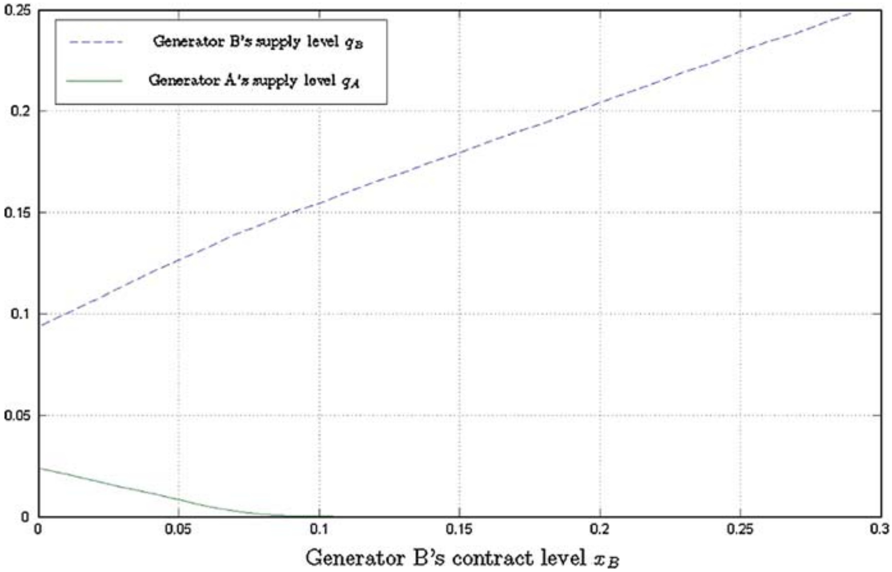

Fig. 1 The average dispatch w.r.t. contract level x_B

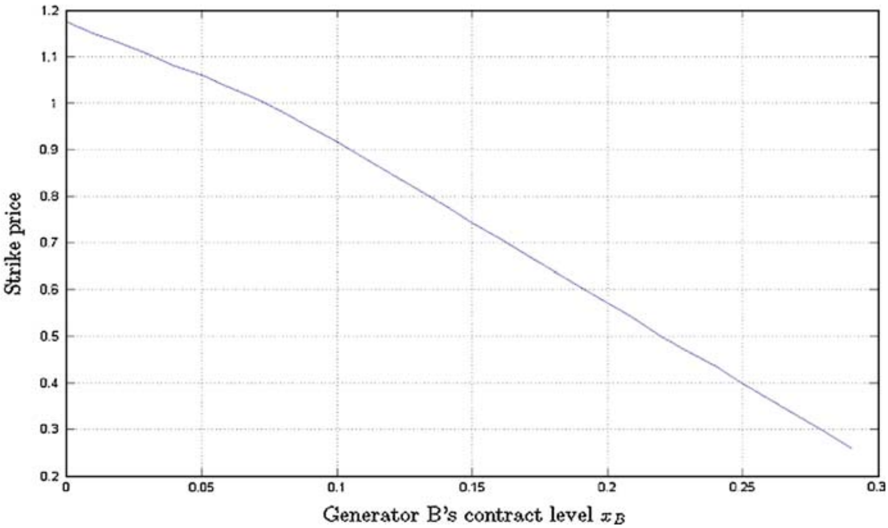

Fig. 2 The strike price w.r.t. contract level x_B

In Fig. 2, we show that the strike price is a piecewise smooth and decreasing function of x_B. Moreover, because $z(x_A + x_B) = \mathbb{E}[\alpha(\xi) - \beta(\xi)Q(x, \xi)]$, and $Q(x, \xi)$ is a convex function of x_B, the strike price is a concave function of x_B.

In Fig. 3, we present some results on the expected profits of generator B, that is $\pi_B(x_A, x_B)$, for various contracts x_A and x_B. We observe that there is a local maximizer of $\pi_B(x_A, x_B)$ w.r.t. $x_B \in [0, 0.2]$ for every fixed x_A. The underlying reason

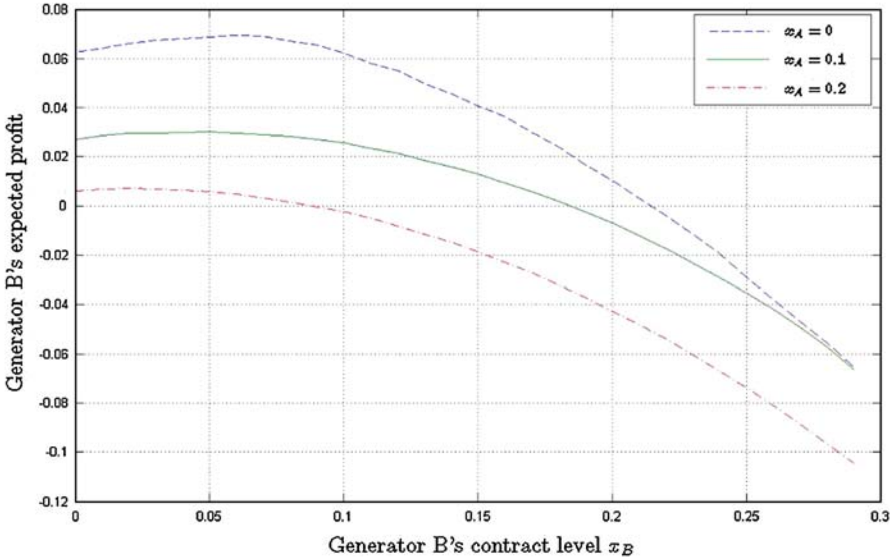

Fig. 3 Generator B's profit w.r.t. contract level x_B

of the results is that by signing more contracts, generator B becomes more incentives to dispatch in the spot market as we have shown in Fig. 1. On the other hand, more contracts result in a lower average spot price and hence contract strike price. Consequently, it results in a lower expected profit for generator B shown in Fig. 3. For its rival, because a greater contract quantity from B leads to a lower average price in the spot market, generator A will lose its profit.

6 Further discussion

In this paper, we have developed an SEPEC model for studying interactions between the forward market and the spot market. The model is essentially an extension of a Nash–Cournot model developed by Gans et al. (1998) for deterministic duopolistic electricity markets. A number of restrictions have been made to simplify the discussions: (a) the spot market competition is assumed to take place in a single node where the network constraints and transmission costs are not considered; (b)one-way contracts such as call options and put options, are not considered; (c) there is no speculator in the forward market; (d) bids in spot market is a single quantity rather than a stack of prices and quantities as in supply function models. We believe that similar equilibrium results can be established by dropping some of the restrictions although we have not attempted. We leave this for our future work.

Acknowledgments The authors would like to thank two anonymous referees for their valuable comments which lead to a significant improvement of the presentation of the paper. They would also gratefully acknowledge helpful comments from Richard Green, David Newbery and Daniel Ralph during the UK operational research annual conference OR50 in York in September 2008.

Appendix

Proof of Lemma 3.1 Part (i) is proved in Proposition 2.4, Xu (2005).

Part (ii). By differentiating the function $R(Q,\xi) = (Q-X)p(Q+K,\xi)$, we have

$$R'_Q(Q,\xi) = p(Q+K,\xi) + (Q-X)p'_Q(Q+K,\xi).$$

Consequently,

$$\begin{aligned} R''_Q(Q,\xi) &= 2p'_Q(Q+K,\xi) + (Q-X)p''_Q(Q+K,\xi) \\ &\leq p'_Q(Q+K,\xi) - Xp''_Q(Q+K,\xi). \end{aligned}$$

From Assumption 2.2 (a) and (b), we have

$$R''_Q(Q,\xi) \leq p'_Q(Q+K,\xi) - Xp''_Q(Q+K,\xi) < 0.$$

Therefore, the function $(Q-X)p(Q+K,\xi)$ is strictly concave. □

Proof of Theorem 3.5 Part (i). The Jacobian of $G(q,x,\xi)$ w.r.t. q can be written explicitly as

$$\nabla_q G(q,x,\xi) = -p'_Q(Q,\xi)ee^T - p''_Q(Q,\xi)(q-x)e^T - p'_Q(Q,\xi)I_M + \nabla^2\mathbf{c}(q)$$

where $Q = q^T e$ is the aggregated supply quantity and $I_M \in \mathbb{R}^{M\times M}$ is an identity matrix. Since $p(Q,\xi)$ is strictly decreasing in q_i, we have $p'_Q(Q,\xi) < 0$ for any $Q \geq 0, \xi \in \Xi$. Moreover, because the terms $-p'_Q(Q,\xi)ee^T$ and $-p''_Q(Q,\xi)(q-x)e^T$ are both rank one matrices, and $-p'_Q(Q,\xi)I_M$ and $\nabla^2\mathbf{c}(q)$ are both diagonal matrices, the eigenvalues of $\nabla_q G(q,x,\xi)$ are lower bounded by

$$-Mp'_Q(Q,\xi) - p''_Q(Q,\xi)(q-x)^T e + \min_{i=1,\ldots,M} c''_i(q_i) - p'_Q(Q,\xi). \quad (6.36)$$

Since Ξ is compact, there exists a constant $C > 0$ such that

$$\min_{\xi\in\Xi} -p'_Q(Q,\xi) \geq C, \quad \text{for} \quad Q \in \left[0, \sum_{i=1}^{M} q_i^u\right].$$

On the other hand, from the convexity of $p(Q,\xi)$ and Assumption 2.2 (ii), we have

$$\begin{aligned} &-p'_Q(Q,\xi) - p''_Q(Q,\xi)(q-x)^T e \\ &\quad = -p'_Q(Q,\xi) - p''_Q(Q,\xi)q^T e + p''_Q(Q,\xi)x^T e \geq 0. \end{aligned} \quad (6.37)$$

By the convexity of cost function under Assumption 2.2 (iii), $c''_i(q_i) \geq 0$, for any $q_i \geq 0, \xi \in \Xi$. Substituting (6.37) into (6.36), we have

$$-(M+1)p'_q(q,\xi) - p''_q(q,\xi) + \min_{i=1,\ldots,M} c''_i(q_i) \geq MC,$$

which implies that $\nabla_q G(q, x, \xi)$ is uniformly positive definite. We now consider (3.11). It can be easily found that function $F(q, x, \xi)$ is Lipschitz continuous and the Clarke generalized Jacobian of F in (q, x, ξ) can be written as

$$\partial F(q, x, \xi) = \left\{ \left(I_M - \Theta \;\; \Theta \right) \begin{pmatrix} I_M \\ \nabla_q G(q, x, \xi) \end{pmatrix} : \theta_i \in [0, 1],\ i = 1, 2, \ldots, M \right\}, \tag{6.38}$$

and

$$\partial_q F(q, x, \xi) = \Theta \nabla_q G(q, x, \xi) + (I_M - \Theta),$$

where $\Theta := \text{diag}(\theta_1, \ldots, \theta_M) \in \mathbb{R}^{M \times M}$, is a diagonal matrix with the (i, i)th entry being θ_i, for $i = 1, 2, \ldots, M$. Thus, by Lemma 3.1 in Xu (2005), $\nabla_q F(q, x, \xi)$ is uniformly non-singular.

Part (ii). The conclusion follows straightforwardly from uniqueness and existence of the Nash–Cournot equilibrium in the spot market in Proposition 3.3 together with the definition of the nonsmooth function F.

Part (iii). From Part (i), $\nabla_q F(q, x, \xi)$ is non-singular. By the proof of Part (ii) and Lemma 3.2 in Xu (2005), there exists a unique Lipschitz continuous and piecewise smooth implicit function $q(x, \xi)$, such that $F(q(x, \xi), x, \xi) = 0$ in a neighborhood of (q, ξ). The domain of implicit function can be extended to $[0, +\infty) \times \Xi$ given the non-singularity of $\nabla_q F(q, x, \xi)$ for all x, ξ and the existence and uniqueness in Proposition 3.3. □

Proof of Proposition 3.7 Part (i). The conclusion follows from Xu and Meng (2007, Lemma 2.1) because F is piecewise smooth in q.
Part (ii). Since $\partial_x Q(x, \xi) \subset e^T \partial_x q(x, \xi)$, we have from (3.13)

$$\partial_x Q(x, \xi) \subset e^T \text{conv}\{-W^{-1}U : (W, U, V) \in \partial F(q, x, \xi)\}.$$

Since, for $i = 1, 2, \ldots, M$, the ith component of $F(q, x, \xi)$, F_i, is a piecewise smooth function, the Clarke subdifferential of $F_i(q, x, \xi)$ can be written as

$$\partial_{(q,x)} F_i(q, x, \xi) = \{\theta_i \nabla G_i(q, x, \xi) + (1 - \theta_i)\mathbf{l}_i\},$$

where $\theta_i \in [0, 1]$, $G_i(\cdot)$ is the ith component of function $G(\cdot)$ and $\mathbf{l}_i$ is a $2M + 1$ dimensional vector with the ith component being 1 and the rest being zero.

Note that $\theta_i = 0$ only when $F_i(q, x, \xi) = q_i$. Let $\Theta = \text{diag}(\theta_1, \ldots, \theta_M)$. First, we show that under condition (3.12), $q(x, \xi) \neq 0$. Let $i_0 \in \{1, \ldots, M\}$ be such that

$$c'_{i_0}(0) < p(Q(x, \xi), \xi), \quad \xi \in \Xi. \tag{6.39}$$

By definition, $q_{i_0}(x, \xi)$ solves the following maximization problem

$$\max_{q_{i_0} \geq 0} R_{i_0}(q_{i_0}, x, Q_{-i_0}, \xi) = q_{i_0} p\left(q_{i_0} + Q_{-i_0}, \xi\right) - c_{i_0}(q_{i_0}) - x_{i_0}\left[p(q_{i_0} + Q_{-i_0}, \xi) - z\right].$$

The first-order necessary condition can be represented as the following complementarity conditions:

$$\begin{aligned}
&q_{i_0}(x,\xi)\frac{dR_{i_0}(q_{i_0},x,Q_{-i_0},\xi)}{dq_{i_0}}\\
&\quad = q_{i_0}(x,\xi)[p(Q(x,\xi),\xi)+(q_{i_0}-x_{i_0})p'_Q(Q(x,\xi),\xi)-c'_{i_0}(q_{i_0}(x,\xi))]\\
&\quad = 0,\ q_{i_0}(x,\xi)\geq 0,\\
&-p(Q(x,\xi),\xi)-(q_{i_0}-x_{i_0})p'_Q(Q(x,\xi),\xi)+c'_{i_0}(q_{i_0},\xi)\geq 0.
\end{aligned}$$

Assume that $q_{i_0}(x,\xi)=0$. Then

$$\begin{aligned}
R'_{i_0}(0,x,Q_{-i_0},\xi) &= p(Q(x,\xi),\xi)-c'_{i_0}(0)-x_{i_0}p'(Q(x,\xi),\xi)\\
&\geq p(Q(x,\xi),\xi)-c'_{i_0}(0)>0.
\end{aligned}$$

The last inequality is due to (6.39). This contradicts the second inequality in the above complementarity conditions. This shows $q_{i_0}>0$ and hence $q(x,\xi)\neq 0$. Moreover, the strict complementarity condition indicates that

$$F_{i_0}(q(x,\xi),x,\xi)=G_{i_0}(q(x,\xi),x,\xi)=0,$$

and hence $\theta_{i_0}=1$. This demonstrates that Θ is not a zero matrix under (3.7). We will use this result in the rest of the proof. By definition,

$$\begin{aligned}
R &= \Theta\nabla_q G(q,x,\xi)+(I_M-\Theta)\\
&= \Theta(-p'_Q e-(q-x)p''_Q)e^T+\Theta(-p'_Q I_M+\nabla^2\mathbf{c}(q))+(I_M-\Theta),
\end{aligned}$$

and

$$U=\Theta\nabla_x G(q,x,\xi)=\Theta p'_Q I_M.$$

Let $D=\Theta(-p'_Q I_M+\nabla^2\mathbf{c}(q))+(I_M-\Theta)$. D is an $M\times M$ diagonal matrix. It is easy to verify that D is non-singular and the inverse of D is

$$D^{-1}=\operatorname{diag}\left(\frac{1}{\theta_1(-p'_Q+c''_1(q_1))+(1-\theta_1)},\ldots,\frac{1}{\theta_M(-p'_Q+c''_M(q_M))+(1-\theta_M)}\right).$$

Let

$$\gamma := e^T D^{-1}\Theta(-p'_Q e-(q-x)p''_Q)=\sum_{i=1}^{M}\frac{\theta_i(-p'_Q-p''_Q(q_i-x_i))}{\theta_i(-p'_Q+c''_i(q_i))+(1-\theta_i)}.$$

By the well known Sherman-Morrison formula in linear algebra, we have

$$R^{-1}=D^{-1}-\frac{1}{1+\gamma}D^{-1}\Theta(-p'_Q e-(q-x)p''_Q)e^T D^{-1}.$$

Let

$$\gamma_i := \frac{\theta_i(-p'_Q - (q_i - x_i)p''_Q)}{\theta_i(-p'_Q + c''_i(q_i)) + (1-\theta_i)} > \frac{\theta_i(-p'_Q - Qp''_Q)}{\theta_i(-p'_Q + c''_i(q_i)) + (1-\theta_i)} \geq 0,$$

where the first inequality is due to the convexity of the inverse demand function and the second inequality is from Assumption 2.2. Because $\gamma_i \geq 0$, for $i = 1, 2, \ldots, M$, we have $\gamma = \sum_{i=1}^{M} \gamma_i \geq \gamma_i$ for any $i = 1, 2, \ldots, M$. Consequently,

$$\begin{aligned}
-e^T R^{-1} U &= -e^T \left[D^{-1} - \frac{1}{1+\gamma} D^{-1} \Theta(-p'_Q e - (q-x)p''_Q) e^T D^{-1} \right] \Theta p'_Q I_M \\
&= -e^T D^{-1} \Theta p'_Q I_M + \frac{1}{1+\gamma} [e^T D^{-1} \Theta(-p'_Q e - (q-x)p''_Q)] e^T D^{-1} p'_Q I_M \\
&= -e^T D^{-1} \Theta p'_Q I_M + \frac{\gamma}{1+\gamma} e^T D^{-1} \Theta p'_Q I_M \\
&= -\frac{1}{1+\gamma} e^T D^{-1} \Theta p'_Q I_M \\
&= \frac{1}{1+\gamma} \left(\frac{-\theta_1 p'_Q}{\theta_1(-p'_Q + c''_1(q_1)) + (1-\theta_1)}, \ldots, \frac{-\theta_M p'_Q}{\theta_M(-p'_Q + c''_M(q_M)) + (1-\theta_M)} \right)^T.
\end{aligned}$$

Let

$$\kappa_i := \frac{-\theta_i p'_Q}{\theta_i(-p'_Q + c''_i(q_i)) + (1-\theta_i)} \geq 0, \quad i = 1, 2, \ldots, M.$$

By Assumption 2.2, $-p'_Q > 0$, $c''_i(q_i) \geq 0$ and $\theta_i \in [0, 1]$, hence we have $\gamma > 0$ and $0 \leq \kappa_i \leq 1$, and

$$\begin{aligned}
-e^T R^{-1} U &\subset \left[0, \frac{\kappa_1}{1+\gamma}\right) \times \left[0, \frac{\kappa_2}{1+\gamma}\right) \times \cdots \times \left[0, \frac{\kappa_M}{1+\gamma}\right) \\
&\subset [0, 1) \times [0, 1) \times \cdots \times [0, 1).
\end{aligned}$$

Hence we have $\partial_x Q(x, \xi) \subset \partial_x q(x, \xi)^T e \subset [0, 1) \times \cdots \times [0, 1)$ and $\partial_{x_i} Q(x, \xi) \subset \partial_{x_i} q(x, \xi)^T e \subset [0, 1)$. Note that $\theta_i = 0$ corresponds to the case when $F_i(q, x, \xi) = q_i(x, \xi) \equiv 0$. In this case, $(q_i)'_{x_i}(x, \xi) = 0$. Also, by Remark 2.5, $(q_j)'_{x_i}(x, \xi) = 0$. This shows $\partial_{x_i} Q(x, \xi) = \{0\}$.

Part (iii). From the proof of Part (ii), $q_i(x, \xi) > 0$, therefore from the complementarity condition

$$-p(Q, \xi) - (q_i - x_i)p'_Q(Q, \xi) + c'_i(q_i) = 0.$$

By using Clarke's generalized implicit function theorem (Xu 2005, Lemma 2.2), we obtain

$$\partial_{x_i} q_i(x,\xi) \subset \frac{-p'_Q + (p'_Q + (q_i - x_i)p''_Q)\partial_{x_i} Q(x,\xi)}{-p'_Q + c''_i(q_i)}. \tag{6.40}$$

From the proof of Part (ii), the subdifferential $\partial_{x_i} Q(x,\xi)$ is in the set $[0, \frac{\kappa_i}{1+\gamma})$. By the property of γ, $\gamma \geq \gamma_i > 0$, we have

$$\partial_{x_i} Q(x,\xi) \subset \left[0, \frac{\kappa_i}{1+\gamma}\right) \subset \left[0, \frac{\kappa_i}{\gamma_i}\right) = \left[0, \frac{-\theta_i p'_Q}{\theta_i(-p'_Q - (q_i - x_i)p''_Q)}\right).$$

Consequently, the subdifferential of q_i w.r.t. x_i is,

$$\partial_{x_i} q_i(x,\xi) \subset \frac{-p'_Q + (p'_Q + (q_i - x_i)p''_Q)\left[0, \frac{\kappa_i}{\gamma_i}\right)}{-p'_Q + c''_i(q_i)} \subset \left[0, \frac{-p'_Q}{-p'_Q + c''_i(q_i)}\right) \subset [0,1). \tag{6.41}$$

Part (iv). As discussed in Part (iii), we have the following equation,

$$-p(Q,\xi) - (q_i - x_i)p'_Q(Q,\xi) + c'_i(q_i) = 0.$$

By using Clarke's generalized implicit function theorem (Xu 2005, Lemma 2.2), we obtain

$$\partial_{x_j} q_i(x,\xi) \subset \frac{p'_Q + (q_i - x_i)p''_Q}{-p'_Q + c''_i(q_i)} \partial_{x_j} Q(x,\xi).$$

Similarly as the proof of Part (iii), we have

$$\partial_{x_j} q_i(x,\xi) \subset \frac{p'_Q + (q_i - x_i)p''_Q}{-p'_Q + c''_i(q_i)} \left[0, \frac{\kappa_i}{\gamma_i}\right) \subset \left(\frac{p'_Q}{-p'_Q + c''_i(q_i)}, 0\right] \subset (-1, 0],$$

which implies that every element of $\partial_{x_j} q_i(x,\xi)$ is negative. This shows $q_i(\cdot,\xi)$ is strictly decreasing w.r.t. x_j where $q_i(\cdot,\xi) > 0$.
Part (v). From the formulation of $G(q,x,\xi)$, we have

$$\nabla_\xi G(q,x,\xi) = -p'_\xi(q^T e,\xi)e - p''_{Q,\xi}(q^T e,\xi)q.$$

By the assumption that $p''_{Q,\xi}(q^T e,\xi) = 0$ in Proposition 3.7 (v), the above equation can be written as

$$\nabla_\xi G(q,x,\xi) = -p'_\xi(q^T e,\xi)e.$$

The rest of the proof is similar to that of Part (ii). We include it for completeness. Let

$$V := \Theta\nabla_\xi G(q, x, \xi) = -\Theta p'_\xi e,$$

and

$$-e^T R^{-1} V = p'_\xi (e^T D^{-1})\Theta e - \frac{\gamma}{1+\gamma} p'_\xi (e^T D^{-1})\Theta e = \frac{1}{1+\gamma} p'_\xi (e^T D^{-1})\Theta e.$$

By the assumption in statement (v) of this proposition, we have

$$\begin{aligned}\gamma &= \sum_{i=1}^{M} \frac{\theta_i(-p'_Q - (q_i - x_i)p''_Q)}{\theta_i(-p'_Q + c''_i(q_i)) + (1-\theta_i)} \\ &> C\sum_{i=1}^{M} \frac{\theta_i}{\theta_i(-p'_Q + c''_i(q_i)) + (1-\theta_i)} = Ce^T D^{-1}\Theta e.\end{aligned}$$

Hence,

$$-e^T R^{-1} V \le \frac{p'_\xi e^T D^{-1}\Theta e}{1 + Ce^T D^{-1}\Theta e} \le \frac{1}{C} p'_\xi.$$

□

Proof of Theorem 4.3 Part (i). Recall that in Theorem 3.5 (iii), we have shown that the solution $q(x_i, x_{-i}, \xi)$ to equation $F(q, x, \xi) = 0$ is a Lipschitz continuous, piecewise smooth function of x_i for $i = 1, 2, \ldots, M$. By Proposition 3.7 (iv), $\partial_{x_i} q_i(x_i, x_{-i}, \xi) \subset [0, 1)$, which means that $q_i(x_i, x_{-i}, \xi)$ is increasing in x_i. Moreover, from (6.40), we have

$$\partial_{x_i} q_i(x_i, x_{-i}, \xi) \subset \frac{-p'_Q + (p'_Q + (q_i - x_i)p''_Q)\partial_{x_i} Q(x, \xi)}{-p'_Q + c''_i(q_i)}.$$

Note that at a point where $Q(x, \xi)$ is continuously differentiable, both $\partial_{x_i} q_i(x_i, x_{-i}, \xi)$ and $\partial_{x_i} Q(x, \xi)$ reduce to a singleton. Thus, $q_i(x_i, x_{-i}, \xi)$ is a piecewise differentiable function of x_i.

Part (ii). Let $x_i^{(1)}, x_i^{(2)} \ge 0$ be any two positive numbers. From the proof above, we know that $q_i(x_i, x_{-i}, \xi)$ is piecewise smooth in x_i. At a point where the function is not differentiable, we have from (6.40)

$$(q_i)'_{x_i}(x_i, x_{-i}, \xi) = \frac{-p'_Q + (p'_Q + (q_i - x_i)p''_Q)Q'_{x_i}(x, \xi)}{-p'_Q + c''_i(q_i)}.$$

Since, $q_i \in [0, \max\{q_i^u, x_i\}]$ in Proposition 3.3, $Q'_{x_i}(x, \xi) \in [0, 1)$ and $(q_i)'_{x_i}(x_i, x_{-i}, \xi) \in [0, 1)$ for any feasible q_i and x_i, $(q_i)'_{x_i}(x_i, x_{-i}, \xi)$ is bounded by a positive value as

$$(q_i)'_{x_i}(x_i, x_{-i}, \xi) \leq \frac{-p'_Q + (\max\{q_i^u, x_i\} - x_i)p''_Q}{-p'_Q + c''_i(q_i)}. \tag{6.42}$$

By the mean value theorem,

$$\begin{aligned} & q_i(x_i^{(1)}, x_{-i}, \xi) - q_i(x_i^{(2)}, x_{-i}, \xi) \\ & = \int_0^1 (q_i)'_{x_i} \left(x_i^{(2)} + \theta(x_i^{(1)} - x_i^{(2)}), x_{-i}, \xi\right)(x_i^{(1)} - x_i^{(2)})d\theta, \end{aligned}$$

and from (6.42), we have

$$\left|(q_i)'_{x_i} \left(x_i^{(2)} + \theta(x_i^{(1)} - x_i^{(2)}), x_{-i}, \xi\right)\right| \leq (1 + \max\{q_i^u, x_i\})L_1(\xi)/\sigma.$$

The conclusion follows by taking $L_2^i(\xi) := (1 + \max\{q_i^u, x_i\})L_1(\xi)/\sigma$. □

Proof of Theorem 4.4 From Lemma 4.3, for any fixed x_{-i}, we know that $q_i(x_i, x_{-i}, \xi)$ is nondecreasing on x_i and thus $x_i(\xi)$ is unique. Moreover, from the definition of $x_i(\xi)$, we have that $q_i(x_i(\xi), x_{-i}, \xi) = 0$. By Part (v) of Proposition 3.7, $q_i(x_i(\xi), x_{-i}, \xi)$ is increasing in both ξ and x_i, thus $x_i(\xi)$ must decrease at the points where ξ increase to maintain the equation $q_i(x_i(\xi), x_{-i}, \xi) = 0$.

The discussion above shows that for any $x_i \geq 0$ and any fixed x_{-i}, there exists at most one $\xi \in \Xi$, denoted by ξ_i, for generator i such that $q_i(x_i, x_{-i}, \xi) \equiv 0$ for $\xi \geq \xi_i$ and $q_i(x_i, x_{-i}, \xi) > 0$ for $\xi \leq \xi_i$. Therefore $\Xi(x)$ is a finite set. □

Proof of Theorem 4.5 The proof can be divided into two steps, where first we show the once continuous differentiability of $\pi_i(x_i, x_{-i})$ w.r.t. x_i and, in the second step, we show that the function is twice continuously differentiable.

In the first step, we show the continuous differentiability, from Proposition 2 in Ruszczyński and Shapiro (2003) and the differentiability of $v_i(\cdot, \xi)$ w.r.t. x_i, we know that it is sufficient to prove that $v_i(\cdot, \xi)$ is Lipschitz continuous with an integral module, that is, there exists a function $L_4(\xi)$ such that

$$\int_\Xi L_4(\xi)\rho(\xi)d\xi < \infty \quad \text{and}$$

$$|v_i(x_i^{(1)}, \xi) - v_i(x_i^{(2)}, \xi)| \leq L_4(\xi)|x_i^{(1)} - x_i^{(2)}|, \ \forall x_i^{(1)}, x_i^{(2)} \geq 0. \tag{6.43}$$

By assumption, at a point where $Q(x, \xi)$ is differentiable w.r.t. x_i, $p(Q, \xi)$ is bounded by $L_3(\xi)$. Furthermore, p'_Q is bounded by $L_1(\xi)$ and $1 + Q'_{x_i}(x, \xi)$ takes its value in

[1, 2). Therefore

$$\begin{aligned}\left|(v_i)'_{x_i}(x_i,\xi)\right| &= |(q_i)'_{x_i}(x_i,x_{-i},\xi)-1)p(Q(x,\xi),\xi)\\&\quad+q_i(x_i,x_{-i},\xi)p'_Q(Q(x,\xi),\xi)Q'_{x_i}(x,\xi)\\&\quad-c'_i(q_i(x_i,x_{-i},\xi))(q_i)'_{x_i}(x_i,x_{-i},\xi)|\\&\le |(q_i)'_{x_i}(x_i,x_{-i},\xi)-1|L_3(\xi)+q_i(x_i,x_{-i},\xi)L_1(\xi)+L_1(\xi)\\&\le L_3(\xi)+(q_i^u+1)L_1(\xi).\end{aligned}$$

Define

$$L_4(\xi):=L_3(\xi)+(q_i^u+1)L_1(\xi),$$

which satisfies the condition (6.43). Then we have, by the mean value theorem

$$\begin{aligned}|v_i(x_i^{(1)},\xi)-v_i(x_i^{(2)},\xi)| &\le \int_0^1 |(v_i)'_{x_i}(x_i^{(2)}+\theta(x_i^{(1)}-x_i^{(2)}),\xi)||x_i^{(1)}-x_i^{(2)}|d\theta\\&\le L_4(\xi)|x_i^{(1)}-x_i^{(2)}|,\end{aligned}$$

which shows the once continuous differentiability of $\pi_i(x_i,x_{-i})$. Then, from Proposition 2 in Ruszczyński and Shapiro (2003), we can prove that

$$\pi_i(x_i,x_{-i}):=\int_{\xi\in\Xi}\{q_i(x_i,x_{-i},\xi)p(Q(x,\xi),\xi)-c_i(q_i)\}\,\rho(\xi)d\xi$$

is differentiable.

Next, we will show that the second derivative of $\pi_i(x_i,x_{-i})$ exists. To show this point, we again need to apply Proposition 2 in Ruszczyński and Shapiro (2003). It is sufficient to show that $(v_i)'_{x_i}(\cdot,\xi)$ is differentiable w.r.t. x_i for almost every $\xi\in\Xi$ and there exists an integrable function $L_5(\xi)\ge 0$ such that

$$\begin{aligned}&\int_\Xi L_5(\xi)\rho(\xi)d\xi<\infty,\quad \text{and}\\&|(v_i)'_{x_i}(x_i^{(1)},\xi)-(v_i)'_{x_i}(x_i^{(2)},\xi)|\le L_5(\xi)|x_i^{(1)}-x_i^{(2)}|,\quad \forall x_i^{(1)},x_i^{(2)}\ge 0\end{aligned}$$

At any point where $Q'_{x_i}(x,\xi)$ is differentiable w.r.t. x_i, we have

$$\begin{aligned}(v_i)''_{x_i}(x_i,\xi) &= (q_i)''_{x_i}(x_i,x_{-i},\xi)p(Q(x,\xi),\xi)\\&\quad+2q_i(x_i,x_{-i},\xi)p'_Q(Q(x,\xi),\xi)Q'_{x_i}(x,\xi)\\&\quad+q_i(x_i,x_{-i},\xi)p''_Q(Q(x,\xi),\xi)(Q'_{x_i}(x,\xi))^2\\&\quad+q_i(x_i,x_{-i},\xi)p'_Q(Q(x,\xi),\xi)Q''_{x_i}(x,\xi)\end{aligned}$$

$$- c_i''(q_i(x_i, x_{-i}, \xi))((q_i)'_{x_i}(x_i, x_{-i}, \xi))^2$$
$$- c_i'(q_i(x_i, x_{-i}, \xi))(q_i)''_{x_i}(x_i, x_{-i}, \xi).$$

Under the assumption of this theorem, it is easy to derive that, for any fixed x_{-i},

$$\begin{aligned}\left|(v_i)''_{x_i}(x_i, \xi)\right| &\le L_5(\xi) := L_3(\xi)L_3(\xi) + 2q_i^u L_1(\xi)\\ &+ q_i^u L_1(\xi) + q_i^u L_1(\xi)L_3(\xi) + L_3(\xi)L_1(\xi)\\ &= 3q_i^u L_1(\xi) + \left(q_i^u L_1(\xi) + L_3(\xi) + L_1(\xi)\right) L_3(\xi).\end{aligned}$$

Because of $L_1(\xi)$ is bounded and $L_3(\xi)$ is integrable, then $L_5(\xi)$ is also integrable. This shows that $\pi_i(x_i, x_{-i})$ is twice differentiable.

Finally, to complete our proof, we investigate the continuity of the second derivative of $\pi_i(x_i, x_{-i})$. For any fixed x_{-i}, to show the continuity of $\pi_i'(x_i, x_{-i})$ and $\pi_i''(x_i, x_{-i})$, we note that $(v_i)'_{x_i}(\cdot, \xi)$ is a continuous function of x for almost every $\xi \in \Xi$ and $(v_i)''_{x_i}$ is dominated by an integrable bound $L_5(\xi)$. By the Lebesgue dominated convergence theorem, for any fixed x_{-i}

$$\begin{aligned}\lim_{z\to x_i} \pi_i''(z, x_{-i}) &= \int_\Xi \lim_{z\to x_i} (v_i)''_{x_i}(z, x_{-i}, \xi)\rho(\xi)d\xi\\ &= \int_\Xi (v_i)''_{x_i}(x_i, x_{-i}, \xi)\rho(\xi)d\xi\\ &= \pi_i{}''(x_i, x_{-i}).\end{aligned}$$

This completes the proof. □

Proof of Proposition 4.9 We use a methodology analogous to that in DeMiguel and Xu (2008, Proposition 4.2) to prove the results. That is, we show the derivative of Q w.r.t. x_i is non-decreasing. Note that Assumptions 2.2 and 4.2 are satisfied by the inverse demand function and the type of cost functions considered in this proposition. Together with Assumptions 2.1, 2.6 and 3.6, this guarantees the existence and uniqueness of equilibrium in the spot market by Proposition 3.3. We proceed the proof in two steps: Step 1, we consider points where Q is not differentiable w.r.t. x_i; Step 2, we consider points where Q is continuously differentiable w.r.t. x_i. Note that by Theorem 3.5, $q(x, \xi)$ is piecewise smooth w.r.t. x_i for $i = 1, 2, \ldots, M$, and so is Q.

Step 1. Let $i \in \{1, \ldots, M\}$ and let $x_i(\xi)$ denote the point at which $q_j(x_i, x_{-i}, \xi)$ turns from strictly positive to zero at the point, for some $j \in \{1, \ldots, M\}\backslash\{i\}$.

From the discussion above, we have that, for $i = 1, 2, \ldots, M$, $Q(x, \xi)$ and $q_j(x_i, x_{-i}, \xi)$ are all piecewise smooth w.r.t. x_i.

For fixed x_{-i}, as stated in Section 3, we first show that $Q(x_i, x_{-i}, \xi)$ is a piecewise smooth and convex function of x_i at a point where $q_i(x_i, x_{-i}, \xi)$ turns from zero to strictly positive and the points where $q_j(x_i, x_{-i}, \xi)$, for $j \neq i$, turns from strictly positive to zero. At all other points, the function is smooth.

Let $\mathcal{I}(x_i, \xi)$ denote the index set of the generators with $q_j(x_i, x_{-i}, \xi) > 0$ for fixed x_{-i} and $j \neq i$. Then $\mathcal{I}(x_i(\xi)_-, \xi)\backslash\mathcal{I}(x_i(\xi)_+, \xi)$ is the index set of generator i's rivals which turn from a positive supply to zero at $x_i(\xi)$, where

$$\mathcal{I}(x_i(\xi)_-, \xi) = \lim_{\delta \to 0} \mathcal{I}(x_i(\xi) - \delta, \xi), \quad \mathcal{I}(x_i(\xi)_+, \xi) = \lim_{\delta \to 0} \mathcal{I}(x_i(\xi) + \delta, \xi),$$

for any $j \neq i$.

Because $q_i(x_i, x_{-i}, \xi)$ is piecewise smooth in x_i (nonsmooth only at a finite number of points), we may assume that in a neighborhood of $x_i(\xi)$, the function $q_i(x_i, x_{-i}, \xi)$ is differentiable except at $x_i(\xi)$. Since $p(Q, \xi)$ is linear in Q, it follows from the complementary equation (3.9) that $G(q(x, \xi), x, \xi) = 0$ for x in a left neighborhood of $x_i(\xi)$ where $q_i(x_i, x_{-i}, \xi) > 0$, we have, for generator i,

$$(q_i)'_{x_i}(x_i, x_{-i}, \xi) = \frac{1}{-p'_Q + c''_i(q_i)} \left[-p'_Q + \left(p'_Q + (q_i - x_i) p''_Q \right) Q'_{x_i}(x, \xi) \right], \tag{6.44}$$

and

$$(q_j)'_{x_i}(x_i, x_{-i}, \xi) = \frac{1}{-p'_Q + c''_j(q_j)} \left[p'_Q + (q_j - x_j) p''_Q \right] Q'_{x_i}(x, \xi). \tag{6.45}$$

We consider two cases: Case (i) $q_i(x_i, x_{-i}, \xi) > 0$, and Case (ii) $q_i(x_i, x_{-i}, \xi) = 0$.

Case (i). Adding the Eq. (6.45) for all $j \in \mathcal{I}(x_i, \xi)$ and subtracting (6.44), we have that

$$\begin{aligned} Q'_{x_i}(x, \xi) - (q_i)'_{x_i}(x_i, x_{-i}, \xi) &= \sum_{j \in \mathcal{I}(x_i, \xi)} (q_j)'_{x_i}(x_i, x_{-i}, \xi) \\ &= \sum_{j \in \mathcal{I}(x_i, \xi)} \frac{1}{-p'_Q + c''_j(q_j)} \left[p'_Q + (q_j - x_j) p''_Q \right] Q'_{x_i}(x, \xi) \end{aligned}$$

Under the assumptions of this proposition, we have either $c''_j = 0$ or c''_j are identical (in which we denote the derivative by c'' for the cost functions defined in Condition 1 and 2 in Proposition 4.9). Consequently, we have

$$Q'_{x_i}(x, \xi) - (q_i)'_{x_i}(x_i, x_{-i}, \xi) = \frac{1}{-p'_Q + c''} \sum_{j \in \mathcal{I}(x_i, \xi)} \left[p'_Q + (q_j - x_j) p''_Q \right] Q'_{x_i}(x, \xi). \tag{6.46}$$

Since $p(Q, \xi) = \alpha(\xi) - \beta(\xi) Q$, (6.46) is equivalent to

$$\begin{aligned} &Q'_{x_i}(x, \xi) - \frac{1}{\beta(\xi) + c''} \left[\beta(\xi) - \beta(\xi) Q'_{x_i}(x, \xi) \right] \\ &= \frac{1}{\beta(\xi) + c''} \sum_{j \in \mathcal{I}(x_i, \xi)} \left[-\beta(\xi) Q'_{x_i}(x, \xi) \right]. \end{aligned}$$

Let $|\mathcal{I}|$ denote the cardinality of $\mathcal{I}(x_i, \xi)$. Then we can reformulate (6.46) as

$$Q'_{x_i}(x, \xi) = \frac{1}{2 + |\mathcal{I}| + \frac{c''}{\beta(\xi)}}. \tag{6.47}$$

Since $(q_j)'_{x_i}(x_i, x_{-i}, \xi) \in (-1, 0]$, $|\mathcal{I}(x_i, \xi)|$ is a decreasing function of x_i, $\frac{1}{2+|\mathcal{I}|+c''/\beta(\xi)}$ is an increasing function of x_i. This implies the convexity of Q in x_i at the point $x_i(\xi)$.

Case (ii). Since $q_i(x, \xi) = 0$, by the proof in Proposition 3.7, $Q'_{x_i} = 0$ at the left side of the neighborhood of $x_i(\xi)$. At the right side of the neighborhood of $x_i(\xi)$, $Q'_{x_i} > 0$. This shows the convexity of Q in x_i at the point $x_i(\xi)$.

Step 2. Let us consider the points x_i at which both $q_i(x_i, x_{-i}, \xi)$ and $q_j(x_i, x_{-i}, \xi)$ are continuously differentiable w.r.t. x_i. In this case,

$$\mathcal{I}(x_i - \delta, \xi) = \mathcal{I}(x_i + \delta, \xi) = \mathcal{I}(x_i, \xi),$$

for $\delta > 0$ sufficiently small. We can establish (6.47) and the rest of arguments are similar to Step 1 except that $|\mathcal{I}|$ is a constant. □

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