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NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMAL OFFERS IN ELECTRICITY MARKETS*

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Abstract. In this paper, we consider the optimal policy for a generator offering power into a wholesale electricity market operating under a pool arrangement. Anderson and Philpott [*Math. Oper. Res.*, 27 (2002), pp. 82–100] recently discussed necessary conditions for an optimal offer curve when there is uncertainty in the demand and in the behavior of other participants in the market. They show that the objective function in these circumstances can be expressed as a line integral along the offer curve of a profit function integrated with respect to a market distribution function. In this paper, we prove the existence of an optimal offer stack, and we extend the analysis of [*Math. Oper. Res.*, 27 (2002), pp. 82–100] to include necessary conditions of a higher order in the presence of horizontal and/or vertical sections in an offer curve. Finally, we establish sufficient conditions for an offer curve to be locally optimal.

Key words. electricity markets, optimal offer, necessary conditions, sufficient conditions

AMS subject classifications. 90C46, 65K10, 49K30

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1. Introduction.

In the past few years, there has been an enormous change in the way that wholesale prices for electricity are determined in many parts of the world. Increasingly, market mechanisms are being set up in which the clearing price for electricity is determined by some sort of auction process. The book by Chao and Huntington [3] gives a useful overview of the nature of electricity markets, and the working paper by von der Fehr and Harbord [4] is another useful starting point as it reviews the form of a variety of markets as they existed in 1998. In this paper, we consider a model of the operation of such a market which captures some important features of the electricity markets which operate in the UK, Australia, New Zealand, and parts of the US. The model was introduced in a recent paper by Anderson and Philpott [1]. We begin by reviewing this model before going on to extend the results of Anderson and Philpott on the form of the optimal solutions for generators operating in a market of this sort.

Generators in an electricity market offer energy into the market at prices that they determine. We take this offer as having the form of an offer curve linking quantity and price. We can write the quantity of electricity offered as a nondecreasing function of price, $S(p)$. $S(p)$ is the quantity of electricity that will be delivered by the generator in question if the clearing price is p. In some markets, power is offered in blocks, which will imply that $S(p)$ has the form of a step function. The clearing price is determined by a market mechanism that incorporates consideration of the offers made by all the generators, transmission constraints operating within the electricity network, and the demand (which may have price dependence for some large consumers).

The model we consider represents a market operating under pool arrangements. In some markets, there are more bilateral trading arrangements, and our model will

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not apply in these cases. In a pool arrangement, some form of independent system operator (ISO) is responsible for determining which generators are dispatched and will do this in a way which satisfies demand at least cost, using the generator bids as proxies for cost. If the spot market consists of a pool located at a single node, then the price of electricity can be computed by successively dispatching generation from the offers with the lowest price until all of the demand is met. The price of the marginal offer, the system marginal price, is then the price that is paid for all the electricity dispatched. Thus a low cost generator can choose to offer power at its real marginal cost and will be very likely dispatched and paid a substantial premium over its marginal generating cost.

Following Anderson and Philpott [1], we will be interested in finding an optimal offer for a generator when there is uncertainty about the demand and the behavior of the other generators. This takes a different approach than that considered by Gross and Finlay [6], who assume perfect competition so that the clearing price is unaffected by any single generator's offer. In our model, the offer that a generator makes has a direct influence on the clearing price, but we will not consider the equilibrium framework that would arise if we were to consider the optimal response of other generators to our offer curve. In fact, the majority of papers dealing with electricity markets have studied equilibria, often with the aim of assessing the degree of market power implied by different market structures. See, for example, the papers by Hobbs [7], Bolle [2], Green and Newbery [5], Rudkevich, Duckworth, and Rosen [8], and Wei and Smeers [9].

Our work is more directly concerned with the problem faced by a generator in deciding on an offer. In this case, the generator might take an equilibrium solution as an indication of where things may end up in the long term, but the immediate problem will be to respond to the current environment, which will involve uncertainty in demand and in other generators' offers. Such uncertainty arises partly because of possible outages and partly because generators' bidding behavior is not stable. Concentration only on an equilibrium solution is problematic: this assumes that other generators are behaving in a fully rational way and that we have access to all relevant information. Moreover, in many circumstances, there will be more than one equilibrium possible; which one should guide the offer behavior for a generator? Finally, we should observe that the actual computation of equilibria in these markets with many participants is extremely hard. Of course, since offers are repeated many times a day and there is often considerable similarity between outcomes at the same hour on different days, we should expect electricity markets to move toward some sort of equilibrium behavior. Thus we are not arguing here against the importance of understanding equilibria but merely that the problem we address is of importance. Indeed, a good understanding of this problem will be helpful in moving forward in the analysis of equilibrium models.

Anderson and Philpott have explored the problem of finding an offer curve that maximizes the expected value of the profit made by an individual generator. The offer curve is simply a monotonic continuous curve in the two-dimensional (quantity, price) space. This curve need not be smooth; indeed, it will often in practice take the form of a series of steps. Anderson and Philpott show how the problem of maximizing expected profit is, in some circumstances, equivalent to maximizing the line integral along the offer curve of a market distribution function defined in the (quantity, price) space. The market distribution function captures all the elements of uncertainty in either demand or the other players' behavior.

Anderson and Philpott give necessary conditions for an optimal offer curve in this framework. In this paper, we make a number of contributions. First, in section 2, we demonstrate the existence of an optimal solution. Second, in section 3, we extend the necessary conditions given previously. In section 4, we give sufficient conditions for an offer curve to be locally optimal. Finally, in section 5, we give an example to demonstrate the application of these conditions.

The analysis we give is quite general and has some interest apart from its electricity market context. The necessary and sufficient conditions apply to the problem of finding a choice of monotonic curve within a bounded region in order to maximize a line integral along the curve. In the absence of monotonicity constraints, this is a problem in the calculus of variations. However, the requirement for monotonicity is fundamental and implies additional conditions that apply on sections of the curve which are either horizontal or vertical (and hence at points where monotonicity acts as a binding constraint). The results we give are proved by relatively direct and elementary methods involving consideration of perturbations of an offer curve. It is interesting that sufficient conditions for optimality can be obtained in this way.

2. Problem formulation and fundamentals. In this section, we will introduce some notation and formulate the problem that we shall consider. Let $R(q,p)$ be the return function: that is, $R(q,p)$ denotes the profit we make if we are dispatched an amount q at a clearing price p. We will assume that R has continuous partial derivatives. This function captures not only the cost of generating an amount q and the proceeds pq which arise from the sale of this electricity but also the effects of any hedging contracts the generator holds which depend on the market clearing price.

Rather than dealing with a supply function $S(\cdot)$ directly, it is convenient to model the offer using a continuous curve $\mathbf{s} = \{(x(t), y(t)), 0 \le t \le T\}$, in which the components $x(t)$ and $y(t)$ are continuous monotonic increasing functions of t, and $x(t)$ and $y(t)$ trace, respectively, the quantity and price components. Without loss of generality we may take $x(0) = y(0) = 0$ and $y(T) \le p_M$, where p_M is a bound on the price of any offer. We also assume that q_m is a bound on the generation capacity of the generator, so $x(T) \le q_M$.

We use a single *market distribution function* $\psi(q,p)$ to describe the uncertainty in the market. $\psi(q,p)$ is defined as the probability of not being fully dispatched if we offer generation q at price p. It turns out that knowledge of the single function $\psi(q,p)$ is enough to determine the expected profit for a generator. In practice, a generator will estimate the market distribution function from knowledge of the distribution of demand and from repeated observations of the behavior of other generators. This estimation problem will depend on the information released to market participants about other players' bids (something which varies between different markets). Another issue will be to decide the class of functions from which an estimate is to be chosen. However, subject to these considerations, either Bayesian or maximum likelihood estimation techniques can be used.

Since $\psi(q,p)$ is a probability, it takes values between 0 and 1. Let $\Psi = \{(q,p), 0 < \psi(q,p) < 1\}$. Throughout, we assume that ψ is continuously differentiable on $\Psi \cap \{(q,p), 0 \le q \le q_M, 0 \le p \le p_M\}$. The first key result, which we repeat from Anderson and Philpott [1], demonstrates that the expected return if a generator offers in a supply curve $\mathbf{s}$ can be expressed as a line integral along $\mathbf{s}$. This can be established by showing that a generator can only be dispatched at price-quantity points lying on its offer curve and observing that the derivative of the market distribution function ψ captures the appropriate probability density.

LEMMA 1 (see [1, Theorem 2]). *If a generator offers in a curve* $\mathbf{s}$ *and the market distribution function* ψ *is continuous, then the expected return is the line integral*

$$v(\mathbf{s}) = \int_{\mathbf{s}} R(q,p) d\psi(q,p).$$

Anderson and Philpott [1] treat $v(\mathbf{s})$ as an objective function and investigate the necessary conditions for an offer curve $\mathbf{s}$ to be a local maximizer. However, there is an important question that they do not address explicitly: does there exist a maximum over the set of curves which are considered? We begin by answering this question before discussing optimality conditions.

A generator need not offer all of its generation capacity into the market; the offer curve will start at some point $(0, y(0))$ and finish at $(x(T), y(T))$. However, the clearing price is determined as though the offer curve began with a vertical segment from the origin to $(0, y(0))$ and finished with a vertical segment from $(x(T), y(T))$ to $(x(T), p_M)$. Hence we assume that Ω, the set of curves, has these characteristics.

LEMMA 2. *Let* Ω *be the set of monotonic continuous curves starting at the origin and ending on the closed line segment,* $\mathcal{L}$*, from* $(0, p_M)$ *to* (q_M, p_M)*. Then* Ω *is compact under the Hausdorff metric:*

$$|\mathbf{s}_1 - \mathbf{s}_2|_H = \max_{(x_1,y_1)\in\mathbf{s}_1} \min_{(x_2,y_2)\in\mathbf{s}_2} \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

Proof. Let $\mathbf{s} \in \Omega$ and $L(\mathbf{s})$ be the arc length of $\mathbf{s}$. By the monotonicity of $\mathbf{s}$,

$$p_M \le L(\mathbf{s}) \le p_M + q_M.$$

For any $\mathbf{s} \in \Omega$, we may use the arc length measured from the origin as a parameter and write $\mathbf{s} = \{(x(t), y(t)), 0 \le t \le L(\mathbf{s})\}$. For $0 \le t_1 < t_2 \le L(\mathbf{s})$, we have

$$t_2 - t_1 \ge \max\left(x(t_2) - x(t_1), y(t_2) - y(t_1)\right).$$

Thus both x and y are Lipschitz with respect to t. By replacing t with $\frac{T}{L(\mathbf{s})}t$, we obtain a representation of $\mathbf{s}$ with both x and y defined on $[0, T]$. The scaled x and y are still Lipschitz and monotonic. It is well known that the class of monotonic Lipschitz functions defined on $[0, T]$ forms a compact set. Thus the set of curves Ω, when represented as pairs of (Lipschitz) functions, is compact with the metric

$$|\mathbf{s}_1 - \mathbf{s}_2| = \max\left(\sup_{0\le t\le T} |x_1(t) - x_2(t)|, \sup_{0\le t\le T} |y_1(t) - y_2(t)|\right).$$

Since $|s_1 - s_2|_H \le \sqrt{2}|s_1 - s_2|$, this implies compactness of Ω with the Hausdorff metric. ☐

Anderson and Philpott [1] introduced the notation

$$Z(q,p) \equiv \begin{cases} R_q\psi_p - R_p\psi_q, & (q,p) \in \Psi, \\ 0 & \text{otherwise} \end{cases}$$

and observed that

$$\int\int_{\mathcal{S}} Z(q,p) dp dq = \int_{\mathcal{C}} R(q,p) d\psi(q,p), \tag{1}$$

where $\mathcal{S}$ is a region enclosed by a curve $\mathcal{C}$. Relation (1) follows immediately from Green's theorem and plays an important role in the investigation of optimality conditions. The scalar function Z can be thought of as indicating the direction in which the offer curve needs to move to produce an improvement in expected profit. If $Z > 0$, then a move of the offer curve down and to the left in the (q, p) plane will produce an improvement, while $Z < 0$ indicates that the offer curve should move up and to the right. Consequently, when monotonicity constraints are not binding, it will be optimal to follow a $Z = 0$ curve.

Using Lemma 2 and (1), we are able to obtain the following result.

THEOREM 3. *Let Ω be defined as above, and let v be the expected return function given in Lemma* 1. *Then v achieves its maximum on Ω.*

Proof. By Lemma 2, Ω is a compact set. It suffices to prove that v is continuous with the Hausdorff metric. Let $\tilde{\mathbf{s}} \in \Omega$, and assume without loss of generality that $\mathcal{S}$ is the region surrounded by $\tilde{\mathbf{s}}$, $\mathbf{s}$. Thus, as $\tilde{\mathbf{s}}$ is close to $\mathbf{s}$, the area $\mathcal{S}$ is small. By (1), the boundedness of Ψ, and the continuous differentiability ψ and R, we know that the integral of Z over $\mathcal{S}$ will be arbitrarily small when the area of $\mathcal{S}$ shrinks to zero. On the other hand, the line integral at the right side of (1) over the segment of $\mathcal{L}$ between $\mathbf{s}$ and $\tilde{\mathbf{s}}$ tends to zero as $\tilde{\mathbf{s}} \to \mathbf{s}$. This implies that $v(\tilde{\mathbf{s}})$ tends to $v(\mathbf{s})$. ◻

Given this theorem, the maximization problem

$$\text{maximize } v(\mathbf{s}), \ \text{ subject to } \ \mathbf{s} \in \Omega$$

is well defined. In the rest of this paper, we will discuss the necessary and sufficient conditions for an offer curve $\mathbf{s}$ to be a local maximizer.

3. Necessary conditions. We turn now to necessary conditions for optimality. Anderson and Philpott [1] establish a set of conditions which we will extend. Such conditions are important in the development of algorithms to solve the generator's maximization problem. Later we will illustrate their use by considering a simple example. In general, more complete optimality conditions will serve to eliminate more potential candidate optimal offer curves. It is convenient to restate the result of [1] in a slightly different form in order to show how it is related to the results that we prove in this paper.

Throughout, we need to use the line integral of Z along a curve $\{(x(t), y(t)) : 0 \le t \le T\}$ which is defined by

$$w(t) = \int_0^t Z(x(\tau), y(\tau))(x'(\tau) + y'(\tau))d\tau.$$

In some cases, we will write $w(x(t), y(t))$ to denote $w(t)$ for clarity.

THEOREM 4. *Suppose that $\mathbf{s} = \{x(t), y(t), 0 \le t \le T\}$ is an increasing continuous offer curve. Suppose that there exist m numbers $0 \le t_1 < t_2 < \cdots < t_m \le T$ with $0 < x(t) < q_M$ and $0 < y(t) < p_M$ for $t_1 < t < t_m$ and such that, on each section (t_{i-1}, t_i), $i = 2, \ldots, m$, $\mathbf{s}$ is either strictly increasing in both components or horizontal or vertical, with different characteristics in successive segments. If $\mathbf{s}$ is optimal, then each of the $w(t_i)$, $i = 1, 2, \ldots, m$, takes the same value, say, w_0, and, for each interval I, being one of (t_{i-1}, t_i), $i = 2, \ldots, m$, $(0, t_1)$, or (t_m, T), one of the following holds:*

(i) $\mathbf{s}$ *is strictly increasing in both components, and $Z(x(t), y(t)) = 0$ for $t \in I$;*

(ii) $\mathbf{s}$ *is horizontal on I, and $w(t) \le w_0$ for $t \in I$;*

(iii) $\mathbf{s}$ *is vertical on I, and $w(t) \ge w_0$ for $t \in I$.*

Anderson and Philpott [1] show, in addition, that, when $\mathbf{s}$ is neither horizontal nor vertical, then there are sign constraints on the partial derivatives of Z if they exist. In fact, the existence of partial derivatives for Z will allow us to extend the results of Theorem 4 provided that Z is well behaved enough.

We will need to make an assumption about the partial derivatives of Z. We give the weakest form of this required for our results; it is stronger than continuous differentiability, requiring also a uniformity condition on horizontal and vertical sections of $\mathbf{s}$. This condition will be implied, for example, by Z having continuous second derivatives.

Assumption 1. Z is continuously differentiable on Ψ. If $\mathbf{s}$ is horizontal on $[t_{i-1}, t_i]$, then, for every $\eta > 0$, there is a $\tau_0 > 0$ with

$$|Z(x(t), y(t_i) + \tau) - Z(x(t), y(t_i)) - Z_p(x(t), y(t_i))\tau| \leq \eta|\tau|$$

for every $t \in [t_{i-1}, t_i]$ and $|\tau| < \tau_0$. Similarly, if $\mathbf{s}$ is vertical on $[t_{i-1}, t_i]$, then, for every $\eta > 0$, there is a $\tau_0 > 0$ with

$$|Z(x(t_i) + \tau, y(t)) - Z(x(t_i), y(t)) - Z_q(x(t_i), y(t))\tau| \leq \eta|\tau|$$

for every $t \in [t_{i-1}, t_i]$ and $|\tau| < \tau_0$.

The theorem below extends Theorem 4, but there is a key difference that is worth pointing out before stating the result. In the previous theorem, the values t_i are defined as corresponding to points where the curve changes characteristic, say, from horizontal to vertical. In the theorem we give next, we will define the values t_i in terms of the w values instead. Thus suppose we have a horizontal segment within which $w(t) \leq w_0$: then, in Theorem 4, the t_i values mark either end of this segment, but in the next theorem we add to this any other values of t at which $w(t) = w_0$, between the end points. Thus we may have a point t_i, with $\mathbf{s}$ horizontal on either side of it. It will be convenient to distinguish those values of t_i such that the curve $\mathbf{s}$ changes its characteristics at $(x(t_i), y(t_i))$, for instance, from vertical to horizontal or strictly increasing in both components. For convenience, we call both the parameter t_i and the point $(x(t_i), y(t_i))$ a *turning point*.

THEOREM 5. *Suppose that* $\mathbf{s} = \{x(t), y(t), 0 \leq t \leq T\}$ *is an increasing continuous offer curve. Suppose that there exist* m *numbers* $0 \leq t_1 < t_2 < \cdots < t_m \leq T$ *such that each of the* $w(t_i)$, $i = 1, 2, \ldots, m$, *takes the same value, say,* w_0, *and, on each section* $[t_{i-1}, t_i]$, $i = 2, \ldots, m$, $\mathbf{s}$ *is either strictly increasing in both components or horizontal or vertical. Suppose that* $\mathbf{s}$ *is an optimal offer stack. Then, under Assumption* 1, *for each section* (t_{i-1}, t_i), *the following hold:*

(i) *if* $\mathbf{s}$ *is strictly increasing on* (t_{i-1}, t_i), *then* $Z_p(x(t), y(t)) \geq 0$ *and* $Z_q(x(t), y(t)) \leq 0$ *for* $t \in (t_{i-1}, t_i)$;

(ii) *if* $\mathbf{s}$ *is horizontal on* (t_{i-1}, t_i) *and one of* t_{i-1} *or* t_i *is a turning point, then*

$$\int_{x(t_{i-1})}^{x(t_i)} Z_p(x, y(t_i))dx \geq 0; \tag{2}$$

further, if $\mathbf{s}$ *turns from horizontal to vertical at* t_i, *then*

$$\int_{x(t_{i-1})}^{x(t_i)} Z_p(x, y(t_i))dx - \int_{y(t_i)}^{y(t_{i+1})} Z_q(x(t_i), y)dy \geq 2Z(x(t_i), y(t_i)); \tag{3}$$

(iii) *if* $\mathbf{s}$ *is vertical on* (t_{i-1}, t_i) *and one of* t_{i-1} *or* t_i *is a turning point, then*

$$\int_{y(t_{i-1})}^{y(t_i)} Z_q(x(t_i), y)dy \leq 0; \tag{4}$$

further, if $\mathbf{s}$ *turns from vertical to horizontal at* t_i*, then*

$$\int_{y(t_{i-1})}^{y(t_i)} Z_q(x(t_i), y)dy - \int_{x(t_i)}^{x(t_{i+1})} Z_p(x, y(t_i))dx \leq 2Z(x(t_i), y(t_i)). \tag{5}$$

Before giving a proof of this result, it may be helpful to discuss the conditions (2), (3), (4), and (5).

If $\mathbf{s}$ is horizontal on (t_{i-1}, t_i), then, since $w(t_{i-1}) = w(t_i) = w_0$,

$$\int_{x(t_{i-1})}^{x(t_i)} Z(x, y(t_i))dx = 0.$$

This integral captures the first order effect of a move of the horizontal segment up or down by a small amount. Given that the integral is zero, we can look at the second order effects of the same move, and these are given by the integral in (2). The same argument applied to a vertical segment leads to the integral in (4).

The stronger conditions (3) and (5), which apply when there is a turn from horizontal to vertical (or vice versa), arise from considering a move of a horizontal segment at the same time as a vertical segment. Notice that, if $\mathbf{s}$ turns from horizontal to vertical at t_i, then $Z(x(t_i), y(t_i)) \geq 0$. This follows from the observation that Theorem 4 implies that $w(t)$ is increasing at $t = t_i$. In the same way, $Z(x(t_i), y(t_i)) \leq 0$ if there is a turn from vertical to horizontal at t_i.

Proof. Part (i) was already established by Anderson and Philpott [1]. We will prove part (ii); part (iii) follows in exactly the same way. For the sake of contradiction, assume that (2) does not hold and $\mathbf{s}$ is horizontal on (t_{i-1}, t_i) but not horizontal on (t_i, t_{i+1}). Let $\delta > 0$ be sufficiently small, and consider a vertical perturbation of $\mathbf{s}$ by an amount δ between t_{i-1} and t_i. We can make this explicit as follows. Define $t_i(+\delta) = y^{-1}(y(t_i) + \delta)$, and let

$$\tilde{\mathbf{s}}(t) = \begin{cases} (x(t), y(t)), & t \leq t_{i-1}, \\ (x(t_{i-1}), t - t_{i-1} + y(t_{i-1})), & t_{i-1} \leq t \leq t_{i-1} + \delta, \\ (x(t-\delta), y(t_{i-1}) + \delta), & t_{i-1} + \delta \leq t \leq t_i(+\delta) + \delta, \\ (x(t-\delta), y(t-\delta)), & t_i(+\delta) + \delta \leq t \leq T + \delta, \end{cases}$$

be a perturbation of $\mathbf{s}$. This is illustrated in Figure 1. For $0 \leq \tau \leq \delta$, we consider the line integral

$$I(\tau) = \int_{x(t_{i-1})}^{x(t_i)} Z(x, y(t_i) + \tau)dx.$$

Let $\epsilon > 0$ be such that $\int_{x(t_{i-1})}^{x(t_i)} Z_p(x, y(t_i))dx < -\epsilon < 0$. By Assumption 1, for τ sufficiently small and $x \in [x(t_{i-1}), x(t_i)]$,

$$Z(x, y(t_i) + \tau) \leq Z(x, y(t_i)) + Z_p(x, y(t_i))\tau + \frac{\tau\epsilon}{x(t_i) - x(t_{i-1})}.$$

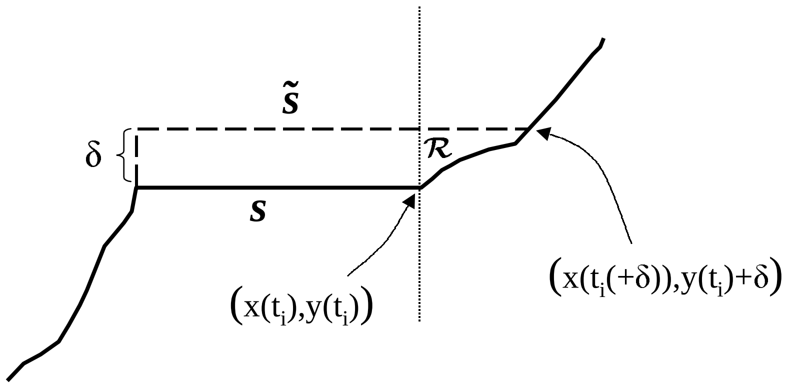

FIG. 1. *Perturbation of a horizontal section of* $\mathbf{s}$.

Thus

$$I(\tau) \leq w(t_i) - w(t_{i-1}) + \tau \int_{x(t_{i-1})}^{x(t_i)} Z_p(x, y(t_i))dx + \tau\epsilon.$$

Consequently, we have, for all τ, $I(\tau) < 0$ when δ is sufficiently small. The area integral of Z over the region surrounded by $\mathbf{s}$, $\tilde{\mathbf{s}}$, $x = x(t_{i-1})$, and $x = x(t_i)$ can be written as $\int_0^\delta I(\tau)d\tau$, which is not larger than $\frac{\delta^2}{2}[\int_{x(t_{i-1})}^{x(t_i)} Z_p(x, y(t_i))dx + \epsilon]$.

Now we need to consider the region, namely, $\mathcal{R}$, to the right of $x = x(t_i)$ surrounded by $\mathbf{s}$ and $\tilde{\mathbf{s}}$. This area exists only when $\mathbf{s}$ is strictly increasing in each component on the interval (t_i, t_{i+1}). In this case, the area is of order $o(\delta)$. On the other hand, since $Z(x(t), y(t)) = 0$ along the lower boundary of the region,

$$Z(x(t), y(t_i) + \tau) = Z_p(x(t), y(t))(y(t_i) + \tau - y(t)) + o(\tau) < Z_p(x(t), y(t))\tau + o(\tau)$$

for all $t_i \leq t \leq t_i(+\delta)$, $0 \leq \tau \leq \delta$, such that $(x(t), y(t_i) + \tau) \in \mathcal{R}$. Note that Z_p is continuous, and the distance between any point $(x, y) \in \mathcal{R}$ and $(x(t_i), y(t_i))$ tends to 0 as $\delta \to 0$. Consequently, the integral of Z over $\mathcal{R}$ is at most of order $o(\delta^2)$. Thus we have

$$v(\mathbf{s}) - v(\tilde{\mathbf{s}}) \leq \int_0^\delta I(\tau)d\tau + o(\delta^2) = \frac{\delta^2}{2} \int_{x(t_{i-1})}^{x(t_i)} Z_p(x, y(t_i))dx + \frac{\delta^2}{2}\epsilon + o(\delta^2)$$

for δ small enough. From the choice of ϵ, this contradicts the optimality of $\mathbf{s}$ for δ small enough.

In the case that $\mathbf{s}$ is not horizontal on (t_{i-2}, t_{i-1}), we can obtain the same result by considering a vertical perturbation of $\mathbf{s}$ between t_{i-1} and t_i downward by an amount δ.

Now suppose that t_i is a corner where $\mathbf{s}$ turns from horizontal to vertical. We will assume that (3) does not hold and derive a contradiction. Let η be a scalar such that

$$\int_{x(t_{i-1})}^{x(t_i)} Z_p(x, y(t_i))dx - \int_{y(t_i)}^{y(t_{i+1})} Z_q(x(t_i), y)dy - 2Z(x(t_i), y(t_i)) < -2\eta < 0. \tag{6}$$

We consider a perturbation that moves the horizontal section between t_{i-1} and t_i upward by a small amount δ and the vertical section between t_i and t_{i+1} to the left by the same amount δ. We can make this explicit as follows. Define $t_i(-\delta) = x^{-1}(x(t_i) - \delta)$, and let

$$\tilde{\mathbf{s}}(t) = \begin{cases} (x(t), y(t)), & t \le t_{i-1}, \\ (x(t_{i-1}), t - t_{i-1} + y(t_{i-1}), & t_{i-1} \le t \le t_{i-1} + \delta, \\ (x(t-\delta), y(t_{i-1}) + \delta), & t_{i-1} + \delta \le t \le t_i(-\delta) + \delta, \\ (x(t_i) - \delta, y(t + k - \delta)), & t_i(-\delta) + \delta \le t \le t_{i+1} - k + \delta, \\ (x(t_{i+1}) + t - 2\delta + k - t_{i+1}, y(t_{i+1})), & t_{i+1} - k + \delta \le t \le t_{i+1} - k + 2\delta, \\ (x(t + k - 2\delta), y(t + k - 2\delta))), & t_{i+1} - k + 2\delta \le t \le T - k + 2\delta, \end{cases}$$

be a perturbation of $\mathbf{s}$, where $k = t_i(+\delta) - t_i(-\delta)$.

First observe that

$$\begin{aligned} w(t_i(-\tau)) - w(t_i(+\tau)) &= -\int_{x(t_i)-\tau}^{x(t_i)} Z(x, y(t_i))dx - \int_{y(t_i)}^{y(t_i)+\tau} Z(x(t_i), y)dy \\ &\le -2\tau Z(x(t_i), y(t_i)) + \eta\tau/2 \end{aligned} \tag{7}$$

for τ sufficiently small, using the mean value theorem and the continuity of Z. On the other hand, under Assumption 1, for the given η, there exists $\delta > 0$ sufficiently small such that, for $0 < \tau \le \delta$, the line integral

$$\begin{aligned} I(\tau) \equiv& \int_{x(t_{i-1})}^{x(t_i)-\tau} Z(x, y(t_i) + \tau)dx + \int_{y(t_i)+\tau}^{y(t_{i+1})} Z(x(t_i) - \tau, y)dy \\ &\le w(t_i(-\tau)) - w(t_{i-1}) + w(t_{i+1}) - w(t_i(+\tau)) \\ &+ \tau \int_{x(t_{i-1})}^{x(t_i)-\tau} Z_p(x, y(t_i))dx - \tau \int_{y(t_i)+\tau}^{y(t_{i+1})} Z_q(x(t_i), y)dy + \eta\tau/2. \end{aligned} \tag{8}$$

Since Z_p and Z_q are continuous, it follows from (6) that, for δ sufficiently small and $\tau \le \delta$,

$$\int_{x(t_{i-1})}^{x(t_i)-\tau} Z_p(x, y(t_i))dx - \int_{y(t_i)+\tau}^{y(t_{i+1})} Z_q(x(t_i), y)dy - 2Z(x(t_i), y(t_i)) < -\eta. \tag{9}$$

Combining (7)–(9) and noticing that $w(t_{i-1}) = w(t_{i+1}) = w_0$, we have $I(\tau) < 0$ for δ sufficiently small and all $0 \le \tau \le \delta$.

Now the area integral of Z over the region surrounded by $\mathbf{s}$ and $\tilde{\mathbf{s}}$ can be written as $\int_0^\delta I(\tau)d\tau$ and hence is strictly negative. Thus we have constructed a perturbed curve which leads to a larger value for v, which is a contradiction. This completes the proof. ∎

4. Sufficient conditions. In this section, we discuss sufficient conditions for an offer curve $\mathbf{s}$ to be locally optimal. This involves a more complex set of conditions than were required for the necessary conditions. To establish this result, we will have to consider all possible monotonic perturbations around the offer curve $\mathbf{s}$. As we shall see, considerable care is needed in the argument required to prove this result.

THEOREM 6. *Let* $\mathbf{s} = \{x(t), y(t), 0 \le t \le T\}$ *be an increasing and continuous offer stack. Suppose that there exist finite numbers* $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ *such that, for* $i = 1, \ldots, M-1$, $w(t_i)$ *takes a common value, say,* w_0*, and on each*

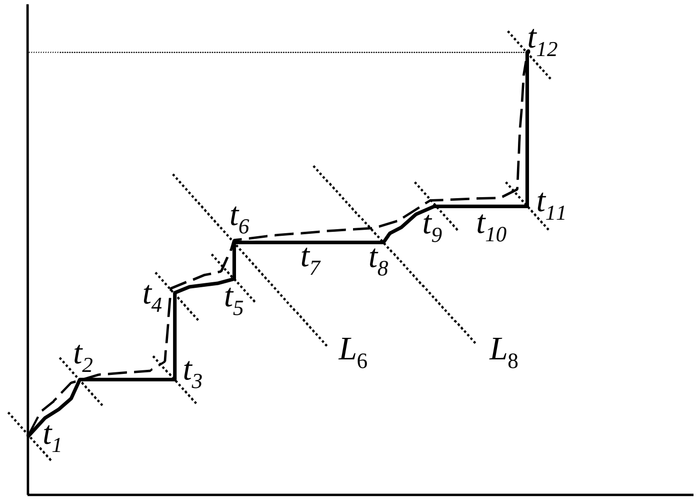

FIG. 2. *Area of perturbation split into subregions.*

section (t_{i-1}, t_i)*,* $i = 1, \ldots, M$*,* $\mathbf{s}$ *is either strictly increasing in both components or horizontal or vertical. Suppose also that Assumption* 1 *and the following hold:*

(i) *if* $\mathbf{s}$ *is increasing in both components on* (t_{i-1}, t_i)*, then, for* $t \in (t_{i-1}, t_i)$*,* $Z(x(t), y(t)) = 0$*,* $Z_p(x(t), y(t)) > 0$*, and* $Z_q(x(t), y(t)) < 0$*;*

(ii) *if* $\mathbf{s}$ *is horizontal on* (t_{i-1}, t_i)*, then, for* $t \in (t_{i-1}, t_i)$*,* $w(t) < w_0$*; moreover, for any* $j < k$ *such that* $\mathbf{s}$ *is horizontal from* t_j *to* t_k *with at least one of* t_j *and* t_k *a turning point,*

$$\int_{x(t_j)}^{x(t_k)} Z_p(x, y(t_j))dx > Z(x(t_k), y(t_k)) - Z(x(t_j), y(t_j)); \tag{10}$$

(iii) *if* $\mathbf{s}$ *is vertical on* (t_{i-1}, t_i)*, then, for* $t \in (t_{i-1}, t_i)$*,* $w(t) > w_0$*; moreover, for any* $j < k$ *such that* $\mathbf{s}$ *is vertical from* t_j *to* t_k *with at least one of* t_j *and* t_k *a turning point,*

$$\int_{y(t_j)}^{y(t_k)} Z_q(x(t_j), y)dy < Z(x(t_k), y(t_k)) - Z(x(t_j), y(t_j)); \tag{11}$$

(iv) *if* $(x(t_i), y(t_i))$ *is a point where the curve turns from horizontal to vertical, then either* $Z(x(t_i), y(t_i)) > 0$ *or* $Z(x(t_i), y(t_i)) = 0$ *and* $Z_p(x(t_i), y(t_i)) > 0$*,* $Z_q(x(t_i), y(t_i)) < 0$*; if* $(x(t_i), y(t_i))$ *is a point where the curve turns from vertical to horizontal, then either* $Z(x(t_i), y(t_i)) < 0$ *or* $Z(x(t_i), y(t_i)) = 0$ *and* $Z_p(x(t_i), y(t_i)) > 0$*,* $Z_q(x(t_i), y(t_i)) < 0$*.*

Then s is a locally optimal offer stack.

Proof. In order to prove that $\mathbf{s}$ is locally optimal, it suffices to prove that, for any local perturbation $\tilde{\mathbf{s}}$ which is sufficiently close to $\mathbf{s}$, $v(\tilde{\mathbf{s}}) < v(\mathbf{s})$. By Green's theorem, this is equivalent to proving that the area integral of Z over any region surrounded by $\tilde{\mathbf{s}}$ and $\mathbf{s}$ is positive if the region is above or to the left of the curve $\mathbf{s}$ and negative if under or to the right of $\mathbf{s}$. We discuss only the case that the perturbed region is above or to the left of the curve $\mathbf{s}$, and the other case can be dealt with similarly.

We define a line $y = y(t_i) - (x - x(t_i))$ at every turning point and call this line L_i. Figure 2 illustrates this with $\mathbf{s}$ shown bold and $\tilde{\mathbf{s}}$ as a dashed line. Assume the

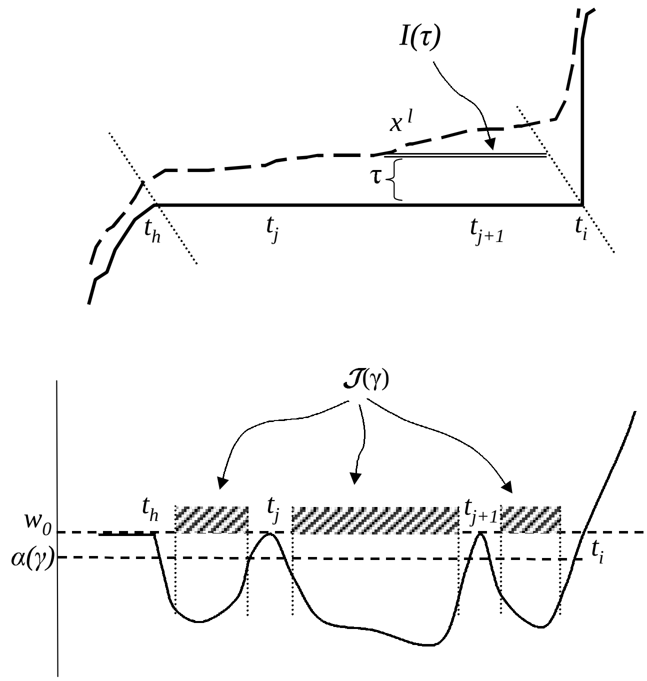

FIG. 3. *Comparison between w and* $\mathbf{s}$.

maximum distance between $\mathbf{s}$ and $\tilde{\mathbf{s}}$ is no larger than δ_0. The idea of the proof is to deal separately with different parts of the region between $\mathbf{s}$ and $\tilde{\mathbf{s}}$. In fact, we divide this region into subregions using the lines L_i and show that the area integral of Z is positive over each nonempty subregion. To accomplish this, we will consider integrals along horizontal (or vertical) segments with an end point on one of the lines L_i and show that each of these has positive value. One of these integrals $I(\tau)$ is illustrated in Figure 3. Since the perturbation is monotonic, the horizontal line segments will finish on an L_i, while the vertical line segments will start on an L_i.

Suppose first that $\mathbf{s}$ is strictly increasing in both components on (t_{i-1}, t_i). By assumption (i), it is easy to verify that $Z(x, y) > 0$ for any (x, y) within the region surrounded by $\mathbf{s}$, $\tilde{\mathbf{s}}$, and lines L_{i-1} and L_i provided δ_0 is chosen small enough. Consequently, the area integral of Z over the region surrounded by $\mathbf{s}$, $\tilde{\mathbf{s}}$, and these lines, if not empty, is positive.

Now we turn to the central part of the proof, and we consider the case that $\mathbf{s}$ is horizontal between t_h and t_i, and $(x(t_h), y(t_h))$ and $(x(t_i), y(t_i))$ are turning points. We observe first from (i) and (iv) that

$$Z(x(t_h), y(t_h)) \le 0 \tag{12}$$

and

$$Z(x(t_i), y(t_i)) \ge 0. \tag{13}$$

We will need to keep track of the parameter t at points on $\mathbf{s}$ which are a distance γ to the right or left of one of the t_i values in this horizontal segment. We let $t_k(-\gamma)$ denote $x^{-1}(x(t_k) - \gamma)$ for $k = h+1, \dots, i$, and $t_j(+\gamma) = x^{-1}(x(t_j) + \gamma)$ for $j = h, \dots, i-1$.

The proof proceeds in two major steps. In step 1, we construct a value of δ sufficiently small for the $I(\tau)$ integrals to be positive when $\delta_0 < \delta$; then, in step 2, we demonstrate this inequality.

Step 1. To define δ appropriately, we need first to define some intermediate quantities γ and ϵ. Choose $\gamma > 0$ small enough so that, for all $x \in (x(t_i) - \gamma, x(t_i))$,

$$Z(x, y(t_i)) > 0,\ Z_p(x, y(t_i)) > 0, \tag{14}$$

and for $x^u \in [x(t_i) - \gamma, x(t_i)]$, $x^l \in [x(t_j) - \gamma, x(t_j) + \gamma]$, $j = h+1, \ldots, i-1$,

$$\int_{x^l}^{x^u} Z_p(x, y(t_i))dx - Z(x(t_i), y(t_i)) > 0, \tag{15}$$

and for $x^l \in [x(t_h), x(t_h) + \gamma]$,

$$\int_{x^l}^{x^u} Z_p(x, y(t_i))dx + Z(x(t_h), y(t_h)) - Z(x(t_i), y(t_i)) > 0. \tag{16}$$

The existence of a γ satisfying (14) is guaranteed by conditions (i) and (iv) and (13), while the existence of a γ satisfying (15) and (16) follows from (10) after observing that $Z(x(t_j), y(t_j)) = 0$ since $w(t_j)$ is a local maximum of $w(\cdot)$ for $j = h+1, \ldots, i-1$.

Given such a γ, choose $\epsilon > 0$ small enough so that

$$w_0 - \alpha(\gamma) > \epsilon,$$

where

$$\alpha(\gamma) = \sup_{x(t) \in \mathcal{J}(\gamma)} w(t),$$

and

$$\mathcal{J}(\gamma) = \bigcup_{h \le j \le i-1} [x(t_j) + \gamma, x(t_{j+1}) - \gamma];$$

and for $x^u \in [x(t_i) - \gamma, x(t_i)]$, $x^l \in [x(t_j) - \gamma, x(t_j) + \gamma]$, $j = h+1, \ldots, i-1$,

$$\int_{x^l}^{x^u} Z_p(x, y(t_i))dx - Z(x(t_i), y(t_i)) > 2\epsilon, \tag{17}$$

and for $x^l \in [x(t_h), x(t_h) + \gamma]$,

$$\int_{x^l}^{x^u} Z_p(x, y(t_i))dx + Z(x(t_h), y(t_h)) - Z(x(t_i), y(t_i)) > 3\epsilon. \tag{18}$$

The definitions of $\alpha(\gamma)$ and $\mathcal{J}(\gamma)$ are illustrated in Figure 3.

Having chosen ϵ and γ, now let $\delta > 0$ be chosen small enough such that the following six conditions (a)–(f) hold.

(a)

$$\delta < \frac{1}{2\epsilon}\left(w_0 - \alpha(\gamma) - \epsilon\right). \tag{19}$$

(b) For $x^l \in \mathcal{J}(\gamma)$ and $x^u \in [x(t_i) - \gamma, x(t_i)]$,

$$\delta \left| \int_{x^l}^{x^u} Z_p(x, y(t_i))dx - Z(x(t_i), y(t_i)) \right| < \epsilon. \tag{20}$$

This follows from the fact that Z_p is continuous.

(c) For $0 \le \tau \le \delta$,

$$w_0 - w(t_i(-\tau)) \le \tau Z(x(t_i), y(t_i)) + \epsilon\tau. \tag{21}$$

This can be proved using continuity of Z and the mean value theorem.

(d) For $0 \le \tau \le \delta$ and $t \in [t_h, t_i]$,

$$Z(x(t), y(t_i) + \tau) \ge Z(x(t), y(t_i)) + Z_p(x(t), y(t_i))\tau - \frac{\epsilon\tau}{(x(t_i) - x(t_h))}. \tag{22}$$

This follows from Assumption 1.

(e) For $x \in (x(t_i) - \gamma, x(t_i))$, $0 \le \tau \le \delta$,

$$Z(x, y(t_i) + \tau) > 0. \tag{23}$$

This follows from (14) and Assumption 1.

(f) For $0 \le \tau_1, \tau_2 \le \delta$,

$$Z(x(t_h) - \tau_1, y(t_h) + \tau_2) > Z(x(t_h), y(t_h)) - \epsilon. \tag{24}$$

This follows from the continuity of Z.

Step 2. For $0 < \tau \le \delta$, we consider the integral

$$I(\tau) = \int_{x^l}^{x(t_i)-\tau} Z(x, y(t_i) + \tau)dx.$$

We shall prove that $I(\tau) > 0$ for all $x^l \in [x(t_h) - \tau, x(t_i) - \tau]$. To do this, we need to consider four cases depending on the position of x^l. When $x^l \in \mathcal{J}(\gamma)$, we will show that the value of the integral $I(\tau)$ is dominated by $I(0)$, which is a similar integral of Z but shifted down by an amount τ. Now $I(0) = w_0 - w(x^l, y(t_i))$, which is positive. However, when x^l is near t_h, t_i, or one of the intermediate t_j, then $I(0)$ is near zero, and we need to consider more precisely the difference between $I(\tau)$ and $I(0)$. This difference will be determined by the integral of Z_p along the line segment.

Suppose first that $x^l \ge x(t_h)$. By (22), we have

$$I(\tau) \ge w(x(t_i) - \tau, y(t_i)) - w(x^l, y(t_i)) + \tau \int_{x^l}^{x(t_i)-\tau} Z_p(x, y(t_i))dx - \epsilon\tau. \tag{25}$$

Case A. For $x^l \in [x(t_h), x(t_i) - \gamma] \backslash \mathcal{J}(\gamma)$, since $w(x^l, y(t_i)) \le w(x(t_i), y(t_i)) = w_0$, it follows from (21) and (25) that

$$I(\tau) \ge \tau \left(\int_{x^l}^{x(t_i)-\tau} Z_p(x, y(t_i))dx - Z(x(t_i), y(t_i)) \right) - 2\epsilon\tau. \tag{26}$$

Thus (using (17), (18), and (12)), $I(\tau) > 0$.

Case B. For $x^l \in \mathcal{J}(\gamma)$, combining (19), (20), (21), and (25), we have

$$\begin{aligned} I(\tau) &\geq w_0 - \alpha(\gamma) + \tau \left(\int_{x^l}^{x(t_i)-\tau} Z_p(x, y(t_i))dx - Z(x(t_i), y(t_i)) \right) - 2\epsilon\tau \\ &\geq w_0 - \alpha(\gamma) - \epsilon - 2\tau\epsilon \\ &> 0. \end{aligned}$$

Case C. For $x^l \in [x(t_i) - \gamma, x(t_i) - \tau]$, $0 \leq \tau \leq \delta$, it follows immediately from (23) that $I(\tau)$ is positive.

Case D. For $x^l < x(t_h)$, we must have $x^l \in [x(t_h) - \tau, x(t_h)]$, $0 \leq \tau \leq \delta$, and

$$I(\tau) = \int_{x^l}^{x(t_h)} Z(x, y(t_i) + \tau)dx + \int_{x(t_h)}^{x(t_i)-\tau} Z(x, y(t_i) + \tau)dx.$$

By (26),

$$\int_{x(t_h)}^{x(t_i)-\tau} Z(x, y(t_i) + \tau)dx \geq \tau \left(\int_{x(t_h)}^{x(t_i)-\tau} Z_p(x, y(t_i))dx - Z(x(t_i), y(t_i)) \right) - 2\epsilon\tau.$$

On the other hand, it follows from (24) that

$$\int_{x^l}^{x(t_h)} Z(x, y(t_i) + \tau)dx \geq (Z(x(t_h), y(t_h)) - \epsilon)\tau.$$

Thus

$$I(\tau) \geq \tau \left(\int_{x(t_h)}^{x(t_i)-\tau} Z_p(x, y(t_i))dx - Z(x(t_i), y(t_i)) + Z(x(t_h), y(t_h)) - 3\epsilon \right) > 0.$$

The last inequality is due to (18).

Now that we have established that $I(\tau)$ is positive, we are almost done. Let $\mathcal{L}$ be any horizontal line segment with a distance less than δ_0 above $\mathbf{s}$ and lying between the lines L_h and L_i, and with its right-hand end on the line L_i. The above discussion shows that, when $\delta_0 \leq \delta$, the line integral of Z along $\mathcal{L}$ is positive. Let $\tilde{\mathcal{R}}$ denote the region surrounded by $\tilde{\mathbf{s}}$, $\mathbf{s}$, L_h, and L_i. Since $\tilde{\mathbf{s}}$ is monotonic increasing and the maximum distance between $\tilde{\mathbf{s}}$ and $\mathbf{s}$ is not larger than δ_0, any intersection of a horizontal line and $\tilde{\mathcal{R}}$ will take the form $\mathcal{L}$. This implies that the area integral of Z over $\tilde{\mathcal{R}}$ is positive.

The argument for a vertical section is similar, but the integral $I(\tau)$ takes the form

$$I(\tau) = \int_{y(t_i)+\tau}^{y^u} Z(x(t_i) - \tau, y)dy.$$

This completes the proof. $\square$

The conditions of Theorem 6 are stronger than the necessary conditions of Theorems 4 and 5, and it is worth discussing the differences. First observe that the necessary conditions of Theorem 4 (i) and Theorem 5 (i) carry over as we would expect after a change to strict inequalities for Z_p and Z_q.

When we come to consider conditions (ii) and (iii) of Theorems 4 and 5, the position is more complex. We replace the condition $w(t) \leq w(t_1)$ on a horizontal section with the condition $w(t) < w(t_1)$ except at identified points among the t_j, $j = 1, \ldots,$ where $w(t_j) = w(t_1)$. The same thing happens on a vertical section. However, there is no direct equivalence of conditions (3) and (5) (though these inequalities can be derived from adding (10) and (11) in an appropriate way). Consider the inequality (10). As we have already observed, $Z(x(t_k), y(t_k)) \geq 0$, and $Z(x(t_j), y(t_j)) \leq 0$. So inequality (10) is strictly stronger than inequality (2). Similarly, inequality (11) is stronger than inequality (4).

Condition (iv) of Theorem 6 strengthens the inequality $Z(x(t_i), y(t_i)) \geq 0$ (or $Z(x(t_i), y(t_i)) \leq 0$), which can be derived from the necessary conditions at a turning point from horizontal to vertical (or vertical to horizontal).

5. An example. To illustrate the application of these necessary and sufficient conditions, we consider a small example based on one given in [1]. We define the market distribution function ψ via an intermediate function ϕ, which is defined as

$$\phi(q, p) = ((q - p)^2 - 1)((q - p)^2 - 0.7) - 1.59p^2 - 1.11q^2.$$

Then we set $\psi(q, p) = pq + 0.045\phi(q, p) - 0.1$. We suppose that the cost function is given by the quadratic $C(q) = 0.08q^2$. We also suppose that the generator has a two-way hedging contract for a quantity 0.15 and thus makes payments under these contracts of $0.15(f - p)$ where f is the contract price. Since f is fixed, we can ignore this term in seeking an optimal solution, and so we can take the profit function as $R(q, p) = (q - 0.15)p - C(q)$.

The first step in understanding the behavior of this example is to look at the values of the function Z over the region Ψ. This is shown in Figure 4, where the dashed lines show that $Z = 0$ and divide Ψ into regions where Z is either positive or negative. Also shown in the figure are three solid lines AB, CD, and EF, which connect the lower boundary of Ψ, $\psi = 0$, with its upper boundary, $\psi = 1$. These are candidate offer curves. AB runs along a $Z = 0$ line, and EF runs along another $Z = 0$ line for most of its length. It is clear that AB will satisfy all of the conditions used in this paper and is a local optimum, but the position is less clear for the other two curves.

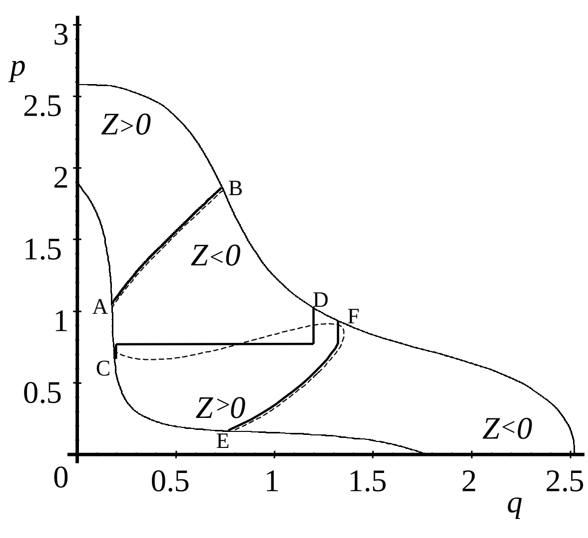

FIG. 4. *Candidate supply curves for the example.*

First we look at the CD offer curve. This starts with a vertical section from $(0.1858, 0.68244)$ to $(0.1858, 0.75685)$, then has a horizontal section to the point $(1.1972, 0.75685)$, and then finishes with a vertical section to hit the boundary of Ψ at the point $(1.1972, 1.0245)$. These points have been chosen so that the solution satisfies all of the conditions of Theorem 4. Each of the three sections has the property that the integral of Z along the section is zero, which is what is required for w to take the same value w_0 at the endpoints of each section. Moreover, the fact that the two vertical sections move from $Z > 0$ to $Z < 0$, while the horizontal section does the reverse, will ensure that w is no less than w_0 on the vertical sections and no more than w_0 on the horizontal section. Once it is decided to search for an offer curve of this general form, these conditions can be used to find the exact curve. Starting from different points on the $\psi = 0$ curve, we can let the w condition determine when to switch from vertical to horizontal and then back to vertical. We then iterate amongst possible starting positions to search for a solution which achieves a zero Z integral on the final vertical section; i.e., it makes $w = w_0$ at the point where the vertical section crosses the $\psi = 1$ curve. (All of the numerical calculations for this example were performed using Maple.)

The next step is to check the second order conditions of Theorem 5. We require that the integral of Z_q on the first vertical section be no greater than zero, and, in fact,

$$\int_{0.68244}^{0.75685} Z_q(0.1858, y)dy = -5.408 \times 10^{-3},$$

so this condition is satisfied. We also require that the integral of Z_p along the horizontal section be no less than zero, but

$$\int_{0.1858}^{1.1972} Z_p(x, 0.75685)dx = -0.16526,$$

so this condition fails. Moreover, the integral of Z_q on the last vertical section is greater than zero so that this condition fails as well. Finally, both the conditions (3) and (5) involving the value of Z at the corner points fail. So we know from the theorem that this solution is not a local optimum.

Next we consider the EF solution. The final vertical section of this is chosen in such a way that the integral of Z on this vertical section is zero. It starts at the point $(1.365, 0.82561)$ and moves vertically until meeting the $\psi = 1$ boundary at $(1.365, 0.90056)$ Since the rest of the curve is on the $Z = 0$ curve, the conditions of Anderson and Philpott will be satisfied. We check the conditions of Theorem 5. We have

$$\int_{0.82561}^{0.90056} Z_q(1.365, y)dy = -9.4287 \times 10^{-3} < 0$$

as required.

The next step is to check the sufficient conditions of Theorem 6. Most of the conditions of this theorem will hold trivially, but we need to check that

$$\int_{0.82561}^{0.90056} Z_q(1.365, y)dy < Z(1.365, 0.90056) - Z(1.365, 0.82561).$$

Now $Z(1.365, 0.82561) = 0$, and $Z(1.365, 0.90056) = -1.3136 \times 10^{-3}$, so this condition will hold.

Hence both the curves AB and EF are locally optimal: to choose between them, we must evaluate the objective function for each. In fact, the objective function value along the curve AB is 0.5183, while the value along EF is 0.4857. So the curve AB is the (global) optimum for this problem.

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