SIAM J. CONTROL OPTIM.
Vol. 44, No. 4, pp. 1391–1418

ε-OPTIMAL BIDDING IN AN ELECTRICITY MARKET WITH DISCONTINUOUS MARKET DISTRIBUTION FUNCTION*

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Abstract. This paper investigates the optimal bidding strategy (supply function) for a generator offering power into a wholesale electricity market. The model has three characteristics: the uncertainties facing the generator are described by a single probability function, namely the market distribution function; the supply function to be chosen is nondecreasing but need not be smooth; the objective function is the expected profit which can be formulated as a Stieltjes integral along the generator's supply curve. In previous work the market distribution function has been assumed smooth, but in practice this assumption may not be satisfied. This paper focuses on the case that the market distribution function is not continuous, and hence an optimal supply function may not exist. We consider a modified optimization problem and show the existence of an optimal solution for this problem. Then we show constructively how such an optimum can be approximated with an ε-optimal supply function by undercutting when the generator does not hold a hedging contract (and possibly overcutting when the generator has a hedging contract). Our results substantially extend previous work on the market distribution model.

Key words. electricity markets, discontinuous market distribution function, R-semicontinuity, ε-optimal supply function

AMS subject classifications. 90C46, 65K10, 49K30

DOI. 10.1137/S0363012903425556

1. Introduction.

In recent years many countries have carried out substantial restructuring of their electricity industries. Though each country has adopted its own solution, the trend has been towards increased market mechanisms, particularly at the wholesale level. It is important to understand the operation of these electricity markets, and yet the special features that are characteristic of wholesale electricity markets make this a challenging task.

We begin by sketching the fundamentals of the way that a wholesale market for electricity works. Generators compete to supply electricity to users (primarily retailers providing electricity to consumers). The price paid fluctuates as demand (and supply) varies. The price is determined through a process that is a type of sealed bid auction. In each time period each generator submits a bid, which we refer to as a supply function $S(p)$, which gives the quantity of electricity that the generator is willing to supply for any price p (strictly, this is a price per megawatt hour and the quantity is measured in megawatts). The supply function is increasing (not necessarily strictly) and often has to satisfy other restrictions imposed by the market operator. The spot price is determined from the combined supply functions of all the generators, and is such that supply at the spot price is just sufficient to meet demand. In practice this has to take account of the location of both the generators and the demand within the network, but we will ignore location effects in this paper. A generator needs to decide on the supply function to offer into the market in order to maximize profits. The

*Received by the editors April 8, 2003; accepted for publication (in revised form) March 24, 2005; published electronically November 4, 2005.
http://www.siam.org/journals/sicon/44-4/42555.html

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demand at any time is uncertain and the offers of other generators are also unknown.

A number of authors have used equilibrium concepts to look at the operation of an electricity market. An important set of papers is that by Green and Newbery [8], Green [9], Newbery [12], and Green [7], in which they analyze the experience in the pool market of England and Wales using the concept of an equilibrium in supply functions (see Klemperer and Meyer [10]). The concept of supply function equilibria has also been applied in this context by Rudkevich [13, 14], Anderson and Philpott [1], and Baldick, Grant, and Kahn [5]. The equilibrium models usually assume the smoothness or piecewise linearity of generators' supply functions and consequently it becomes relatively easy to find the optimal choice of strategy by a single generator. However, this may not be the case when we allow general supply functions.

Some recent papers have looked in detail at the optimal strategy for a generator offering power in an electricity spot market. The conclusions depend largely on the models that are used to describe both the generator's objective and constraints, and the market mechanisms. Anderson and Philpott [2] study strategies for generators making offers into an electricity market when either or both of the demand and the offers of competing generators are stochastic. They introduce the *market distribution function* and use it to describe the residual demand for a generator. The market distribution function ψ is a function of price p and quantity q and the value $\psi(q, p)$ represents the probability that a generator is not fully dispatched if it offers a quantity q of electricity at price p. The advantage of this approach is that in many circumstances a single function $\psi(q, p)$ is enough to determine a generator's expected profit given any particular offer curve.

Anderson and Philpott [2] explore the problem of finding an offer curve that maximizes the expected value of the profit made by an individual generator. The offer curve is simply a monotonic continuous curve in the two-dimensional (quantity, price) space. This curve need not be smooth; indeed, in practice it will often take the form of a series of steps. Anderson and Philpott show that the problem of maximizing expected profit is, in some circumstances, equivalent to maximizing a line integral along the offer curve of the market distribution function and they derive necessary conditions for a supply offer curve to be optimal. Anderson and Xu [4] study the same model and extend the analysis to include necessary conditions of a higher order in the presence of horizontal and/or vertical sections in an offer curve. They also derive sufficient conditions for an offer curve to be locally optimal. Neame, Philpott, and Pritchard [11] use Anderson and Philpott's model to study a generator's optimal choice of supply offer curve under the assumption that the generator is a price taker.

All of this work has been carried out under the assumption that the market distribution function $\psi(q, p)$ is continuously differentiable in both price p and quantity q. In this paper we address the problem of finding optimal offer strategies when ψ may be discontinuous in price.

In many electricity wholesale markets, a generator's offer curve consists of a finite number of steps. For instance, in Australia generator bids are restricted to have no more than ten prices. We can model a generator's offer strategy as a step function of price. In this case the market clearing price will not have a continuous distribution; instead, the distribution of clearing price will be concentrated at certain prices. The consequence of this is that the market distribution function will be discontinuous at these prices. The probability of not being fully dispatched if an offer is made of a quantity q at a price p (i.e., the value of ψ) will increase discontinuously as the price moves from just below the offer of another generator to the same price as the other generator, when the dispatch will be shared between the two generators. In fact this discontinuity happens in both directions of price movement, since as the price moves from being the same as the other generator to just above this level, there is another discontinuous jump in the probability of not being fully dispatched. Previous work in this area has generally made the assumption that the market has a large number of participants, with offer prices well diversified, and so the market distribution function for a generator is nearly continuous and can be approximated with a continuous function.

In this paper we look in more detail at what happens with a discontinuous market distribution function. In the most natural models for this, the existence of an optimal solution for a generator will not be guaranteed (and indeed will occur only rarely). The lack of an optimal solution just reflects the actual difficulty associated with undercutting that can occur in practice. Suppose we know that another generator is offering power at \$30 per MWh, and we have to choose our best offer curve. For example, we might suppose that the other generator is nonstrategic and always submits the same offer. Power we offer at any price up to \$30 will be used in preference to the other generator, power at \$30 will involve a sharing out of the demand between us and the other generator, and power offered at any price higher than \$30 will be dispatched only when the power from the other generator is insufficient to meet demand. This leads to a profit function that is discontinuous in price (in both directions) if we choose to offer power at a single price. A typical good solution to this problem involves offering some power at a price just below the \$30 mark. The closer to \$30 the better for us, but the price must remain below \$30 in order to avoid having to share dispatch. In other words, as we indicated, there is no optimal solution unless we take explicit account of the discretization that may be forced on us by market rules such as, for example, the restriction that we use whole numbers of cents as prices.

In theory at least, this type of undercutting behavior, when translated into the framework of a Nash equilibrium, with different generators all engaged in the same process, will lead to very competitive outcomes as generators repeatedly lower their offer prices towards their true marginal costs. This is essentially the same kind of argument that gives rise to low consumer prices in the Bertrand equilibrium of classical microeconomics. But it can be argued that this is misleading, since markets usually operate in the form of a sealed bid auction, with participants unaware of the bids of other generators. This leads to the possibility of less competitive outcomes through the use of strategies which randomize over the prices offered (see von der Fehr and Harbord [15]).

In this paper, however, we will not discuss equilibrium solutions. Instead we seek to characterize solutions which approach optimality in the undercutting case. In practice it is not unusual for a generator to know the prices at which one or more of the other generators will offer power. For example, there may be nonstrategic generators who offer some quantity of energy at fixed prices which do not vary from day to day. It may be surprising that this occurs, since it is clear that this policy will not in general be optimal for the nonstrategic generator. One explanation is that more complex randomized policies may offer only a limited improvement in profit. As a concrete example of this behavior, consider the Australian market, in which a single generation unit offers power at 10 different price points, set for a 24-hour period (with quantities offered at each price point set separately for each half-hour). These price points are not usually varied from one day to the next; moreover, complete information on all bids is freely available one day after the event (see the web site www.nemmco.com).

For example, power from the Bayswater (coal-fired) power station in New South Wales is offered at a number of price points, but these have included the price \$22.89 for many months on end.

In order to deal with the undercutting behavior, our approach is to alter the model to ensure that the limit of undercutting solutions is a solution with the limiting value. The fundamental idea is to suppose that the generator we are interested in has automatic priority of dispatch when there are other generators offering at the same price. This will ensure that the market distribution function has sufficient continuity properties to guarantee that there will be an optimal solution. Though the exact optimal value is unachievable, the generator can operate in a way that gets as close as it likes to this value. From a practical point of view, establishing the supremum value, and the limiting solution which achieves this, is useful since it enables the generator to find a good solution near the limit, and also bounds the opportunity cost of accepting a suboptimal outcome. In practice market rules imply further restrictions on bids offered, but knowledge of the best possible limit solution will help to guide the selection of a suitable bidding policy.

There is another complication we need to consider. In most cases a generator will have hedging contracts for a significant part of its output. As we shall see, this can have the effect of reversing some of the incentives for the generator. If the generator has contracted for a larger quantity than will actually be dispatched in a certain period, then the generator will benefit from lower prices. The consequence is that, in this case, it will usually be optimal to "overcut" another generator's offer. Hence, to use the same example as above, if another generator has offered some quantity at \$30 per MWh, then we could well decide to make an offer of some amount of power a little above this level (and the closer to \$30 the better). In the case that offers have to be made in whole numbers of cents this would lead to an offer at \$30.01.

We can summarize this paper as follows. We first demonstrate the existence of an optimal solution for a modified problem (section 2). The modified problem differs from the original problem through the method used to determine the sharing of dispatch when two generators offer power at the same price: essentially, the optimizing generator is given the ability to choose its best sharing rule. Care is needed in determining the precise form of the objective function in these circumstances (Theorem 2.9) and in establishing the appropriate form of continuity in order to show the existence result (Lemma 2.7). Then we explore the necessary conditions for an optimal solution to this modified problem (section 3). Next we show how to use an optimal solution for the modified problem to generate an ε-optimal solution for the original problem (section 4). Finally, we illustrate all this with an example (section 5).

2. Problem formulation and fundamentals.

In this section we will introduce some notation and formulate the problem that we shall consider.

We consider the behavior of a single generator, which we call A, and we let $R(q, p)$ be the profit for generator A if it is dispatched q at a clearing price p. Usually $R(q, p)$ has three components. First there is the cost, $C(q)$, of generating a quantity q of electricity, which is often taken to be an increasing convex function. Second there is the money, pq, paid to the generator through the market clearing mechanism. Finally, we must also consider the hedging contracts entered into by the generator. These are financial instruments which do not involve the actual generation of electricity; the money paid under the contract is tied to the pool price. If the generator enters into a contract at a strike price f for a quantity Q, and the actual spot price is p, then the generator will pay an amount $Q(p - f)$ to the other party in the contract. The contracts we consider are two-way contracts for differences, so if the spot price is lower than the contract strike price, then the generator will receive an amount $Q(f-p)$. Contracts of this sort are a common feature of electricity markets operating with a "pool" structure in which prices for all traded electricity are determined through a combined pricing and dispatch mechanism (such as the markets operating in Australia and New Zealand and the old pool arrangements in England and Wales). Note that this is a different environment than that of markets which are based on bilateral contracts, such as in the new trading arrangements in England and Wales.

Thus we arrive at the following expression for the profit to generator A as a function of spot price p and dispatched quantity q:

$$R(q,p) = pq - C(q) + Q(f-p). \tag{2.1}$$

We will not assume any particular functional form for the function R. However, throughout this paper we will assume that R has continuous bounded partial derivatives, R_q and R_p, and is strictly concave in q for fixed p. Thus we have $R_q(\cdot,p)$ strictly decreasing for each fixed p. In the case that (2.1) holds, this assumption will be satisfied provided that the marginal cost of generation is strictly increasing, since $R_q(q,p) = p - C'(q)$.

Next we consider the market dispatch mechanism. We will restrict attention to the case where there is a single node. We consider this from the point of view of generator A. We model the sequence of events in this way. First generator A submits a supply function $S_A(p)$, which gives the total amount of power that generator A is prepared to supply as a function of the price p. Then all the other generators submit their supply functions, which we collectively write as $S_B(p)$—this is the total amount of power that all the other generators are prepared to supply as a function of price. We take both S_A and S_B as right-continuous increasing functions (not necessarily strictly increasing). Where there is a discontinuity in S_A, a jump up occurs at a certain price p, and this corresponds to a certain quantity of power being offered at p and all available at that price. Hence right-continuity is a natural assumption here.

Finally, a demand occurs, where demand at this node is given by a function $D(p)$ of price. We suppose that from the point of view of generator A, both $S_B(p)$ and $D(p)$ are uncertain and must be modeled as stochastic. The market clears at the lowest price p for which $S_A(p) + S_B(p) \geq D(p)$.

In the model we are considering, which corresponds to the most common type of pool market, all generators are paid this clearing price for all the electricity that they are dispatched. This is a type of uniform price auction mechanism. There are other (discriminatory) auction price mechanisms that have been proposed.

Throughout this paper, we assume that a generator's supply function (or equivalently supply curve) is nondecreasing. It can be step-like or strictly increasing, or both. Rather than dealing with a supply function $S_A(\cdot)$ directly, it is convenient to model the offer using a continuous curve $\mathbf{s} = \{(\hat{q}(\tau), \hat{p}(\tau)), 0 \leq \tau \leq T\}$, in which the components $\hat{q}(\tau)$ and $\hat{p}(\tau)$ are continuous monotonic increasing function of τ, and $\hat{q}(\tau)$ and $\hat{p}(\tau)$ trace, respectively, the quantity and price components. Without loss of generality we may take $\hat{q}(0) = \hat{p}(0) = 0$ and $\hat{p}(T) \leq p_M$, where p_M is a bound on the price of any offer. It is quite common for electricity markets to have a cap on prices; for example, this is \$10,000 per MWh in Australia. We also assume that q_M is a bound on the generation capacity of generator A, and thus $\hat{q}(T) \leq q_M$.

In all markets there are restrictions on the form of offers made into the market; we have already mentioned the need for offers to consist of step functions in many cases. But in this paper we will not include any constraints on the form of offers. Our perspective is that a generator which has a specific optimal offer curve will usually be able to approximate this within the rules of the market. Owners of generators will often be offering power from more than one generation set in a coordinated way, and this can also increase their flexibility.

We use a single market distribution function $\psi(q,p)$ to describe the uncertainty in the market. Following Anderson and Philpott [2], $\psi(q,p)$ is defined as the probability of generator A not being fully dispatched if it offers an amount of generation q at a price p. Different generators will have different market distribution functions, but we just write ψ rather than ψ_A for this function. It turns out that when $\psi(q,p)$ is continuous, knowledge of the single function ψ is enough to determine the expected profit for a generator. When ψ is continuous, Anderson and Philpott [2] have demonstrated that the expected profit if a generator offers in a supply curve $\mathbf{s}$ can be expressed as a Stieltjes integral along the line $\mathbf{s}$:

$$v(\mathbf{s}) = \int_{\mathbf{s}} R(q,p)\ d\psi(q,p). \tag{2.2}$$

The generator's aim is to choose an optimal supply curve $\mathbf{s}$ so that $v(\mathbf{s})$ is maximized. Note that the market distribution function ψ is assumed to be known. Anderson and Philpott [3] have proposed a Bayesian inference method to estimate ψ given data on the market behavior in previous days. Note that although the setting is a stochastic one, this formulation of the problem of maximizing expected profit has converted the objective function into a deterministic optimization problem.

When the function ψ is discontinuous we need a different form of the fundamental relationship (2.2), and this will be derived in Theorem 2.9 below.

2.1. Discontinuous ψ function. Previous work in this area has assumed that the market distribution function is continuous. In this paper, rather than requiring ψ to be continuous, we assume that ψ may be discontinuous at a finite number of prices. Since ψ is a type of probability distribution function, a discontinuity in its value corresponds to a single price at which there is a jump in the probability of being fully dispatched. For this to occur two things have to happen. First some other generator has to make an offer which contains a "step," a distinct tranche of energy at a given price, and second this price has to be determined in advance (in other words, it cannot be drawn from a continuous distribution). The first condition may be met because of market rules which only allow step function offers, but for a discontinuity in ψ it is also necessary to be able to predict the prices at which other generators make offers.

We illustrate this with an example.

Example 2.1. Suppose that just two generators A and B are offering power into the spot market. Generator B is nonstrategic: its offer does not vary and is known in advance from previous market data. Thus the only uncertainty is in relation to the level of demand. Suppose that the generator B offers 200 MW at a price of \$10 per MWh, 300 MW at a price of \$14, and 300 MW at a price of \$18. Thus generator B's supply function is

$$S_B(p) = \begin{cases} 0 & \text{for } 0 \le p < 10, \\ 200 & \text{for } 10 \le p < 14, \\ 500 & \text{for } 14 \le p < 18, \\ 800 & \text{for } p \ge 18. \end{cases}$$

Consider generator A offering 100 MW at price \$10 per MWh. We suppose that demand, which can be a function of price, is uncertain. If the market clears at price \$10 with a total demand of 300 MW, then all the power offered at this price is dispatched. However, if the demand is below 300 MW at price \$10, then market rules will impose some sharing of dispatch should the market clear at this price. Suppose that the market rules share dispatch proportionately to the quantity offered at that price, so that one third of demand is met from generator A and two thirds from generator B. Thus neither of the generators gets fully dispatched at price \$10. On the other hand, if generator A offers 100 MW at price $\$10-\varepsilon$, where $\varepsilon > 0$ is small, then it is fully dispatched provided that the demand at price \$10 is greater than or equal to 100 MW. Therefore the probability of not being fully dispatched if generator A offers at price \$10 with a quantity of 100 MW is strictly greater than the probability of not being fully dispatched if A offers at price $\$10-\varepsilon$ with a quantity of 100 MW; i.e.,

$$\lim_{\epsilon\downarrow 0}\psi(100, 10-\varepsilon) < \psi(100, 10).$$

This example motivates us to consider discontinuities in the functions $S_B(p)$. We write $\mathcal{P}$ for the entire set of prices at which the other generators may make significant offers, and hence at which there may be discontinuities in $S_B(p)$. Let $\mathcal{P}\equiv \{p^1, \ldots, p^n\}$, where $0 < p^1 < \cdots p^n \le p_M$. We assume that the prices in $\mathcal{P}$ are known in advance.

For clarity we write $\omega_1 \in \Omega_1$ for the realizations of the demand, and $\omega_2 \in \Omega_2$ for realizations of the other generator offers. More formally, we assume a probability space $(\Omega_1\times\Omega_2, \mathcal{F}, \Pr)$. The demand need not be independent of other generator offers. We shall assume that for every realization ω_1, the demand, $D(p, \omega_1)$, is a continuously differentiable decreasing function of p. Moreover we assume that for every realization ω_2, the total of the other generator offers, $S_B(p, \omega_2)$, is a continuously differentiable increasing function of p except at points in $\mathcal{P}$. We will normally omit the explicit dependence on ω_1 and ω_2, and write $D(p)$ and $S_B(p)$.

Observe that in any realization (of demand and other generator offers) for which $D(p) < q + \lim_{\varepsilon\downarrow 0} S_B(p-\varepsilon)$, an offer of q at price p cannot be fully dispatched (since if it were fully dispatched, then the price is at least p and so the other generators would be dispatched at least $\lim_{\varepsilon\to 0} S_B(p-\varepsilon)$, giving a contradiction). Hence

$$\psi(q, p) \ge \Pr(D(p) < q + \lim_{\varepsilon\downarrow 0} S_B(p-\varepsilon)). \tag{2.3}$$

If $p \notin\mathcal{P}$, then for every realization of other generator offers $\lim_{\varepsilon\downarrow 0} S_B(p-\varepsilon) = S_B(p)$ and thus $\psi(q, p) \ge \Pr(D(p) < q + S_B(p))$.

On the other hand, in any realization for which an offer of q at price p is not fully dispatched we can show that $D(p) < q + \lim_{\varepsilon\downarrow 0} S_B(p+\varepsilon)$ (since in this case the clearing price is p or less, and so the maximum quantity dispatched from the other generators is $\lim_{\varepsilon\downarrow 0} S_B(p+\varepsilon)$). Thus

$$\psi(q, p) \le \Pr(D(p) < q + \lim_{\varepsilon\downarrow 0} S_B(p+\varepsilon)) \tag{2.4}$$

and $\psi(q, p) \le \Pr(D(p) < q + S_B(p))$ when $p \notin\mathcal{P}$. Hence, except at points of discontinuity in S_B,

$$\psi(q, p) = \Pr(D(p) < q + S_B(p)). \tag{2.5}$$

We may use this as a definition of $\psi(q,p)$ for $p \notin \mathcal{P}$, but for $p \in \mathcal{P}$ the value of ψ depends on the sharing rule.

Since in any realization of demand and other generator offers in which $D(p) < q + S_B(p)$ the same inequality holds for any higher value of p or q, we can deduce that $\psi(q,p)$ is increasing in both its arguments at prices $p \notin \mathcal{P}$. Moreover we can use (2.3) and (2.4) to show that $\psi(q,p)$ is also increasing in p at prices $p \in \mathcal{P}$.

Note that since ψ is monotonic increasing in p and bounded, the two limits $\lim_{\delta\downarrow 0}\psi(q,p^j+\delta)$ and $\lim_{\delta\downarrow 0}\psi(q,p^j-\delta)$ will both exist. For convenience, we will use the following notation: for $j = 1,\ldots,n$,

$$\psi_+(q,p^j) = \lim_{\delta\downarrow 0}\psi(q,p^j+\delta),$$

$$\psi_-(q,p^j) = \lim_{\delta\downarrow 0}\psi(q,p^j-\delta),$$

$$\Phi(q,p^j) \equiv \psi_+(q,p^j) - \psi_-(q,p^j). \tag{2.6}$$

Thus $\Phi(q,p^j)$ is the jump in the probability of dispatch that takes place if the generator offers an amount q at a price just below p^j in comparison with what happens if the price is increased to be just above p^j.

It is important to consider the expected return for a generator offering a curve $\mathbf{s}$ when the market distribution function is discontinuous: in the continuous case we have the expression (2.2). In general we would expect to have, in addition to an integral, a sum of discrete values $R(q,p)$ at points (q,p) on $\mathbf{s}$ at which there is a jump in the value of ψ. This is indeed what happens when the curve $\mathbf{s}$ is strictly increasing. We will show later that if we define $q^j(\mathbf{s})$ as the point at which the curve $\mathbf{s}$ crosses the discontinuity p^j, then

$$v(\mathbf{s}) = \int_{\mathbf{s}^C} R(q,p)\ d\psi(q,p) + \sum_{j=1}^{n} R(q^j(\mathbf{s}),p^j)\Phi(q^j(\mathbf{s}),p^j), \tag{2.7}$$

where $\mathbf{s}^C$ is the part of curve $\mathbf{s}$ excluding the points $(q^j(\mathbf{s}),p^j)$. However, when the curve $\mathbf{s}$ has a horizontal section at one of the prices p^j, things are more complex.

2.2. Sharing rules. If we suppose that the generator is offering power at the same price p^j as another generator, then we cannot calculate the expected profit without knowledge of the market rules concerning the sharing of dispatch between two generators offering at the same price. Moreover, the value of ψ at p^j gives just the probability of complete dispatch, whereas the sharing rules imply more information than this. Specifically the values of ψ might not be enough to determine a generator's expected profit. It may be that two different sharing rules give the same ψ values but different expected profit. To illustrate this we return to Example 2.1.

Example 2.2. Suppose as before that generator B offers 200 MW at price \$10 and 300 MW at \$14, while generator A offers 100 MW at \$10. Suppose that sharing of dispatch between two generators offering at the same price is proportional to the offers made at that price. Suppose now that generator A has costs of \$8 per MWh and demand is uniformly distributed between 0 MW and 500 MW. Thus with probability 0.4 demand is greater than 300 MW, the market clears at \$14, and the profit to generator A is \$600 per hour. On the other hand, with probability 0.6 the market will clear at \$10 and generator A will be only partially dispatched. It is not hard to see that the total expected profit per hour is given by

$$v = 0.6\int_0^{100} 2\frac{x}{100}dx + 0.4 \times 600 = 300.$$

Consider now a sharing rule which gives priority to generator B. In this case with probability 0.4 the demand is less than 200 MW and generator A is not dispatched at all (while generator B is partially dispatched). We have the following expression for expected profit:

$$v = 0.2 \int_0^{100} 2\frac{x}{100} dx + 0.4 \times 600 = 260.$$

Notice that in this second case, too, generator A is not fully dispatched unless demand is greater than 300 MW. So both these sharing rules have the same value for $\psi(100, 10)$, which is the probability of generator A not being fully dispatched with this offer. Indeed the two rules will give the same value of $\psi(q, 10)$ for any value of q.

In order to make further progress we need to consider specific sharing rules. We will write $v(\mathbf{s},\mathcal{L})$ for the expected profit when an offer curve of $\mathbf{s}$ is used together with market sharing rules defined by $\mathcal{L}$. We will investigate the particular choice of sharing rule which is best for generator A.

Suppose that generator A uses the supply function $S_A(p)$ and the other generators use the supply function $S_B(p)$. We write $S_B(p_-)$ for the limit $\lim_{\varepsilon \downarrow 0} S_B(p - \varepsilon)$ and $S_A(p_-)$ for the limit $\lim_{\varepsilon \downarrow 0} S_A(p - \varepsilon)$.

We are interested in the sharing rule to be applied when the market clears at price p^j. The market clears at this price if and only if $D(p^j)$ satisfies

$$S_A(p^j_-) + S_B(p^j_-) \le D(p^j) \le S_A(p^j) + S_B(p^j). \tag{2.8}$$

A sharing rule $\mathcal{L}$ is any method for determining the dispatch quantity $\gamma_A(\mathcal{L})$ for generator A in this case. Though this is not made explicit in the notation, the sharing rule is applied at a particular price p^j; and in general we need to define a sharing rule for each price $p \in \mathcal{P}$. Notice that $\gamma_A(\mathcal{L})$ is a function of the demand, but we suppress this dependence in the notation.

A feasible sharing rule has to satisfy the following inequalities:

$$S_A(p^j_-) \le \gamma_A(\mathcal{L}) \le S_A(p^j), \tag{2.9}$$

$$S_B(p^j_-) \le D(p^j) - \gamma_A(\mathcal{L}) \le S_B(p^j). \tag{2.10}$$

The right-hand inequalities correspond to the restriction that no generator can be dispatched more than it offers at price p^j. The left-hand inequalities correspond to the restriction that any power offered at prices less than p^j must be completely dispatched.

More generally, we can make the following definition.

DEFINITION 2.3. *Let the market clear at price $p \in (0, p^M)$, and thus*

$$S_A(p_-) + S_B(p_-) \le D(p) \le S_A(p) + S_B(p).$$

Then $\mathcal{L}$ is a feasible sharing rule if it determines uniquely the respective dispatch quantities $\gamma_A \in [S_A(p_-), S_A(p)]$ for generator A, and $\gamma_B \in [S_B(p_-), S_B(p)]$ for the other generators, such that

$$\gamma_A + \gamma_B = D(p).$$

Notice that unless two generators both offer power at the price p there will only be one possible choice for γ_A and γ_B. Thus the only prices at which the sharing rule needs to be defined are p^j, $j = 1, 2, \ldots, n$.

We let $q^*(p^j)$ be the value of q at which $R(q,p^j)$ achieves its maximum over $[0,q_M]$. Our assumptions on R imply that this is unique. We have $q^*(p^j)=0$ if $R_q(0,p^j)<0$, $q^*(p^j)=q_M$ if $R_q(q_M,p^j)>0$, and $R_q(q^*(p^j),p^j)=0$ otherwise. Notice that $q^*(p^j)$ is not affected by any hedging contracts.

Now we define the sharing rule $\mathcal{L}^*$ as follows. Let

$$q^j=\begin{cases} S_A(p^j) & \text{if } S_A(p^j)<q^*(p^j),\\ q^*(p^j) & \text{if } S_A(p^j_-)\le q^*(p^j)\le S_A(p^j),\\ S_A(p^j_-) & \text{if } q^*(p^j)<S_A(p^j_-).\end{cases}$$

It is easy to see that q^j maximizes $R(q,p^j)$ subject to $S_A(p^j_-)\le q\le S_A(p^j)$. Since we also require that (2.10) be satisfied, we define the dispatch amount γ_A from generator A under $\mathcal{L}^*$ (when the price is p^j) as

$$\gamma_A(\mathcal{L}^*)=\begin{cases} (a) & D(p^j)-S_B(p^j_-) & \text{if } D(p^j)-S_B(p^j_-)<q^j,\\ (b) & D(p^j)-S_B(p^j) & \text{if } D(p^j)-S_B(p^j)>q^j,\\ (c) & q^j & \text{otherwise.}\end{cases}$$

The lemma below demonstrates that $\mathcal{L}^*$ is the *best* choice of sharing rule for generator A, in the sense that no other sharing rule will produce such a large profit for A.

LEMMA 2.4. *$\mathcal{L}^*$ is a feasible sharing rule, and $v(\mathbf{s},\mathcal{L}^*)\ge v(\mathbf{s},\mathcal{L})$ for every feasible sharing rule $\mathcal{L}$.*

Proof. We consider the profit when the clearing price is $p^j\in\mathcal{P}$, since if the clearing price is not in $\mathcal{P}$ no sharing rule will be needed. We write I_A for the interval $[S_A(p^j_-),S_A(p^j)]$ and I_B for the interval $[D(p^j)-S_B(p^j),D(p^j)-S_B(p^j_-)]$. The length of interval I_A is the offer from A at price p^j, while I_B is the range of possible residual demand for A at this price. Therefore a feasible sharing rule has γ_A in both I_A and I_B.

We wish to establish that γ_A is the unique optimal solution to

$$\max_{q\in I_A\cap I_B} R(q,p^j). \tag{2.11}$$

From (2.8) we observe that I_A and I_B will overlap, so the feasible set for the maximization problem is nonempty. Observe also that q^j is the unique optimal solution to the problem

$$\max_{q\in I_A} R(q,p^j),$$

which implies that, within interval I_A, $R(\cdot,p^j)$ is strictly increasing for $q\le q^j$ and strictly decreasing for $q>q^j$.

We consider the three cases in the definition of $\gamma_A(\mathcal{L}^*)$. In case (a) q^j falls to the right of the interval I_B, hence the right end point of I_B, $D(p^j)-S_B(p^j_-)$, is the optimal solution of (2.11). Similarly in case (b) q^j falls to the left of the interval I_B, and hence the left-hand end point of interval I_B, $D(p^j)-S_B(p^j)$, is the optimal solution of (2.11). In case (c) q^j is in I_B and is therefore the optimal solution of (2.11). This shows $\gamma_A(\mathcal{L}^*)$ is the optimal solution of (2.11). ∎

2.3. *R*-semicontinuous *ψ* function. We need to define a specific type of discontinuity behavior for the function ψ. In fact, at some points in the (q,p) plane we need ψ to be continuous from above, and at other points to be continuous from below, depending on the characteristics of the function R.

DEFINITION 2.5. *Suppose that the market distribution function $\psi(q,p)$ is continuous at all prices $p \notin \mathcal{P}$. ψ is called R-semicontinuous if $\psi_-(q,p^j) = \psi(q,p^j)$ when $R_q(q,p^j) \geq 0$ and $\psi_+(q,p^j) = \psi(q,p^j)$ when $R_q(q,p^j) < 0$, $j = 1,\dots,n$.*

With the form of profit function given in (2.1) we can see that an R-semicontinuous market distribution function will have the property of being continuous from below (in the (q,p) plane) when $p > C'(q)$, and will be continuous from above when the reverse inequality holds. We will show that ψ will be R-semicontinuous when the sharing rule $\mathcal{L}^*$ is applied.

Though we have assumed that both demand and other generator offers are well behaved in any given realization, we also need to have the realizations of demand and offers in some sense continuously distributed through the appropriate spaces.

Assumption 2.6 (continuity). The function $q \longmapsto \Pr(D(p) < q + S_B(p))$ is continuous on $[0,q^M]$, and the function $q \longmapsto \Pr(D(p) < q + S_B(p))$ is continuous on $[0,p^M]\backslash\mathcal{P}$.

This assumption implies, from (2.5), that $\psi(q,p)$ is continuous at all prices $p \notin \mathcal{P}$.

LEMMA 2.7. *Under Assumption* 2.6*, if the sharing rule $\mathcal{L}^*$ is used, then the market distribution function ψ is R-semicontinuous.*

Proof. We consider an offer of an amount q by generator A at a price p^j, where $\psi(q,\cdot)$ is discontinuous. We suppose that there is no other offer by generator A. Thus $S_A(p^j) = q$ and $S_A(p^j_-) = 0$.

Suppose first that $R_q(q,p^j) \geq 0$, so $q \leq q^*(p^j)$. From the definition of q^j, we have $q^j = q$, thus $\mathcal{L}^*$ will choose to dispatch an amount q, if this is possible when the constraints due to the demand realization are considered.

Recall that $\psi(q,p^j)$ is defined as the probability of not being fully dispatched when A makes an offer of q at p^j. Under $\mathcal{L}^*$, the probability of not being fully dispatched is the probability of either the market clearing at a price below p^j or clearing at price p^j but $D(p^j) - S_B(p^j_-) < q$, which means generator B's offer at p^j is not dispatched at all, and the residual demand for generator A falls below q. In this case A gets dispatched $D(p^j) - S_B(p^j_-)$. This is exactly the case (a) in the definition of $\gamma_A(\mathcal{L}^*)$.

Define the event

$$H = \{(\omega_1,\omega_2) : D(p^j,\omega_1) < q + S_B(p^j_-,\omega_2)\}.$$

Then

$$\psi(q,p^j) = \Pr(H).$$

In what follows, we show $\psi(q,\cdot)$ is continuous at $p = p^j$ from below. We write G_ε for the event that an offer of q at price $p^j - \varepsilon$ is not fully dispatched; i.e.,

$$G_\varepsilon = \{(\omega_1,\omega_2) : D(p^j - \varepsilon,\omega_1) < q + S_B(p^j - \varepsilon,\omega_2)\}.$$

Then the G_ε are monotonically increasing sets as ε decreases to zero, with, say, a limit G. D is a continuous function of p in each realization, and so if $(\omega_1,\omega_2) \in H$, then for some choice of $\varepsilon_0 > 0$, $D(p^j - \varepsilon,\omega_1) < q + S_B(p^j - \varepsilon,\omega_2)$ for $0 < \varepsilon < \varepsilon_0$. Therefore $H \subset G$. From the axioms of probability, $\Pr(G) = \lim_{\varepsilon\to 0}\Pr(G_\varepsilon)$. Thus $\psi(q,p^j) \leq \lim_{\varepsilon\to 0}\psi(q,p^j - \varepsilon)$. But since ψ is increasing in p, there must be equality here, i.e., $\psi_-(q,p^j) = \psi(q,p^j)$, as required.

Now consider the case that $R_q(q,p^j) < 0$ and we show $\psi(q,\cdot)$ is continuous at $p = p^j$ from above.

In this case $q > q^*(p^j)$. The probability of not being fully dispatched under $\mathcal{L}^*$ is the probability of either the market clearing at a price below p^j or of clearing at p^j with $D(p^j) - S_B(p^j) < q$. Thus $\psi(q, p^j) = \Pr(J)$ where

$$J = \{(\omega_1, \omega_2) : D(p^j, \omega_1) < q + S_B(p^j, \omega_2)\}.$$

We write F_ε for the event that an offer of q at price $p^j + \varepsilon$ is not fully dispatched. Then F_ε is monotonically decreasing as ε decreases to zero, with a limit F, say. So every realization in F is in every F_ε for $\varepsilon < \varepsilon_0$, where ε_0 depends on the realization; i.e., for every $(\omega_1, \omega_2) \in F$, $D(p^j + \varepsilon, \omega_1) < q + S_B(p^j + \varepsilon, \omega_2)$ for $\varepsilon > 0$. Thus from the continuity of D and S_B, $F \subset \{(\omega_1, \omega_2) : D(p^j, \omega_1) \le q + S_B(p^j, \omega_2)\}$. Thus from Assumption 2.6

$$\begin{aligned}\Pr(F) &\le \Pr((\omega_1, \omega_2) : D(p^j, \omega_1) \le q + S_B(p^j, \omega_2))\\ &= \Pr((\omega_1, \omega_2) : D(p^j, \omega_1) < q + S_B(p^j, \omega_2)) = \psi(q, p^j).\end{aligned}$$

Now, since $\lim_{\varepsilon \to 0} \Pr(F_\varepsilon) = \Pr(F)$, we have established that $\psi_+(q, p^j) \le \psi(q, p^j)$, and the monotonicity of ψ shows that these are equal. $\square$

The reverse implication does not hold: we can have an R-semicontinuous ψ without using the sharing rule $\mathcal{L}^*$.

2.4. Expected profit. We let $\Psi = \{(q, p) : 0 < \psi(q, p) < 1\}$. In line with Definition 2.5, we can divide Ψ into two regions Ψ_+ and Ψ_-, where

$$\Psi_+ = \{(q, p) \in \Psi : R_q(q, p) \ge 0\}, \qquad \Psi_- = \{(q, p) \in \Psi : R_q(q, p) < 0\}.$$

In the case that ψ is R-semicontinuous, to calculate the expected profit from a supply curve $\mathbf{s}$ when it has a segment on the horizontal line $\{(q, p^j) : q \in [0, q^M]\}$, we need to think of it as part of the region below that line in the set Ψ_+ and part of the region above that line in the set Ψ_-. This motivates the following definitions:

$$\begin{aligned}\Psi^j &= \{(q, p) \in \Psi : 0 \le q \le q_M,\ p^{j-1} < p < p^j\} \cup \{(q, p^j) \in \Psi_+\}\\ &\quad \cup \{(q, p^{j-1}) \in \Psi_-\}, \qquad j = 2, \dots, n,\\ \Psi^1 &= \{(q, p) \in \Psi : 0 \le q \le q_M,\ 0 \le p < p^1\} \cup \{(q, p^1) \in \Psi_+\},\\ \Psi^{n+1} &= \{(q, p) \in \Psi : 0 \le q \le q_M,\ p^n < p \le p^M\} \cup \{(q, p^n) \in \Psi_-\},\\ \mathbf{s}^j &= \mathbf{s} \cap \Psi^j.\end{aligned}$$

It is not hard to see that the values q^j which we introduced in relation to the sharing rule $\mathcal{L}^*$ also define the points at which an offer curve $\mathbf{s}$ moves from Ψ^j to Ψ^{j+1}. Thus, writing q^j as a function of $\mathbf{s}$,

$$q^j(\mathbf{s}) = \sup\{q : (q, p^j) \in \mathbf{s}^j\}$$

for $j = 1, \dots, n$. From the monotonicity of the offer curve $\mathbf{s}$, and because R_q is decreasing, we can also write

$$q^j(\mathbf{s}) = \inf\{q : (q, p^j) \in \mathbf{s}^{j+1}\}.$$

Under the assumptions of Lemma 2.7, we can take ψ as R-semicontinuous and made up from a number of different pieces ψ^j, where ψ^j is defined on the interval between p^{j-1} and p^j and is well behaved on that interval. Thus we let

$$\psi^j(q, p) = \begin{cases} \psi_+(q, p^{j-1}) & \text{for } p = p^{j-1},\\ \psi(q, p) & \text{for } p^{j-1} < p < p^j,\\ \psi_-(q, p^j) & \text{for } p = p^j \end{cases}$$

for $j = 2, \ldots, n$, and

$$\psi^1(q,p) = \begin{cases} \psi(q,p) & \text{for } \ 0 \le p < p^1, \\ \psi_-(q,p^1) & \text{for } \ p = p^1, \end{cases}$$

$$\psi^{n+1}(q,p) = \begin{cases} \psi_+(q,p^n) & \text{for } \ p = p^n, \\ \psi(q,p) & \text{for } \ p^n < p \le p^M. \end{cases}$$

We need to make an assumption on the behavior of the function ψ.

Assumption 2.8 (continuous differentiability). $\psi(q,p)$ is continuously differentiable for $p \notin \mathcal{P}$ and each ψ^j can be extended to a continuously differentiable function on an open set W^j which contains the closure of the set Ψ^j.

THEOREM 2.9. *Suppose that a generator offers a curve* $\mathbf{s}$ *and Assumptions* 2.6 *and* 2.8 *are satisfied. If sharing rule* $\mathcal{L}^*$ *is used, then the expected profit for the generator is*

$$v(\mathbf{s}) = \sum_{j=1}^{n+1} \int_{\mathbf{s}^j} R(q,p)\ d\psi^j(q,p) + \sum_{j=1}^{n} R(q^j(\mathbf{s}),p^j)\Phi(q^j(\mathbf{s}),p^j), \tag{2.12}$$

where Φ *is defined in* (2.6).

Proof. To simplify our presentation we prove the theorem for $n = 1$ with just one price discontinuity at p^1. The case with $n > 1$ can be dealt with similarly. We suppose that generator A uses an offer curve $\mathbf{s}$ which we take as $\mathbf{s} = \{(x(\tau), y(\tau))\}$ in parameter form.

We write γ_A for the dispatch quantity from generator A given the offer curve $\mathbf{s}$. We start by showing that $\psi(x(\tau), y(\tau))$ is the probability that γ_A is less than $x(\tau)$. In the case that ψ is continuous in a neighborhood of $(x(\tau), y(\tau))$, this is straightforward and is implicitly established in [2]. But when $y(\tau) = p^1$ we need to be more careful. Observe that from the definition of $\mathcal{L}^*$, if $x(\tau) < q^1$, then

$$\Pr(\gamma_A < x(\tau)) = \Pr(D(p^1) - S_B(p^1_-) < x(\tau)).$$

But the probability of an offer of $x(\tau)$ at price p^1 is not fully dispatched under $\mathcal{L}^*$ with the same probability. Hence $\Pr(\gamma_A < x(\tau)) = \psi(x(\tau), y(\tau))$ as required. The case when $x(\tau) \ge q^1$ can be dealt with similarly.

We let τ^1 be such that $y(\tau^1) = p^1$ and $x(\tau^1) = q^1$. We consider the expected profit on a segment, $\mathbf{s}_\delta \equiv \{(x(\tau), y(\tau)) : \tau^1 - \delta < \tau \le \tau^1 + \delta\}$, of curve $\mathbf{s}$. From our observation on $\psi(x(\tau), y(\tau))$ we know that the probability that the market clears at a price p and quantity q on the offer curve in the segment $\mathbf{s}_\delta$ is given by $\psi(x(\tau^1 + \delta), y(\tau^1 + \delta)) - \psi(x(\tau^1 - \delta), y(\tau^1 - \delta))$.

The expected profit from the line segment $\mathbf{s}_\delta$ is bounded above (below) by this probability multiplied by the supremum (infimum) of R over the set $\mathbf{s}_\delta$. Since R is continuously differentiable, for δ small enough, the expected profit from segment $\mathbf{s}_\delta$ is

$$v(\mathbf{s}_\delta) = R(x(\tau^1), y(\tau^1))(\psi(x(\tau^1 + \delta), y(\tau^1 + \delta)) - \psi(x(\tau^1 - \delta), y(\tau^1 - \delta))) + o(\delta).$$

The total expected profit from the offer curve $\mathbf{s}$ can be written as $v(\mathbf{s}) = v(\mathbf{s}^1_\delta) + v(\mathbf{s}_\delta) + v(\mathbf{s}^2_\delta)$, where $\mathbf{s}^1_\delta$, $\mathbf{s}^2_\delta$ are the other components of $\mathbf{s}$ created when $\mathbf{s}_\delta$ is removed. So $\mathbf{s}^1_\delta = \{(x(\tau), y(\tau)) : \tau \le \tau^1 - \delta\}$ and $\mathbf{s}^2_\delta = \{(x(\tau), y(\tau)) : \tau \ge \tau^1 + \delta\}$. These components lie entirely within the regions where ψ is continuously differentiable (and given by ψ^i), using Assumption 2.8. Using the result of Anderson and Philpott [2] we know that

$$v(\mathbf{s}_\delta^i) = \int_{\mathbf{s}_\delta^i} R(q,p)\ d\psi^i(q,p), \qquad i = 1, 2.$$

By driving δ to zero, we have

$$\begin{aligned}\lim_{\delta\to 0} v(\mathbf{s}_\delta) &= R(x(\tau^1), y(\tau^1))(\psi^2(x(\tau^1), y(\tau^1)) - \psi^1(x(\tau^1), y(\tau^1)))\\ &= R(q^1, p^1)\Phi(q^1, p^1),\end{aligned}$$

and

$$\lim_{\delta\to 0} \int_{\mathbf{s}_\delta^i} R(q,p)\ d\psi^i(q,p) = \int_{\mathbf{s}^i} R(q,p)\ d\psi^i(q,p), \qquad i = 1, 2.$$

This completes the proof. $\square$

2.5. Existence of an optimal solution. Having established the objective function formula (2.12), our approach to showing that an optimal solution exists is to concentrate on the formal problem of maximizing (2.12) given an R-semicontinuous market distribution function ψ.

In order to discuss the optimality of a continuous offer curve, we need to compare the line integrals on two distinct curves. When ψ is continuous, Anderson and Philpott [2] use Green's theorem and observe that

$$\int\int_{\mathcal{S}} Z(q,p)\ dpdq = \int_{\mathcal{C}} R(q,p)\ d\psi(q,p),$$

where $\mathcal{S}$ is a region enclosed by a curve $\mathcal{C}$ and

$$Z(q,p) = \begin{cases} R_q\psi_p - R_p\psi_q, & (q,p) \in \Psi, \\ 0 & \text{otherwise.}\end{cases} \tag{2.13}$$

Clearly this result will not hold when the curve $\mathcal{C}$ crosses one of the lines of discontinuity at $p \in \mathcal{P}$.

Our approach will be to calculate the change in v that arises from a change in offer curve $\mathbf{s}$ by applying Green's theorem separately to each region Ψ^j together with a calculation of the change that arises across the lines of discontinuity. We need to start with a lemma that can be established using an integration by parts argument.

LEMMA 2.10. *Suppose $p^j \in \mathcal{P}$ and $0 \le q_1 < q_2 \le q_M$. Then, under Assumption 2.8,*

$$\begin{aligned}&\int_{q_1}^{q_2} R(q,p^j)\ d\psi^j(q,p^j) - \int_{q_1}^{q_2} R(q,p^j)\ d\psi^{j+1}(q,p^j) + R(q_2,p^j)\Phi(q_2,p^j)\\ &\qquad\qquad -R(q_1,p^j)\Phi(q_1,p^j)\\ &= \int_{q_1}^{q_2} \Phi(q,p^j)R_q(q,p^j)\ dq.\end{aligned}$$

Proof. Let

$$v_1 = \int_{q_1}^{q_2} R(q,p^j)\ d\psi^j(q,p^j) + R(q_2,p^j)\Phi(q_2,p^j)$$

and

$$v_2 = \int_{q_1}^{q_2} R(q,p^j)\ d\psi^{j+1}(q,p^j) + R(q_1,p^j)\Phi(q_1,p^j).$$

From Assumption 2.8, both $\psi^j(\cdot,p^j)$ and $\psi^{j+1}(\cdot,p^j)$ are continuously differentiable, and we have already assumed that $R(\cdot,p^j)$ is continuously differentiable. Integrating both v_1 and v_2 by parts, we obtain

$$\begin{aligned} v_1 - v_2 &= R(q_2,p^j)\psi^j(q_2,p^j) - R(q_1,p^j)\psi^j(q_1,p^j) - \int_{q_1}^{q_2} \psi^j(q,p^j)R_q(q,p^j)\ dq \\ &\quad - R(q_2,p^j)\psi^{j+1}(q_2,p^j) + R(q_1,p^j)\psi^{j+1}(q_1,p^j) + \int_{q_1}^{q_2} \psi^{j+1}(q,p^j)R_q(q,p^j)\ dq \\ &\quad + R(q_2,p^j)\Phi(q_2,p^j) - R(q_1,p^j)\Phi(q_1,p^j) \\ &= \int_{q_1}^{q_2} \Phi(q,p^j)R_q(q,p^j)\ dq, \end{aligned}$$

as required. ◻

Note that v_1 is the expected return of the generator for offering $q_2 - q_1$ at price just under p^j, and v_2 is the expected return of the generator for offering $q_2 - q_1$ at price just above p^j. The lemma states that the difference between these two values can be expressed as the integral of $\Phi(q,p^j)R_q(q,p^j)$ with respect to q from q_1 to q_2. Since R_q is the marginal profit, $\Phi(q,p^j)R_q(q,p^j)$ represents the difference of the marginal profits between the offer of q at just above p^j and the offer of q at just below p^j.

Anderson and Philpott [2] and Anderson and Xu [4] treat $v(\mathbf{s})$ in (2.2) as an objective function and investigate the necessary and sufficient conditions for an offer curve $\mathbf{s}$ to be a local maximum. When ψ is continuously differentiable on Ψ, Anderson and Xu prove that there exists a maximum over the set of curves that are considered. However, the existence result is not straightforward when ψ is not continuous, and our first result is to confirm that a maximum does exist provided that ψ satisfies the conditions we have given.

A generator need not offer all its generation capacity into the market; the offer curve will start at some point $(0,\hat{p}(0))$ and finish at $(\hat{q}(T),\hat{p}(T))$. However, the clearing price is determined as though the offer curve began with a vertical segment from the origin to $(0,\hat{p}(0))$ and finished with a vertical segment from $(\hat{q}(T),\hat{p}(T))$ to $(\hat{q}(T),p_M)$. Hence we assume that Λ, the set of possible offer curves, has these characteristics. The following result has been established by Anderson and Xu [4].

LEMMA 2.11. *Let Λ be the set of monotonic continuous curves starting at the origin and ending on the closed line segment from $(0,p_M)$ to (q_M,p_M). Then Λ is compact under the Hausdorff metric:*

$$|\mathbf{s}_1 - \mathbf{s}_2|_H = \max_{(q_1,p_1)\in\mathbf{s}_1} \min_{(q_2,p_2)\in\mathbf{s}_2} \sqrt{(q_1-q_2)^2 + (p_1-p_2)^2}.$$

We need some sort of compactness result such as this to ensure the existence of an optimal solution; once compactness is established in some topology, then the existence result follows provided that we have a suitable continuity property in that topology. Our next result uses compactness to establish that the problem of finding a curve $\mathbf{s}$ which maximizes the expected profit $v(\mathbf{s})$ in (2.12) has an optimal solution. But before we prove this theorem we need to establish a preliminary lemma (which is required because $q^j(\mathbf{s})$ is not a continuous function of $\mathbf{s}$).

LEMMA 2.12. *If* $\mathbf{s}_k \to \mathbf{s}$ *in the Hausdorff metric and for some* j*:* $\lim_{k\to\infty} q^j(\mathbf{s}_k) = q_0$*, then*

$$\int_{q^j(\mathbf{s})}^{q_0} \Phi(q, p^j) R_q(q, p^j)\ dq \le 0. \tag{2.14}$$

Proof. Observe that if $q^j(\mathbf{s}) = q_0$ there is nothing to prove, so we suppose these two are unequal. Since $\mathbf{s}_k \to \mathbf{s}$ in the Hausdorff metric we can deduce that $\min_{(q,p)\in\mathbf{s}} \left((q_0 - q)^2 + (p^j - p)^2\right)^{1/2} = 0$ and hence that $(q_0, p^j) \in \mathbf{s}$. Thus, from monotonicity, all of the line interval $(q^j(\mathbf{s}), p^j)$ to (q_0, p^j) is in $\mathbf{s}$. First we suppose that $q^j(\mathbf{s}) < q_0$. Then, from the definition of q^j, this line interval lies in Ψ^{j+1} and hence is part of Ψ_-, where $R_q < 0$. Since $\Phi(q, p^j) \ge 0$, the inequality (2.14) follows. On the other hand, if $q^j(\mathbf{s}) > q_0$, then the line interval (q_0, p^j) to $(q^j(\mathbf{s}), p^j)$ lies in Ψ^j and hence is part of Ψ_+, where $R_q \ge 0$. Again, we have shown the desired inequality (2.14) after noting that the limits of the integral are reversed. ∎

THEOREM 2.13 (existence). *Let* Λ *be defined as above and let* v *be the expected return function given in* (2.12)*. Under Assumptions* 2.6 *and* 2.8*, if the market distribution function is R-semicontinuous, then* v *achieves its maximum on* Λ*.*

Proof. Let $v^* = \sup_{\mathbf{s}\in\Lambda} v(\mathbf{s})$, which exists since R is bounded and ψ lies between 0 and 1. For every $k > 0$, there exists a supply curve $\mathbf{s}_k \in \Lambda$ such that $v^* - v(\mathbf{s}_k) \le \frac{1}{k}$. Since Λ is a compact set, there exists $\mathbf{s}^* \in \Lambda$ such that $|\mathbf{s}_k - \mathbf{s}^*|_H \to 0$ (we can take a subsequence if necessary). In addition we shall arrange that for each j, $\lim_{k\to\infty} q^j(\mathbf{s}_k)$ exists. We want to prove that $v(\mathbf{s}^*) = v^*$. We will do this by showing that $v(\mathbf{s}_k) \to v(\mathbf{s}^*)$, using Green's theorem on each of the Ψ^j regions together with Lemma 2.10 for the crossovers from one Ψ^j to the next.

We define $\mathbf{s}^{*j} = \mathbf{s}^* \cap \Psi^j$ and $\mathbf{s}_k^j = \mathbf{s}_k \cap \Psi^j$. Thus

$$v(\mathbf{s}^*) = \sum_{j=1}^{n+1} \int_{\mathbf{s}^{*j}} R(q,p)\ d\psi^j(q,p) + \sum_{j=1}^{n} R(q^j(\mathbf{s}^*), p^j)\Phi(q^j(\mathbf{s}^*), p^j),$$
$$v(\mathbf{s}_k) = \sum_{j=1}^{n+1} \int_{\mathbf{s}_k^j} R(q,p)\ d\psi^j(q,p) + \sum_{j=1}^{n} R(q^j(\mathbf{s}_k), p^j)\Phi(q^j(\mathbf{s}_k), p^j).$$

Let $\mathrm{sign}(q,p)$ be a function such that $\mathrm{sign}(q,p) = 1$ if (q,p) is located below the curve $\mathbf{s}^*$; $\mathrm{sign}(q,p) = -1$ if (q,p) is located above the curve $\mathbf{s}^*$; and $\mathrm{sign}(q,p) = 0$ if (q,p) is located on the curve $\mathbf{s}^*$. Now, using Green's theorem

$$\begin{aligned}
&\int_{\mathbf{s}_k^j} R(q,p)\ d\psi^j(q,p) - \int_{\mathbf{s}^{*j}} R(q,p)\ d\psi^j(q,p) \\
&= \int\int_{A_k^j} \mathrm{sign}(q,p) Z(q,p)\ dq\ dp + \int_{q^j(\mathbf{s}^*)}^{q^j(\mathbf{s}_k)} R(q,p^j)\ d\psi^j(q,p^j) \\
&\quad - \int_{q^{j-1}(\mathbf{s}^*)}^{q^{j-1}(\mathbf{s}_k)} R(q,p^{j-1})\ d\psi^j(q,p^j),
\end{aligned}$$

where A_k^j is the area between $\mathbf{s}^{*j}$ and $\mathbf{s}_k^j$, and Z is given by (2.13). Let A_k be the entire area between $\mathbf{s}^*$ and $\mathbf{s}_k$. Then

$$v(\mathbf{s}_k) - v(\mathbf{s}^*) = \int\int_{A_k} \mathrm{sign}(q,p) Z(q,p)\ dq\ dp$$

$$+\sum_{j=1}^{n}\int_{q^j(\mathbf{s}^*)}^{q^j(\mathbf{s}_k)} R(q,p^j)\ d\psi^j(q,p^j) - \sum_{j=1}^{n}\int_{q^j(\mathbf{s}^*)}^{q^j(\mathbf{s}_k)} R(q,p^j)\ d\psi^{j+1}(q,p^j)$$
$$+\sum_{j=1}^{n} R(q^j(\mathbf{s}_k),p^j)\Phi(q^j(\mathbf{s}_k),p^j) - \sum_{j=1}^{n} R(q^j(\mathbf{s}^*),p^j)\Phi(q^j(\mathbf{s}^*),p^j)$$
$$= \int\int_{A_k} \operatorname{sign}(q,p)Z(q,p)\ dq\ dp + \sum_{j=1}^{n}\int_{q^j(\mathbf{s}^*)}^{q^j(\mathbf{s}_k)} \Phi(q,p^j)R_q(q,p^j)\ dq$$

using Lemma 2.10.

Since $|\mathbf{s}_k - \mathbf{s}^*|_H \to 0$ and Z is bounded from Assumption 2.6, the area integral approaches zero as $k \to \infty$. Also from Lemma 2.12, we know that

$$\lim_{k\to\infty}\int_{q^j(\mathbf{s}^*)}^{q^j(\mathbf{s}_k)} \Phi(q,p^j)R_q(q,p^j)\ dq \le 0.$$

Thus $v(\mathbf{s}^*) \ge \lim_{k\to\infty} v(\mathbf{s}_k) = v^*$, but from the definition of v^*, $v(\mathbf{s}^*) \le v^*$, and thus we have established the desired equality. ◻

Using this theorem and the results of Theorem 2.9 and Lemma 2.7, we have the immediate corollary.

COROLLARY 2.14. *Under Assumptions* 2.6 *and* 2.8*, if the sharing rule $\mathcal{L}^*$ is used, then there is an optimal supply curve.*

3. Necessary conditions for optimality. From Theorem 2.13, we know that the problem of maximizing profit using the sharing rule $\mathcal{L}^*$ is well defined. This is equivalent to the following maximization problem:

$$\max_{\mathbf{s}\in\Lambda}\ v(\mathbf{s}) \equiv \sum_{j=1}^{n+1}\int_{\mathbf{s}^j} R(q,p)\ d\psi^j(q,p) + \sum_{j=1}^{n} R(q^j(\mathbf{s}),p^j)\Phi(q^j(\mathbf{s}),p^j). \tag{3.1}$$

In this section, we discuss necessary conditions for an offer curve to be optimal for this problem.

When ψ is continuously differentiable, optimality conditions were derived by Anderson and Philpott [2] and extended by Anderson and Xu [4]. Let $\mathbf{s} = \{(\hat{q}(\tau),\hat{p}(\tau)) : 0 \le \tau \le T\}$ be the offer curve. The main tool that is used in investigating the optimality conditions of $\mathbf{s}$ is the line integral of Z along $\mathbf{s}$, which is defined by

$$w(\tau) = \int_0^{\tau} Z(\hat{q}(t),\hat{p}(t))(\hat{q}'(t)+\hat{p}'(t))\ dt.$$

When ψ is not continuously differentiable, we need to use a different approach.

We take ψ as R-semicontinuous and we define, for $(q,p) \in \Psi^j$, the function $Z^j(q,p) = R_q\psi_p^j - R_p\psi_q^j$. This will make Z^j match Z in the interior of Ψ^j and be defined by continuity for points in Ψ^j that lie on its boundary.

Given a monotonic continuous, piecewise smooth offer curve $\mathbf{s} = \{(\hat{q}(\tau),\hat{p}(\tau)), 0 \le \tau \le T\}$, for each τ we let $J(\tau)$ be the index of the region Ψ^j in which $(\hat{q}(\tau),\hat{p}(\tau))$ lies and we let τ^j be the parameter value at which the curve moves from Ψ^j to Ψ^{j+1}, and thus $\hat{q}(\tau^j) = q^j$. Then we define

$$w(\tau) = \int_0^{\tau} Z^{J(\tau)}(\hat{q}(t),\hat{p}(t))(\hat{q}'(t)+\hat{p}'(t))\ dt + \sum_{j=1}^{J(\tau)-1} \Phi(q^j,p^j)R_q(q^j,p^j).$$

THEOREM 3.1 (first order necessary conditions). *Suppose that* $\mathbf{s} = \{\hat{q}(\tau), \hat{p}(\tau), 0 \leq \tau \leq T\}$ *is an offer curve and Assumptions* 2.6 *and* 2.8 *are satisfied. Suppose that there exist* m *numbers* $0 \leq \tau_1 < \tau_2 < \cdots < \tau_m \leq T$ *with* $0 < \hat{q}(\tau) < q_M$ *and* $0 < \hat{p}(\tau) < p_M$ *for* $\tau_1 < \tau < \tau_m$. *Suppose further that on each section* (τ_{i-1}, τ_i), $i = 2, \ldots, m$, $\mathbf{s}$ *is either strictly increasing in both components, or horizontal, or vertical, with different characteristics in successive segments and with* $\tau_1 (\tau_m)$ *the smallest (largest) parameter value such that* $(\hat{q}(\tau), \hat{p}(\tau)) \in \Psi$. *If* $\mathbf{s}$ *is optimal for* (3.1), *then* $w(\tau_1) = 0$ *and* $w(\tau_m) = w(T)$. *Moreover, for each interval* I *being one of* (τ_{i-1}, τ_i), $i = 2, \ldots, m$, *one of the following holds:*

(i) $\mathbf{s}$ *is strictly increasing in both components and* $w(\tau) = w(\tau_{i-1})$ *for* $\tau \in I$.

(ii) $\mathbf{s}$ *is horizontal on* I. *For* $\tau \in I$ *with* $(\hat{q}(\tau), \hat{p}(\tau)) \in \Psi_+$, *then* $w(\tau) \leq w(\tau_{i-1})$; *for* $\tau \in I$ *with* $(\hat{q}(\tau), \hat{p}(\tau)) \in \Psi_-$, *then* $w(\tau) \leq w(\tau_i)$. *Moreover, if* $\hat{p}(\tau_i) \notin \mathcal{P}$, *then* $w(\tau_{i-1}) = w(\tau_i)$.

(iii) $\mathbf{s}$ *is vertical on* I, $w(\tau_{i-1}) = w(\tau_i)$, *and* $w(\tau) \geq w(\tau_i)$ *for* $\tau \in I$.

Proof. We begin by looking at the w values at τ_1 and τ_m. First, we prove $w(\tau_1) = 0$ (the proof that $w(\tau_m) = w(T)$ is similar). By assumption, for any $\tau < \tau_1$, $(\hat{q}(\tau), \hat{p}(\tau))$ is located outside the Ψ region where Z and $w(\tau)$ are zero. Note that if $\hat{p}(\tau_1) \notin \mathcal{P}$, then w is continuous at τ_1 and $w(\tau_1) = 0$. Thus we only need to consider the case that $\hat{p}(\tau_1) = p^j \in \mathcal{P}$. This means that the lower boundary of Ψ contains a horizontal section $p = p^j$ and the point $(\hat{q}(\tau_1), \hat{p}(\tau_1))$ is located on the horizontal section. We consider three cases according to whether the point $(\hat{q}(\tau_1), \hat{p}(\tau_1))$ is located in Ψ_+, in Ψ_-, or on the line separating these regions. In the latter case $R_q(\hat{q}(\tau_1), \hat{p}(\tau_1)) = 0$, and hence $w(\tau_1) = 0$. If $(\hat{q}(\tau_1), \hat{p}(\tau_1))$ is in Ψ_+, then $J(\tau_1) = j - 1$ and $w(\tau_1) = 0$ by the definition of the w function. Thus we are left with the case when $(\hat{q}(\tau_1), \hat{p}(\tau_1))$ is in Ψ_-, when $J(\tau_1) = j$. By definition, since Z is zero outside Ψ,

$$w(\tau_1) = R_q(\hat{q}(\tau_1), p^j)\Phi(\hat{q}(\tau_1), p^j) \leq 0.$$

Suppose for a contradiction that $w(\tau_1) < 0$. Since $\Phi(q, p^j) = \psi(q, p^j_+)$ for all q with (q, p_j) at the boundary, this implies that $\psi(\hat{q}(\tau_1), p^j_+) > 0$. By Assumption 2.8, $\psi(q, p^j_+)$ is continuous in q, and so there exists $\delta > 0$ such that $\psi(\hat{q}(\tau_1) - \delta, p^j_+) > 0$. Consider another supply curve $\mathbf{r}$ which enters Ψ at a point $(\hat{q}(\tau_1) - \delta, p^j_+)$ and then goes horizontally until it reaches the point $(\hat{q}(\tau_1), p^j_+)$ and then joins $\mathbf{s}$ to the end. Using Lemma 2.10, it is easy to verify the difference between the expected profits of the two supply curves,

$$E(\mathbf{s}) - E(\mathbf{r}) = \int_{\hat{q}(\tau_1)-\delta}^{\hat{q}(\tau_1)} R_q(x, p^j)\Phi(x, p^j)dx = \delta w(\tau_1) + o(\delta) < 0,$$

for δ sufficiently small. This contradicts the optimality of $\mathbf{s}$ and establishes $w(\tau_1) = 0$.

Part (i). This part of the theorem amounts to the statement that if $\hat{q}(\tau)$ and $\hat{p}(\tau)$ are both increasing in an interval $\tau \in (\tau_A, \tau_B)$ and we choose a point $(\hat{q}, \hat{p}) = (\hat{q}(\tau^*), \hat{p}(\tau^*))$ in this interval, then $Z(\hat{q}, \hat{p}) = 0$ if $(\hat{q}, \hat{p})$ is in the interior of a Ψ^j, and $\Phi(\hat{q}, \hat{p})R_q(\hat{q}, \hat{p}) = 0$ if $\hat{p} \in \mathcal{P}$. The first statement is proved in Anderson and Philpott [2], but for convenience we will repeat their argument here. We begin by defining a small perturbation of $\mathbf{s}$ around the point $(\hat{q}(\tau^*), \hat{p}(\tau^*))$. Reparameterizing $\mathbf{s}$ if necessary, we can assume that $\hat{q}'(\tau^*) > 0$. Let

$$\mathbf{r}_\delta(\tau) = \begin{cases} (\hat{q}(2\tau - (\tau^* - \delta)), \hat{p}(\tau^* - \delta)), & \tau^* - \delta \leq \tau \leq \tau^*, \\ (\hat{q}(\tau^* + \delta), \hat{p}(2\tau - (\tau^* + \delta)), & \tau^* \leq \tau \leq \tau^* + \delta, \\ (\hat{q}(\tau), \hat{p}(\tau)), & \text{otherwise.} \end{cases}$$

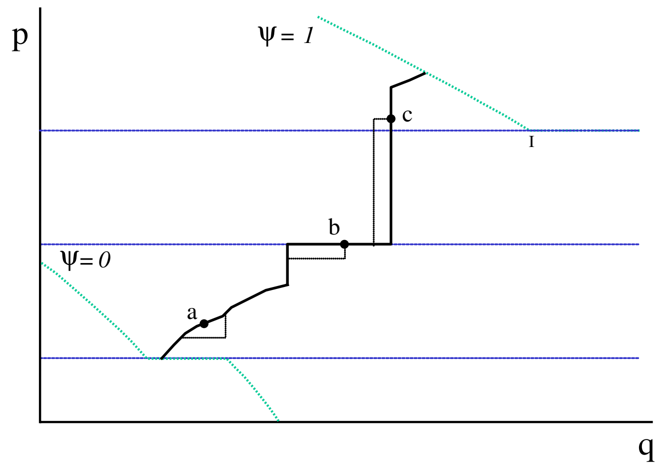

FIG. 1. *Perturbations for first order optimality.*

This perturbation is illustrated in Figure 1 at the point marked a.

In the case that $\hat{p}(\tau^*) \notin \mathcal{P}$ we can avoid any discontinuities within the perturbation by taking δ small enough. In this case

$$v(\mathbf{r}_\delta) - v(\mathbf{s}) = \int\int_{A(\delta)} Z(q,p)\ dqdp,$$

where $A(\delta)$ is the region between the two curves. Since $\mathbf{s}$ is optimal and Z is continuous in this region we obtain the conclusion that $Z(\hat{q}(\tau^*), \hat{p}(\tau^*)) \leq 0$, since otherwise we have $v(\mathbf{r}_\delta) > v(\mathbf{s})$ for δ small enough. If we reverse the direction of the perturbation (going above $(\hat{q}(\tau^*), \hat{p}(\tau^*))$ rather than below it) we can show that $Z(\hat{q}(\tau^*), \hat{p}(\tau^*)) \geq 0$. The two inequalities show that $Z = 0$ as required.

Now suppose that $\hat{p}(\tau^*) = p^j \in \mathcal{P}$. Using Lemma 2.10 and the usual Green's theorem argument we have

$$\begin{aligned} v(\mathbf{r}_\delta) - v(\mathbf{s}) = &\int\int_{A^j(\delta)} Z(q,p)\ dqdp + \int\int_{A^{j+1}(\delta)} Z(q,p)\ dqdp \\ &+ \int_{\hat{q}(\tau^*)}^{\hat{q}(\tau^*+\delta)} \Phi(q,p^j) R_q(q,p^j)\ dq, \end{aligned}$$

where $A^i(\delta) = A(\delta) \cap \Psi^i$. But we have just shown that $Z(q,p) = 0$ along the $\mathbf{s}$ curve (except where $\mathbf{s}$ crosses the p^j line). The continuity of Z implies that both the first two integrals are $o(\delta)$. Thus the continuity of Φ and R_q will imply that

$$v(\mathbf{r}_\delta) - v(\mathbf{s}) = (\hat{q}(\tau^* + \delta) - \hat{q}(\tau^*))\Phi(\hat{q}(\tau^*), p^j) R_q(\hat{q}(\tau^*), p^j) + o(\delta).$$

Since $\hat{q}'(\tau^*) > 0$, $\hat{q}(\tau^*+\delta) - \hat{q}(\tau^*)$ is $O(\delta)$, and thus $\Phi(\hat{q}(\tau^*), p^j) R_q(\hat{q}(\tau^*), p^j) \leq 0$ from the optimality of $\mathbf{s}$. Again reversing the perturbation shows $\Phi(\hat{q}(\tau^*), p^j) R_q(\hat{q}(\tau^*), p^j) \geq 0$, and thus

$$\Phi(\hat{q}(\tau^*), p^j) R_q(\hat{q}(\tau^*), p^j) = 0,$$

as required.

Part (ii). As before we consider a point with parameter $\tau^* \in (\tau_{i-1}, \tau_i)$, a horizontal section. We will need to use two different types of perturbation. Suppose first that $(\hat{q}(\tau_{i-1}), \hat{p}(\tau_{i-1})) \in \Psi_+$. There are two possibilities: $\mathbf{s}$ is vertical immediately before τ_{i-1} and $\mathbf{s}$ is strictly increasing immediately before τ_{i-1}. We only consider the first case in detail.

Let $\delta > 0$ be small and define $\tau_{i-1}(-\delta) = \hat{p}^{-1}(\hat{p}(\tau_{i-1}) - \delta)$ so that this is the parameter value at which $\mathbf{s}$ reaches a price level δ below $\hat{p}(\tau_{i-1})$. Let $\mathbf{r}_\delta$ be the perturbation of $\mathbf{s}$ which moves a horizontal section from $\hat{q}(\tau_{i-1})$ to $\hat{q}(\tau^*)$ down by an amount δ. Thus

$$\mathbf{r}_\delta(\tau) = \begin{cases} (\hat{q}(\tau), \hat{p}(\tau)), & 0 \le \tau < \tau_{i-1}(-\delta), \\ (\hat{q}(\tau_{i-1} - \tau_{i-1}(-\delta) + \tau), \hat{p}(\tau_{i-1}) - \delta), & \tau_{i-1}(-\delta) \le \tau < \tau^* - \tau_{i-1} + \tau_{i-1}(-\delta), \\ (\hat{q}(\tau^*), \hat{p}(\tau + \tau_{i-1} - \tau^*)), & \tau^* - \tau_{i-1} + \tau_{i-1}(-\delta) \le \tau < \tau^*, \\ (\hat{q}(\tau), \hat{p}(\tau)), & \tau^* \le \tau \le T. \end{cases}$$

This perturbation is illustrated in Figure 1, with $(\hat{q}(\tau^*), \hat{p}(\tau^*))$ shown as the point marked b. If $(\hat{q}(\tau^*), \hat{p}(\tau^*)) \in \Psi_+$, then this perturbation does not involve a discontinuity in Z, and thus

$$\begin{aligned} v(\mathbf{r}_\delta) - v(\mathbf{s}) &= \int_{\hat{p}(\tau_{i-1})-\delta}^{\hat{p}(\tau_{i-1})} \int_{\hat{q}(\tau_{i-1})}^{\hat{q}(\tau^*)} Z(q,p)\; dqdp \\ &= \delta \int_{\hat{q}(\tau_{i-1})}^{\hat{q}(\tau^*)} Z(q, \hat{p}(\tau_{i-1}))\; dq + o(\delta). \end{aligned}$$

Since $\mathbf{s}$ is optimal, we obtain the conclusion that $\int_{\hat{q}(\tau_{i-1})}^{\hat{q}(\tau^*)} Z(q, \hat{p}(\tau_{i-1}))dq \le 0$, since otherwise we have $v(\mathbf{r}_\delta) > v(\mathbf{s})$ for δ small enough. Thus $w(\tau^*) \le w(\tau_{i-1})$. In the case that $\mathbf{s}$ is strictly increasing immediately before τ_{i-1}, we need to make a slightly more complex definition for $\mathbf{r}_\delta(\tau)$ and the area over which Z is integrated is no longer rectangular, but the basic argument is the same.

Now if $(\hat{q}(\tau^*), \hat{p}(\tau^*)) \in \Psi_-$, then $(\hat{q}(\tau_i), \hat{p}(\tau_i)) \in \Psi_-$ and we choose a perturbation that moves a horizontal section of $\mathbf{s}$ from $\hat{q}(\tau^*)$ to $\hat{q}(\tau_i)$ upward by an amount δ. If $(\hat{q}(\tau^*), \hat{p}(\tau^*)) \in \Psi_-$, then the continuity of Z for this perturbation implies that $\int_{\hat{q}(\tau^*)}^{\hat{q}(\tau_i)} Z(q, \hat{p}(\tau_i))dq \ge 0$ and hence $w(\tau^*) \le w(\tau_i)$.

When $\hat{p}(\tau_i) \notin \mathcal{P}$ we have continuity for Z without having to restrict ourselves to $(\hat{q}(\tau^*), \hat{p}(\tau^*)) \in \Psi_+$ for a perturbation downwards at the beginning of the horizontal section, or $(\hat{q}(\tau^*), \hat{p}(\tau^*)) \in \Psi_-$ for a perturbation upwards at the end of the horizontal section. Hence we can take $\tau^* = \tau_i$ for the first argument and $\tau^* = \tau_{i-1}$ for the second argument to show that both the inequalities $w(\tau_i) \le w(\tau_{i-1})$ and $w(\tau_{i-1}) \le w(\tau_i)$ hold, and hence that there is equality.

Part (iii). Suppose $\mathbf{s}$ is vertical on the interval between τ_{i-1} and τ_i. We establish the result using perturbations of either end of the interval. We begin with a perturbation that moves the lower part of the interval to the left. There are two possibilities: $\mathbf{s}$ is horizontal immediately before τ_{i-1} or $\mathbf{s}$ is strictly increasing immediately before τ_{i-1}. The first case makes it slightly simpler to give an explicit perturbation, and we restrict ourselves to this.

Let $\delta > 0$ be small and define $\tau_{i-1}(-\delta) = \hat{q}^{-1}(\hat{q}(\tau_{i-1}) - \delta)$ so that this is the parameter value at which $\mathbf{s}$ reaches a quantity $\hat{q}(\tau_{i-1}) - \delta$. Let $\mathbf{r}_\delta$ be the perturbation of $\mathbf{s}$ which moves a vertical section from $\hat{p}(\tau_{i-1})$ to $\hat{p}(\tau^*)$ to the left by an amount δ;

thus:

$$\mathbf{r}_\delta(\tau) = \begin{cases} (\hat{q}(\tau), \hat{p}(\tau)), & 0 \le t < \tau_{i-1}(-\delta), \\ (\hat{q}(\tau_{i-1}) - \delta), \hat{p}(\tau_{i-1} - \tau_{i-1}(-\delta) + t), & \tau_{i-1}(-\delta) \le t < \tau^* - \tau_{i-1} + \tau_{i-1}(-\delta), \\ (\hat{q}(t + \tau_{i-1} - \tau^*), \hat{p}(\tau^*)), & \tau^* - \tau_{i-1} + \tau_{i-1}(-\delta) \le t < \tau^*, \\ (\hat{q}(\tau), \hat{p}(\tau)), & \tau^* \le t \le T. \end{cases}$$

This perturbation is illustrated in Figure 1, with $(\hat{q}(\tau^*), \hat{p}(\tau^*))$ shown as the point marked c. In general this perturbation may involve a number of different regions Ψ^j. Suppose that $(\hat{q}(\tau_{i-1}), \hat{p}(\tau_{i-1})) \in \Psi^f$ and $(\hat{q}(\tau^*), \hat{p}(\tau^*)) \in \Psi^g$. Write $A(\delta)$ for the region between the two curves, $\mathbf{s}$ and $\mathbf{r}_\delta$, and $A^j(\delta) = A(\delta) \cap \Psi^j$. Then

$$\begin{aligned} v(\mathbf{s}) - v(\mathbf{r}_\delta) &= \sum_{j=f}^{g} \int\int_{A^j(\delta)} Z(q,p)\ dqdp + \sum_{j=f}^{g-1} \int_{\hat{q}(\tau_{i-1})-\delta}^{\hat{q}(\tau_{i-1})} \Phi(q, p^j) R_q(q, p^j)\ dq \\ &= \delta \int_{\hat{p}(\tau_{i-1})}^{p^f} Z(\hat{q}(\tau_{i-1}), p)\ dp + \delta \sum_{j=f+1}^{g-1} \int_{p^{j-1}}^{p^j} Z(\hat{q}(\tau_{i-1}), p)\ dp \\ &\quad + \delta \int_{p^{g-1}}^{\hat{p}(\tau^*)} Z(\hat{q}(\tau_{i-1}), p)\ dp + \delta \sum_{j=f}^{g-1} \Phi(\hat{q}(\tau_{i-1}), p^j) R_q(\hat{q}(\tau_{i-1}), p^j) + o(\delta) \\ &= \delta(w(\tau^*) - w(\tau_{i-1})) + o(\delta). \end{aligned}$$

As $\mathbf{s}$ is optimal we obtain the conclusion that $w(\tau^*) \ge w(\tau_{i-1})$, since otherwise we have $v(\mathbf{r}_\delta) > v(\mathbf{s})$ for δ small enough.

The other perturbation to be considered involves a section of the vertical segment from τ^* to τ_i, which moves to the right. The argument in this case is exactly the same and we can show that $w(\tau^*) \ge w(\tau_i)$. Moreover, since these results also apply with $\tau^* = \tau_{i-1}$ and with $\tau^* = \tau_i$, we see that $w(\tau_i) = w(\tau_{i-1})$. ∎

In many cases there will only be a single solution which satisfies the necessary conditions, and hence the optimal solution can be identified without further computation. Later we will illustrate the application of these conditions on an example, but first it is helpful to give some more general discussion.

In practice, the nature of the optimal solution will be quite dependent on the form of the $Z = 0$ curve. If, as is usually the case, this is a monotonic increasing curve, then the optimal solution will typically follow it for much of its length, with some small variations introduced by the discontinuities. We see this behavior in the example we consider in the next section.

On a vertical section of the offer curve we must have w values greater than at the end points of the section. In the case when neither of the end points is on a horizontal price discontinuity, then this will imply that the beginning (bottom) of the vertical section is in a region where $Z > 0$ and the end of the section is in a region where $Z < 0$. If the bottom end point, say $(\hat{q}(\tau_{i-1}), \hat{p}(\tau_{i-1}))$, lies on a horizontal price discontinuity, say p^j, then we need to consider two cases. First suppose that $(\hat{q}(\tau_{i-1}), \hat{p}(\tau_{i-1})) \in \Psi_-$, which in turn implies $(\hat{q}(\tau_{i-1}), \hat{p}(\tau_{i-1})) \in \Psi^{j+1}$. Then $J(\tau_{i-1}) = j + 1$ and $w(\tau_{i-1})$ already incorporates the jump at this discontinuity; thus the vertical section must begin in a region where $Z > 0$ to avoid contradicting the necessary conditions. The other case occurs when $(\hat{q}(\tau_{i-1}), \hat{p}(\tau_{i-1})) \in \Psi_+$, in which case $R_q \ge 0$ and the jump in value is positive. In this case, we can draw no immediate conclusion on the sign of Z at the start of the vertical section. The same kind of argument shows that if the top end of a vertical section is at a price discontinuity, then when this point is in Ψ_+

we can conclude that the vertical section finishes in a region where $Z < 0$, and no conclusion can be drawn when the point is in Ψ_-.

Now consider a horizontal section that does not coincide with a price discontinuity. In this case the condition of the theorem simply says that $w(\tau)$ is less than the w values at both end points. So the left-hand end of the horizontal section must be in a region where $Z < 0$ and the right-hand end in a region where $Z > 0$.

When the horizontal section runs along a price discontinuity the situation is a little more complex. Suppose first that the left-hand end of the horizontal section is in Ψ_+. Then the necessary conditions for optimality imply that w is decreasing and hence that the beginning of the horizontal section is in a region where $Z \le 0$. In the same way, if the right-hand end of the horizontal section is in Ψ_-, then we can deduce that this end point is in a region where $Z \ge 0$.

4. Construction of an approximate optimal supply function through undercutting and overcutting. We have found a form of sharing rule for which there will be an optimal solution. However, this form of sharing rule will not occur in practice. Our eventual aim is to have a way of generating ε-optimal solutions for problems with arbitrary sharing rules.

We suppose that we have found an optimal solution $\mathbf{s}^*$ for the modified problem with sharing rule $\mathcal{L}^*$. The next step is to create an ε-optimal solution for the problem using undercutting and overcutting. We define the solution $\mathbf{s}^*(\delta)$ for any $\delta > 0$ by following $\mathbf{s}^*$ except at the prices in $\mathcal{P}$. In essence, where $\mathbf{s}^* = p^j$ and lies in Ψ_+, we undercut the solution and set $\mathbf{s}^*(\delta) = \mathbf{s}^* - \delta$. Where $\mathbf{s}^* = p^j$ and lies in Ψ_-, we overcut the solution and set $\mathbf{s}^*(\delta) = \mathbf{s}^* + \delta$. To make this definition more precise involves some messy technical details, which we will give below. We shall assume that the equation $R_q(q,p) = 0$ defines a single monotonically increasing line which divides Ψ_+ and Ψ_-, and we write this in the form $p = \Gamma(q)$. In the case in which R is given by (2.1), Γ is just C'. If $\mathbf{s}^*$ moves from the region Ψ_+ to the region Ψ_- at a single value p^j, then $\mathbf{s}^*(\delta)$ follows the line $p = \Gamma(q)$ to join the undercutting section to the overcutting one.

We need to go back to the individual component functions $\hat{q}(\tau)$ and $\hat{p}(\tau)$, say, that define $\mathbf{s}^*$. We let $q_\delta(\tau) = \hat{q}(\tau)$ for every τ. We have the following definitions for $p_\delta(\tau)$:

$$p_\delta(\tau) = \begin{cases} p^j - \delta & \text{for } \{\tau : p^j - \delta \le \hat{p}(\tau) \le p^j, (\hat{q}(\tau), p^j - \delta) \in \Psi_+\}, \\ p^j + \delta & \text{for } \{\tau : p^j \le \hat{p}(\tau) \le p^j + \delta, (\hat{q}(\tau), p^j + \delta) \in \Psi_-\}, \\ \Gamma(\hat{q}(\tau)) & \text{for } \{\tau : p^j - \delta \le \hat{p}(\tau) \le p^j, (\hat{q}(\tau), p^j - \delta) \notin \Psi_+, (\hat{q}(\tau), p^j) \in \Psi_+\}, \\ \Gamma(\hat{q}(\tau)) & \text{for } \{\tau : p^j \le \hat{p}(\tau) \le p^j + \delta, (\hat{q}(\tau), p^j + \delta) \notin \Psi_-, (\hat{q}(\tau), p^j) \in \Psi_-\}, \\ \hat{p}(\tau) & \text{otherwise.} \end{cases}$$

As it stands this defines $p_\delta(\tau)$ in such a way that it may not be continuous. We need to make the definition of p_δ continuous by filling in these (vertical) gaps. Suppose that

$$\hat{p}(\tau_{0-}) = \lim_{\tau \uparrow \tau_0} \hat{p}(\tau) = \lim_{\tau \downarrow \tau_0} \hat{p}(\tau) - \eta$$

for some $\eta > 0$. Then we define

$$(\widetilde{p}_\delta(\tau), \widetilde{q}_\delta(\tau)) = \begin{cases} (p_\delta(\tau), q_\delta(\tau)) & \text{for } \tau < \tau_0, \\ (p_\delta(\tau_{0-}) + \tau - \tau_0, q_\delta(\tau_0)) & \text{for } \tau_0 \le \tau \le \tau_0 + \eta, \\ (p_\delta(\tau - \gamma), q_\delta(\tau - \gamma)) & \text{for } \tau > \tau_0 + \eta. \end{cases}$$

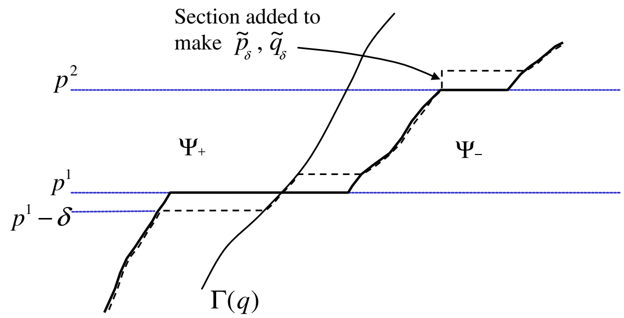

FIG. 2. *Construction of* $\mathbf{s}^*(\delta)$.

This removes one of the discontinuities, and we can continue in the same way to remove each of the other discontinuities (at most one of which is introduced at each p^j). Then $\mathbf{s}^*(\delta)$ is defined by $(\widetilde{p}_\delta(\tau), \widetilde{q}_\delta(\tau))$. Figure 2 illustrates this construction.

Our key result is that $\mathbf{s}^*(\delta)$ is ε-optimal for small enough δ. We are now in a position to prove this.

THEOREM 4.1. *Suppose that* $\mathbf{s}^*$ *maximizes the expected profits of the generator when the sharing rule* $\mathcal{L}^*$ *is used. Then*

$$\lim_{\delta\downarrow 0} u(\mathbf{s}^*(\delta)) = \sup_{\mathbf{s}\in\Lambda} u(\mathbf{s}),$$

where $u(\mathbf{r})$ *denotes the expected profits of the generator given a supply curve* $\mathbf{r}$ *with some other sharing rule* $\mathcal{L}$ *.*

Proof. We write $v(\mathbf{r})$ for the expected profit of the generator given a supply curve $\mathbf{r}$ with the ideal sharing rule $\mathcal{L}^*$. From Lemma 2.4 we know that $v(\mathbf{s}) \geq u(\mathbf{s})$. Thus

$$v(\mathbf{s}^*) \geq \sup_{\mathbf{s}\in\Lambda} u(\mathbf{s}).$$

Moreover, as each $\mathbf{s}^*(\delta) \in \Lambda$, $\lim_{\delta\downarrow 0} u(\mathbf{s}^*(\delta)) \leq \sup_{\mathbf{s}\in\Lambda} u(\mathbf{s})$. So it is enough to show $\lim_{\delta\downarrow 0} u(\mathbf{s}^*(\delta)) = v(\mathbf{s}^*)$.

Now it is not hard to see that $q^j(\mathbf{s}^*(\delta)) = q^j(\mathbf{s}^*)$ for each j. This is a result of the construction we have used for $\mathbf{s}^*(\delta)$. Thus

$$v(\mathbf{s}^*) - v(\mathbf{s}^*(\delta)) = \sum_{j=1}^{n+1} \left(\int_{\mathbf{s}^{*j}} R(q,p)\ d\psi^j(q,p) - \int_{\mathbf{s}^*(\delta)^j} R(q,p)\ d\psi^j(q,p) \right).$$

Since the end points of the segments $\mathbf{s}^{*j}$ and $\mathbf{s}^*(\delta)^j$ coincide within each Ψ^j, we can use Green's theorem within this region to show that the difference between the integrals tends to zero as $\delta \to 0$. So $v(\mathbf{s}^*) = \lim_{\delta\to 0} v(\mathbf{s}^*(\delta))$. But as $\mathbf{s}^*(\delta)$ does not contain a tranche offered at any p^j the sharing rule used will not affect the expected profit, and hence $v(\mathbf{s}^*(\delta)) = u(\mathbf{s}^*(\delta))$ for each δ. ∎

5. An example. In order to illustrate the ideas we have discussed above we return to the example we considered before. We now suppose that the market demand is given by $D(p) + \epsilon$ where $D(p) = 800 - \frac{10}{3}p^2$, and the random shock ϵ ranges uniformly over $[0, 2300]$. We wish to find the optimal offer curve for generator A. For $p \neq 10, 14, 18$, we can derive the market distribution function for generator A:

$$\psi(q,p) = \frac{1}{2300}(q - D(p) + S_B(p))$$

and

$$\Psi = \left\{ (q,p) : 0 < \frac{1}{2300}(q - D(p) + S_B(p)) < 1 \right\}.$$

The values of ψ at $p = 10, 14, 18$ will depend on the market sharing rules, which we do not need to specify.

We need to specify a contract position and the cost of generation. We suppose that generator A has contracts for a total quantity of 800 MW at a strike price of \$15 per MWh (that is, $Q = 800$ MW), and we take the total capacity for the generators to offer into the market as 1100 MW. We also take the costs generator A incurs for generating an amount q MWh as nonlinear and given by $C(q) = 10q + 0.004q^2$. Thus the profit function (in \$ per hour) is

$$R(q,p) = qp - 10q - 0.004q^2 + 800(15 - p).$$

We wish to find an optimal supply curve for generator A so that its expected profit is maximized. However, since ψ here is discontinuous, an optimal supply curve may not exist.

Note that

$$R_q(q,p) = p - 10 - 0.008q.$$

Thus $q^*(p^j) = 125(p^j - 10)$, and

$$\Psi_+ = \{(q,p) \in \Psi : q \le 125(p - 10)\},$$
$$\Psi_- = \{(q,p) \in \Psi : q > 125(p - 10)\}.$$

The optimal sharing rule $\mathcal{L}^*$ is defined according to the rules set out earlier. Essentially, the aim of the sharing rule is to obtain a dispatch of $125(p^j - 10)$ for generator A at each of the prices $p^j = 10, 14, 18$ (or as close to this figure as possible).

In what follows, we derive the optimal supply curve, assuming the sharing rule $\mathcal{L}^*$, using Theorem 3.1. Note that for $p \ne 10, 14$, or 18,

$$Z(q,p) = R_q\psi_p - R_p\psi_q = \frac{1}{2300}\left[(p - 10 - 0.008q)\left(\frac{20}{3}p\right) - q + 800\right].$$

In Figure 3 we show the upper and lower boundaries of the region Ψ (i.e., where $\psi = 0$ and $\psi = 1$) together with the curve (in fact a parabola) where $Z = 0$ and the straight line $p = C'(q) = 10 + 0.008q$.

We will try to identify an optimal offer curve which satisfies the first order necessary conditions that are derived in Theorem 3.1. To do this we consider tracing a curve starting at the lower boundary $\psi = 0$ and finishing at $\psi = 1$. According to part (i) of Theorem 3.1, the optimal offer curve must follow the $Z = 0$ line at any point where it is neither horizontal nor vertical, and it is natural to start by considering a solution which follows this line. The argument given below shows that, for this example, there will be only one solution that satisfies the necessary optimality conditions. This is often the case for this type of problem (but not always). If there is more than one solution satisfying the optimality conditions, then the expected profits for the different offer curves need to be compared directly in order to find a global optimum.

Observe, though, that a solution which follows the line $Z = 0$ from the point $B = (507.2, 9.372)$ where it crosses $\psi = 0$ through the point D cannot be optimal.

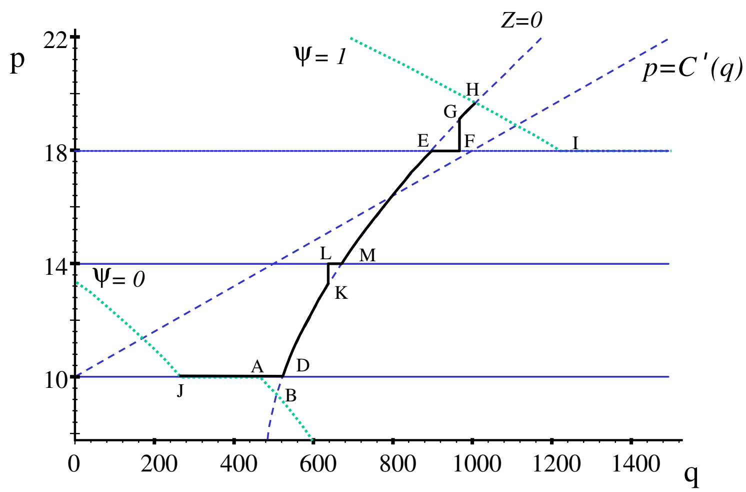

FIG. 3. *An optimal solution for the example.*

The line $Z = 0$ crosses $p = 10$ at $D = (521.7, 10)$, and at this point there is a discontinuity in w which would contradict the necessary condition (i) of the theorem. In fact, $\Phi(521.7, 10)R_q(521.7, 10) = -0.363$.

Thus we need to consider the possibility of a vertical segment finishing at the $p = 10$ line. In order to satisfy the conditions of the theorem, the integral of Z along the vertical section would have to exactly match the jump down that occurs at $p = 10$. For convenience we define

$$W_1(q, a, b) = \int_a^b Z(q, p)dp + \Phi(q, b)R_q(q, b)$$

to be the w integral over a vertical segment at q starting at $p = a$ and finishing at $p = b$ where there is supposed to be a discontinuity. Thus in this case we are interested in finding a starting point (q, a) for which $W_1(q, a, 10) = 0$. The possibilities here are to begin by following the line $Z = 0$ from the point B, but to start a vertical segment before reaching D, or to start from some point between A and B with a vertical section. However, it is not hard to check that in all cases $W_1(q, a, 10)$ will be negative.

Therefore we next consider a horizontal segment starting from some point in JA. This is in the region Ψ_-, and thus the necessary conditions will just imply that $w(\tau)$ is less than the w value at the next corner point. This condition will be satisfied since Z is positive in this region. However, this same condition will ensure that this horizontal section of the optimal offer curve does not go beyond D where Z changes sign. In fact, the necessary conditions imply that the horizontal segment starts at $J = (266.6, 10)$.

Now we consider the possibility of a vertical section which finishes on the horizontal line $p = 14$. Again, using the necessary conditions will imply that we should start this vertical segment at a point (q, a) where $W_1(q, a, 14) = 0$. This equation defines a curve which crosses the $Z = 0$ line at the point $K = (643.7, 13.4)$. Again we can establish that the horizontal section must run from $L = (643.7, 14)$ to the point $M = (671.8, 14)$ on the $Z = 0$ line and no further.

The solution then has to follow the $Z = 0$ line until the point $E = (898.0, 18)$. At this point it starts a horizontal segment. Since the solution is now in the Ψ_+ region, the start of this horizontal section remains in the region "below the line" including $14 < p < 18$. For this reason there is no jump in the w value until the line leaves the horizontal, and there cannot be a solution with a vertical segment ending at the $p = 18$ line which satisfies the necessary conditions.

To make our discussion here easier we define

$$W_2(q,a,b) = \Phi(q,a)R_q(q,a) + \int_a^b Z(q,p)dp,$$

which is the w integral over a vertical segment starting at $(q,a) \in \Psi_+$ when a is a price discontinuity. In this case, we need to continue with a vertical segment starting at a point $(q, 18)$ and ending at a point (q,b), either on the $Z = 0$ line or on the upper boundary $\psi = 1$, with $W_2(q, 18, b) = 0$. Now the curve defined by $W_2(q, 18, b) = 0$ intersects the $Z = 0$ line at $G = (921.2, 18.36)$. The optimal solution then continues along the $Z = 0$ line until crossing the upper boundary at $H = (1009.6, 19.675)$. The complete optimal solution is shown in Figure 1.

Having found the optimal solution to the problem when $\mathcal{L}^*$ is used, it is straightforward to generate ε-optimal solutions for the case where we do not have the ideal sharing rule when prices coincide. We should follow the solution described above, but overcutting slightly for horizontal sections of the offer curve in Ψ_- and undercutting in Ψ_+. It will not matter what the offer looks like outside the region Ψ. If we choose to undercut and overcut by an amount of \$0.01, we end up with the following offer schedule:

(a) an amount of 266.6 MW at price \$0 (or any price below \$10),
(b) an amount of 521.7–266.6 = 255.1 MW at price \$10.01,
(c) an amount of 643.7–521.7 = 122.0 MW in a smooth curve rising to a price of \$13.40,
(d) an amount of 671.8–643.7 = 28.1 MW at a price of \$14.01,
(e) an amount of 898.0–671.8 = 226.2 MW in a smooth curve rising to a price of \$17.99,
(f) an amount of 921.2–898.0 = 23.2 MW at a price of \$17.99,
(g) an amount of 1009.6–921.2 = 88.4 MW in a curve from a price of \$18.36 to \$19.67,
(h) an amount of 1100–1009.6 = 90.4 MW at \$50 (or any price above \$19.67).

The offer schedule above now needs to be altered in line with market rules. In the case that only step functions are allowed as offers, then the smooth curves of (c), (e), and (g) will need to be approximated with step functions. In the case that piecewise linear offers are required, then these curves would be approximated by one or more linear segments.

6. Discussion. Work on optimal offer policies and on Nash equilibria in an electricity market setting has often confronted the issue of undercutting (though our discussion of overcutting solutions is new). The essential problem is that the possibility of undercutting on price will in many models lead to highly competitive (Bertrand-type) equilibrium solutions, with no possibility of supporting an equilibrium in which generators offer at prices above their marginal costs. However, this idealized behavior is very far from that which is observed in actual markets around the world. Different authors have suggested a variety of methods to address the issue.

Using supply functions as a model for the offer procedure is one approach which avoids the difficulty of undercutting. In this framework we usually assume that there is no single price at which a generator offers a significant quantity of power, and this allows us to formulate models in which Nash equilibria exist for supply functions.

An alternative approach which has been suggested by von der Fehr and Harbord [15], and which has been used by Wolfram [16] and Brunekreeft [6], is to assume that offers are made at one or a small number of prices but that these prices are not revealed in advance to the other players. This allows a type of mixed strategy to be played which chooses prices according to a continuous distribution. This clearly rules out the possibility of undercutting, since even though we know the strategy of the other player, we do not know a price which we can then undercut.

In this paper we do not try to establish equilibrium conditions; instead we concentrate on the question of evaluating the optimal (or ε-optimal) offer strategy. From a generator's viewpoint this is valuable if the generator wishes to achieve the maximum one period profit. The analysis we give can then point to the best possible policy which is likely to involve some part of the offer either just below or just above other players' prices. But the analysis is also valuable if the generator decides to adopt a less aggressive policy, since it indicates the degree of suboptimality involved in adopting any other (nonundercutting) solution. In practice generators will also need to build an offer according to specific market rules: again we can think of the optimal supply function strategy as setting a benchmark against which other policies can be compared. Sometimes, as in Australia and New Zealand, these market rules imply that a step function is used, in which case a step function approximation to the type of policy shown in Figure 3 should be constructed. Other markets allow an offer to be piecewise linear, which will enable a much closer approximation to be achieved.

It is natural to ask whether the type of analysis we give here could be extended to an equilibrium analysis. In fact it is not possible to construct an exact Nash equilibrium in offers with the type of undercutting behavior we have analyzed. However, an interesting area for further research is the existence of an ε-equilibrium, in which player i submits an offer S_i (being a step function satisfying the market rules) in such a way that the expected profit for player i, $v_i(S_i)$, is within ε of the best possible expected profit for player i given the offers of the other generators. Such a step function ε-equilibrium will not be unique, and so there will be the usual conceptual problems of coordination on nonunique equilibria, coupled here with additional difficulties in coordinating on an appropriate value of ε. Nevertheless, such ε-equilibrium might be arrived at in practice through repeated adjustment of generator offers in response to the other generators, but where generators prefer not to change their offer strategy unless this will lead to an increase in expected profit of at least ε.

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