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ORIGINAL ARTICLE

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Optimal supply functions in electricity markets with option contracts and non-smooth costs

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Abstract In this paper we investigate the optimal supply function for a generator who sells electricity into a wholesale electricity spot market and whose profit function is not smooth. In previous work in this area, the generator's profit function has usually been assumed to be continuously differentiable. However in some interesting instances, this assumption is not satisfied. These include the case when a generator signs a one-way hedge contract before bidding into the spot market, as well as a situation in which a generator owns several generation units with different marginal costs. To deal with the non-smooth problem, we use the model of Anderson and Philpott, in which the generator's objective function is formulated as a Stieltjes integral of the generator's profit function along his supply curve. We establish the form of the optimal supply function when there are one-way contracts and also when the marginal cost is piecewise smooth.

1 Introduction

In this paper we discuss the following profit optimization model for a generator who sells electricity into a wholesale market:

$$\max_{\mathbf{s}} v(\mathbf{s}), \tag{1}$$

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where we define $v(\mathbf{s}) = \int_{\mathbf{s}} R(q, p) d\psi(q, p)$, and $\mathbf{s}$ is a non-decreasing supply curve. This is a type of constrained calculus of variations problem. Here $v(\mathbf{s})$ is the generator's expected profit when $\mathbf{s}$ is the graph of a supply function which describes the prices and quantities that the generator offers; R is the generator's profit function, that is, if the market clears at price p and the generator gets dispatched a quantity q, then $R(q, p)$ is the total profit for the generator; and $\psi(q, p)$ is the probability of the generator not being fully dispatched if it offers at price p with a quantity of q (i.e. the probability of being dispatched a quantity strictly less than q). We investigate the solution of (1) when R is piecewise continuously differentiable (piecewise C^1 in short).

This model was first introduced by Anderson and Philpott (2002). It applies in a pool system in which generators bid different quantities of power into the market at different prices, and then an independent system operator decides how actual demand is to be met by dispatching cheaper power first. In the simplest case, when all power is offered at the same node, there is a single price at which the market clears and this is the price paid to each generator for all the power it supplies. Generators compete with each other, with each generator offering a supply schedule, usually made up of tranches of power at different prices. Actual electricity markets have to operate within the constraints of an electricity network, and moreover have a number of features designed to ensure continuity of supply.

By assuming the continuous differentiability of the profit function R and of the ψ-function, Anderson and Philpott investigated first order necessary conditions for a general supply curve to be locally optimal. Anderson and Xu (2002) extended the discussion to consider both second order conditions and sufficient conditions for optimality.

In this model the generator aims to maximize $v(\mathbf{s})$, the expected profit. In some circumstances a stronger type of optimality can be achieved in which $\mathbf{s}$ is chosen so that the generator's profit is maximized for every possible realization of the uncertainties in the market. This idea was first used by Klemperer and Meyer (1989) to derive a Nash supply function equilibrium (SFE) in an oligopoly where every player faces uncertainty in demand. Green and Newbery (1992) noted that in electricity markets the uncertainty of demand is equivalent to daily time-varying demand, when a single supply schedule is applied over a whole day. They used the SFE model to investigate optimal supply functions in the electricity market in England and Wales. Since then, the SFE model has been widely used to study bidding behaviour in an electricity spot market (see for instance Bolle 1992; Baldick et al. 2004; Baldick and Hogan 2001; Rudkevich 1999; Rudkevich et al. 1998 and the references therein). In the original work on SFE (Rlemperer and Meyer 1989; Green and Newbery 1992), supply functions were required to be monotonically strictly-increasing and continuously differentiable. In recent work the SFE approach has been extended to non-decreasing and discontinuous supply functions (Bablick and Hogan 2001; Baldick et al. 2004; Rudkevich 2005).

In this paper we investigate the optimization problem (1) when the profit function R is piecewise continuously differentiable. We consider two ways in which the non-smoothness in R can occur.

The first case is associated with the contracts that a generator may sign before bidding in the spot market. Such contracts are used to hedge risks in the spot market; they are financial instruments and do not involve an actual transfer of power.

Some further information on different kinds of electricity derivatives can be found in Deng and Oren (2005). The most common types of contract that a generator may sign are: (a) a one-way (call option) contract which commits the generator to pay any positive difference between the pool price and the strike price to the holder of the contract; (b) a one-way (put option) contract which gives the generator the right to claim any positive difference between the pool price and the strike price; (c) a two-way contract that establishes a strike price for both generator and contract holder. If the pool price is above the strike price, the generator compensates the contract holder the difference, otherwise he claims the difference. Two-way contracts were discussed by Green (1999) and Newbery (1998) in the framework of SFE. More recently, Anderson and Xu used the model (1) to investigate the impact of two-way contracts on the generator's optimal supply function. The one-way contracts (both put and call options) add non-smoothness to the profit function R, while a two-way contract does not. In this paper we derive globally optimal supply curves that solve (1) when there are one-way contracts.

The other case which leads to piecewise smoothness of the profit function occurs when the generator's cost function is piecewise smooth. A generator may own a number of generation units and each unit has its own marginal cost. Consequently the generator's cost function becomes piecewise C^1. Baldick et al. (2004) discussed a linear SFE model under the circumstance of piecewise C^1 cost functions. In this paper, we show how the optimization approach can deal with a generator having a cost function with a general piecewise structure.

Much of the previous work in this area has been concerned with equilibrium analyses, with the aim of assessing the market behaviour when generators compete. Our analysis considers the problem from the point of view of a single generator, with only limited knowledge of the other generators. This is a significant limitation. Electricity markets operate as repeated games and we should expect that, in the absence of any form of collusion, market outcomes would approach a Nash equilibrium for this game. Nevertheless we claim our analysis is of interest.

Firstly we note that a game theory analysis carried out in the usual way assumes common knowledge of generator characteristics. Often, exact knowledge about cost structures for other generators is unlikely, and in particular in most markets generators will not know or be able to deduce the terms of contracts signed by other generators. Often such contracts are signed for short periods of time and the contract position of a generator varies from day to day. This makes it unlikely that a generator can learn about the contract position of another generator by observing their bidding strategy over time.

Second we observe that the analysis we give is valuable in ascertaining the difficulties involved in a full equilibrium analysis. With full information, some of the optimal policies that we derive have prices at which a non-zero quantity is offered, which means the supply function jumps in quantity at a particular price. This will have the effect of making the market distribution function discontinuous for other players. In this case an optimal solution may not exist – so that a Nash equilibrium cannot be found. Anderson and Xu (2005) show how ε-optimal solutions can be calculated in these circumstances.

We claim that the form of the optimal policies that we derive is of interest, even if it is not easy to translate this work into an equilibrium model.

2 Problem formulation and fundamentals

We need to fill in some background on the formulation of (1). For convenience, we will write generator A for the generator whose optimal policy we wish to find. We begin by considering generator A's profit function $R(q, p)$. Usually $R(q, p)$ has three components. First there is the cost of generating a quantity q of electricity which we write as $C(q)$, and which we shall assume is an increasing convex function. Second there is the amount pq that is paid to the generator through the market clearing mechanism. Third, there are payments that may be made as a result of hedging contracts entered into by the generator. The hedging contracts are financial instruments with the money paid under the contract tied to the pool price. We write $H(p)$ for the money added to (or deducted from, when $H(p) < 0$) the generator's profit when the market clears at price p. Thus

$$R(q, p) = qp - C(q) + H(p).$$

Throughout this paper, we assume that spot market demand is given by $D(p)+\epsilon$ where p is the price and ϵ is a random shock. We denote by g the density function of the distribution of the random shock and assume that g is well defined and has an interval support set $[\epsilon_1, \epsilon_2]$. We take demand to be continuously differentiable and strictly decreasing in price, that is, $D'(p) < 0$.

In the spot market each generator makes an offer into the market: in many cases this takes the form of an offer *stack*, being a set of quantities at increasing prices. We use a supply function, S, to describe the price–quantity relationship, so that $S(p)$ denotes generator A's offer of quantity at price p. The way that the market operates means that we must restrict supply functions to be increasing (though not necessarily strictly). The supply function is defined on $[0, p^M]$ where p^M is the price ceiling that operates in the spot market. In many markets it is possible to bid at a negative price, and spot prices in the middle of the night can occasionally be negative. This occurs when demand is very low and generators have to carry the cost of turning equipment on and off: sometimes it makes sense for a generator to pay for the privilege of being left on. However in our model, for simplicity, we normalize prices so that the lower limit is zero. There is no loss of generality in doing this so long as there is a finite price floor.

Generators in an electricity market offer energy into the market at prices they determine. We can express the quantity of electricity offered as a function of price p, $S(p)$ say, which is the amount to be delivered if the market clearing price is p. Note that in electricity markets, offers are required to be put in ascending order of price with those offered at lower price getting dispatched first. Therefore we require $S(p)$ to be non-decreasing. In some cases, we will use, for convenience, the graph of a supply function which is defined as: $\mathbf{s} = \{(S(p), p) : 0 \le p \le p^M\}$. The graph $\mathbf{s}$ is a curve in the quantity–price plane which we call the *supply curve*.

Anderson and Philpott (2002) prove that the expected profit for generator A if it offers a supply function $S(p)$ can be expressed as a Stieltjes line integral over the supply curve $\mathbf{s}$:

$$v(\mathbf{s}) = \int_{\mathbf{s}} R(q, p)\, d\psi(q, p),$$

where ψ is a continuous *market distribution function* for generator A.

To define the function ψ we need to introduce the residual demand function $D_{\mathrm{A}}(p)$ where the subscript A indicates that this is the residual demand for generator A. The residual demand at a price p is the demand which occurs at this price (assuming no demand shock) which will not be met by dispatch from the other generators. If we let $S_{\mathrm{B}}(p)$ be the aggregate supply function offered by the other generators, then $D_{\mathrm{A}}(p) = D(p) - S_{\mathrm{B}}(p)$. The residual demand function, given a particular random demand shock ϵ, is $D_{\mathrm{A}}(p) + \epsilon$.

The market distribution function $\psi(q, p)$ represents the probability that the residual demand function passes below the point (q, p) and is defined as

$$\psi(q, p) = \Pr(D_{\mathrm{A}}(p) + \epsilon < q) = G(q - D_{\mathrm{A}}(p)),$$

where $G(\cdot)$ is the distribution function of the random shock ϵ. An alternative definition of ψ arises from the observation that $\psi(q, p)$ is the probability that generator A is not fully dispatched if it offers a quantity q of power at price p.

Consequently

$$\psi'_p(q, p) = g(q - D_{\mathrm{A}}(p))(-D'_{\mathrm{A}}(p)),$$

and

$$\psi'_q(q, p) = g(q - D_{\mathrm{A}}(p)).$$

Clearly, provided that D_{A} is continuously differentiable, then ψ will also be continuously differentiable.

Note that $v(\mathbf{s})$ only depends on the part of curve $\mathbf{s}$ where $d\psi(S(p), p) \neq 0$ or equivalently $g(q - D_A(p)) \neq 0$. This defines a set of points (q, p) which we write as Ψ. Thus

$$\Psi = \{(q, p) : \epsilon_1 \leq q - D_A(p) \leq \epsilon_2\}.$$

In choosing a supply curve, generator A need only consider the part of this curve located in Ψ. We call Ψ the *effective response region* for generator A.

There are essentially three kinds of contracts that a generator may sign, and hence three forms for the function $H(p)$.

A two-way contract: a contract for differences. This is the type of contract which has been widely discussed in the literature (Anderson and Xu 2005; Green 1999; Gans et al. 1998; Newbery 1998). If the generator enters into a contract at a strike price f for a quantity Q, and the market clearing price in the spot market is p, then the generator will pay an amount $Q(p - f)$ to the contract holder if $p > f$. Conversely, if $p < f$, then the generator will get paid from the contract holder an amount of $Q(f - p)$. Consequently the profit function for the generator can be written as:

$$R_Q(q, p) \equiv qp - C(q) - Q(p - f). \tag{2}$$

A one-way contract: a call option. Generators may also enter into one-way contracts. If generator A sells a call option with strike price f, then generator A will pay $Q(p - f)$ to the contract holder if $p > f$, but no payment is made if $p < f$ (see e.g. von der Fehr and Harbord 1992). Consequently the generator's profit function can be written as:

$$R^1(q, p) \equiv qp - C(q) - Q \max(p - f, 0). \tag{3}$$

Obviously this function is not differentiable at $p = f$. Von der Fehr and Harbord (1992) investigated the impact of these contracts on a generator's bidding behaviour in a Bertrand type game.

A one-way contract: a put option. The other option for a one-way contract is that generator A buys a put option with strike price f. In this case generator A will be paid $Q(f - p)$ from the other party to the contract if $p < f$, but no payment is made if $p > f$ (see Von der fehr and Harbord 1992). Consequently, the generator's profit function takes the following form:

$$R^2(q, p) \equiv qp - C(q) - Q \min(p - f, 0). \tag{4}$$

As with R^1, R^2 will not be differentiable at $p = f$.

Notice that all of these contracts are financial instruments alone and do not involve the delivery of power directly from the generator to the retailer, say. Thus we assume, as is the case in New Zealand and Australia, for example, that all physical power is sold through the spot market. Note that in practice, a generator may have different types of contracts at different prices, but we simplify our discussion by considering a generator who holds a single contract.

All of the contracts we describe are hedging contracts, which are designed to protect the market participants from fluctuations in the spot market price. When there are no contracts a higher price in the spot market leads to a higher profit for the generator; with any of the contract positions above there is some reduction in the extent to which higher prices imply higher generator profits. Indeed when the contract quantity Q is sufficiently above the capacity of the generator, a higher spot market price might even lead to a lower generator profit.

There are alternative cases: for example a generator might *buy* a call option and hence receive $Q \max(p - f, 0)$, rather than pay this amount. But this would serve to *increase* the effect of market price fluctuations on the participants, and so would not serve to hedge the price risk. It is less likely that a generator would enter such a contract unless it were seeking to magnify its market power. Similarly a generator selling a put option, rather than buying it, would not obtain any protection from price movements. We will not analyse these alternative one-way contracts here, though in fact the techniques of this paper can be applied.

Our focus in this section is to derive optimal supply functions for generator A in the case that it signs one of the one-way contracts before bidding into the spot market. The main difficulty here is that the profit functions (both R^1 and R^2) for the generator are not differentiable at the strike price of a contract f. Note that both R^1 and R^2 are a combination (through maximum or minimum) of R_Q and R_0 (i.e. R_Q with $Q = 0$). This motivates us to construct an optimal supply function for the problem with a one-way contract through combining the optimal supply functions with respect to R_0 and R_Q.

3 Optimal supply function with two-way contracts

We begin by considering the case when generator A holds a two-way contract and wishes to submit an optimal supply curve to the spot market in order to maximize its expected profit. Thus the generator needs to solve the following problem

$$\max_{\mathbf{s}} v(\mathbf{s}) \equiv \int_{\mathbf{s}} R_Q(q, p) d\psi(q, p), \tag{5}$$

where **s** is a non-decreasing curve. This problem was investigated by Anderson and Xu (2005) and we review some of their results.

Let q^M denote denote A's generation capacity. Suppose that the generator has signed contracts with a total quantity of Q and then offers into the spot market a supply curve s. To ease the analysis, we assume that $0 < Q \leq q^M$, in other words the generator does not speculate beyond his capacity. Since generator A cannot offer more than q^M and the highest offer price cannot exceed p^M, any supply curve from generator A must be located in the region $[0, q^M] \times [0, p^M]$. On the other hand, as we discussed above, the expected profit $v(\mathbf{s})$ only depends on the part of curve **s** located within Ψ. Therefore, our discussion of optimality conditions for **s** is focused on the region $\Psi \cap [0, q^M] \times [0, p^M]$. For convenience, let

$$\Phi \equiv \Psi \cap [0, q^M] \times [0, p^M].$$

Let Φ^o denote the interior of the set Φ.

We shall assume that $D_A(p)$ is strictly decreasing and continuously differentiable on $(0, p^M)$ and $C(q)$ is continuously differentiable on $(0, q^M)$.

The characteristics of the residual demand, $D_A(p)$, depend on the nature of the supply curves offered by the other generators. We do not assume that these other offer curves are known to generator A. There is uncertainty day-to-day as circumstances change, and, for most markets of this sort, full information on competitor's bids is not available to participants. In these circumstances a generator needs to estimate the residual demand function from limited information and it is natural to use a smooth estimate for $D_A(p)$. One could go further and make use of Bayesian estimation techniques to update estimates on the basis of information observed. Anderson and Philpott (2003) adopt this approach in making smooth estimates of the market distribution function ψ.

Write a candidate supply curve for generator A as $\mathbf{s} = \{(S(p), p) : 0 \leq p \leq p_M, 0 \leq S(p) \leq q_M\}$. Anderson and Xu (2005) prove that if S is optimal and strictly increasing and continuously differentiable, then

$$(p - C'(S(p)))(-D'_A(p)) - S(p) + Q = 0, \tag{6}$$

for $(S(p), p) \in \Phi^o$. The equation coincides with those obtained by Green (1999) and Newbery (1998) using a SFE model in their investigation of the impact of two-way contracts on a generator's bidding strategy in the spot market.

This equation can be derived as follows: let **s** be the supply curve corresponding to supply function $S(p)$ and let $\widetilde{\mathbf{s}}$ be a perturbed supply curve of **s**. The difference between the expected profits based on these two curves is

$$v(\widetilde{\mathbf{s}}) - v(\mathbf{s}) = \int_{\widetilde{\mathbf{s}}} R_Q(q, p) d\psi(q, p) - \int_{\mathbf{s}} R_Q(q, p) d\psi(q, p).$$

Using the classical Green's theorem (Kaplan 1962), we can show that the right hand side of this expression equals $\int\int_{\mathcal{A}} Z_Q(q,p)dqdp$, where

$$\begin{aligned} Z_Q(q,p) &\equiv (R_Q)'_q(q,p)\psi'_p(q,p) - (R_Q)'_p(q,p)\psi'_q(q,p) \\ &= g(q - D_{\mathrm{A}}(p))\left[(p - C'(q))(-D'_{\mathrm{A}}(p)) - q + Q\right] \end{aligned}$$

and $\mathcal{A}$ is the area within the effective response region Ψ surrounded by $\widetilde{\mathbf{s}}$ and $\mathbf{s}$. From this observation it is not hard to see that a necessary condition for $\mathbf{s}$ to be optimal (given that it is strictly increasing) is that $Z_Q(q,p) = 0$, and this gives (6).

We will also be interested in constructing the unique global optimal solution. To achieve this we need to make the following assumption for particular values of Q.

Assumption 3.1 For a given Q, equation (6) defines a unique strictly increasing function $S(p)$ for $p \in [0, p_M]$.

This assumption might hold for one value of Q, but not for another. We will need to use the assumption for values of Q in the range $0 \le Q \le q^M$.

This assumption is equivalent to the function Z_Q taking the value zero on a single strictly increasing curve in the q,p-plane (the curve $q = S(p)$). Moreover since

$$\left(\frac{\partial}{\partial q}\right)((p - C'(q))(-D'_{\mathrm{A}}(p)) - q + Q) = C''(q)D'_{\mathrm{A}}(p) - 1 < 0$$

we can see that the Z_Q is positive to the left of the $q = S(p)$ curve and negative to the right of it (using the fact that $g(.)$ is non-negative). Hence under this assumption the offer curve $S(p)$ divides the region Φ into two with $Z_Q > 0$ above and to the left of the offer curve, and $Z_Q < 0$ below and to the right of the offer curve.

Assumption 3.1 is not very restrictive in practice and can be easily checked for specific examples. Anderson and Xu (2005) have shown that this assumption will hold if the estimate of residual demand D_{A} is sufficiently close to affine. In particular Assumption 3.1 is guaranteed if both D_{A} and C are twice continuously differentiable, $(Q, C'(Q)) \in [0, q^M] \times (0, p^M)$, and

$$\frac{-D'_{\mathrm{A}}(p)^2}{Q} \le D''_{\mathrm{A}}(p) \le \frac{-D'_{\mathrm{A}}(p)}{p}. \tag{7}$$

In the case when $Q = 0$, we only need the second inequality.

Lemma 3.1 *(Anderson and Xu, 2005, Proposition 3.1) Suppose that Assumption 3.1 holds for a given contract quantity $Q \in [0, q^M]$. Let $(q_{\mathrm{L}}, p_{\mathrm{L}})$ and $(q_{\mathrm{U}}, p_{\mathrm{U}})$ denote respectively the points at which the graph of $S(p)$ intercepts the lower and upper boundaries of Φ. Then we can represent the curve as a supply function $S(p)$ defined on $(p_{\mathrm{L}}, p_{\mathrm{U}})$, or equivalently an offer function $T(q) = S^{-1}(q)$ defined on $(q_{\mathrm{L}}, q_{\mathrm{U}})$. We extend T by defining $T(q) = 0$, for $0 \le q \le q_{\mathrm{L}}$, and $T(q) = p^M$, for $q_{\mathrm{U}} \le q \le q^M$. Then T is the unique optimal offer function up to changes outside Φ.*

We have slightly extended the result of Anderson and Xu by allowing Q to take the extreme values 0 and q^M. A careful reading of (Anderson and Xu, 2005, Proposition 3.1) shows that this change does not affect the proof of the result. We will make use of the result for $Q = 0$ in our discussion below.

It is convenient to write the optimal supply function defined in Lemma 3.1 as $S(p, Q)$ to show explicitly its dependence on the contract quantity Q. Similarly we write p_L, q_L, p_U, q_U as $p_L(Q), q_L(Q), p_U(Q), q_U(Q)$ to show explicitly their dependence on Q.

Based on Lemma 3.1, we can obtain two optimal supply functions $S(p, Q)$ and $S(p, 0)$ corresponding respectively to the profit functions $R_Q(q, p)$ and $R_0(q, p)$. We are interested in the case that the corresponding supply curves cross the $p = f$ line in the region Ψ at point $(S(f, 0), f)$ and point $(S(f, Q), f)$. Obviously $S(f, Q) \neq S(f, 0)$ when $Q > 0$. The following lemma shows that $S(f, Q) > S(f, 0)$ and the two supply functions do not cross within the Φ region. All proofs are given in the Appendix.

Lemma 3.2 *If $Q > 0$ and both $S(p, Q)$ and $S(p, 0)$ are strictly increasing functions then*

$$S(p, Q) > S(p, 0), \text{ for all } p \in [p_L(Q), p_U(Q)] \bigcap [p_L(0), p_U(0)].$$

The result shows that the optimal supply curve when there is a contract is located strictly above (in the p, q-plane) the optimal supply curve when there is no contract. From this it follows that, for a given quantity, the generator should sell at a higher price when there is no two-way contract. Results of this form are well known in this literature (e.g. Newbery 1998).

Example 3.1 We give an example to compare optimal supply function with two way contracts and without contract. The example is a variation of (Anderson and Xu, 2005, Example 3.3). We suppose that a generator has a two way contract of quantity 1.5 at strike price $f = 1$.

Suppose that the generator faces a residual demand $D_A(p)+\epsilon$, where $D_A(p) = 0.5 \log(1 + p) - p$, and ϵ has a uniform distribution over the interval [0.5, 4]. The generator's cost for producing a quantity q of electricity is given by $C(q) = q^2/2$. There is a price cap of $p^M = 5$. We use Lemma 3.1 to work out the optimal supply function for the generator.

The effective response region is

$$\Psi = \big\{(q, p) : q \geq 0, 0 \leq p \leq 5 : 0.5 - p \\ + 0.5 \log(1 + p) \leq q \leq 4 - p + 0.5 \log(1 + p)\big\}.$$

The first step is to calculate $S(p, 0)$ and $S(p, 1.5)$. We have,

$$D'_A(p) = \frac{1}{2(1 + p)} - 1.$$

Substituting $D'_A(p)$, $C'(q) = q$ and $Q = 1.5$ into (6), we have

$$(p - q)\left(1 - \frac{1}{2(1 + p)}\right) - q + 1.5 = 0.$$

Solving for q in terms of p in this equation, we obtain

$$S(p, 1.5) = \frac{2p^2 + 4p + 3}{4p + 3}.$$

Similarly, we can obtain

$$S(p, 0) = \frac{2p^2 + p}{4p + 3}.$$

Since the equations for $S(p, 0)$ and $S(p, 1.5)$ have unique solutions in increasing curves, Assumption 3.1 holds for both $Q = 0$ and $Q = 1.5$.

Next we identify optimal supply curves for $Q = 0$ and $Q = 1.5$. We consider the $Q = 0$ case first.

We calculate $p_L(0)$ and $q_L(0)$. We solve

$$0.5 - p + 0.5 \log(1 + p) = \frac{2p^2 + p}{4p + 3},$$

and obtain $p_L(0) = 0.50247$; hence $q_L(0) = S(p_L(0), 0) = 0.20109$. This identifies the point where the supply curve $S(p, 0)$ crosses the lowest residual demand curve at the boundary of the effective region Ψ. Similarly we can obtain $p_U(0) = 1.5055$; hence $q_L(0) = S(p_U(0), 0) = 3.2137$.

Based on these data, we can describe the optimal curve. It starts from (0.2019, 0) and goes vertically to the point (0.2019, 0.50247) to enter the Ψ region. Then it follows the $S(p, 0)$ curve until it crosses the upper boundary of Ψ at (1.5055, 3.2137). Finally it goes vertically up to the price cap at (1.5055, 5). We can express it as follows

$$S_0(p) = \begin{cases} 0.20109, & p \in [0, 0.50247), \\ (2p^2 + p)/(4p + 3), & p \in [0.50427, 3.2137), \\ 1.5055, & p \in [3.2137, 5]. \end{cases}$$

We now consider the $Q = 1.5$ case. Since $S(p, 1.5)$ equals 0 at $p = 0$, the curve $S(p, 1.5)$ does not cross the lower boundary of Ψ. The optimal supply curve starts from (1, 0) and follows the $S(p, 1.5)$ curve until it crosses the upper boundary of Ψ at point (2.0201, 2.6236). Finally it goes vertically to the price cap at (2.0201, 5). The optimal supply function can be written as

$$S_2(p) = \begin{cases} (2p^2 + 4p + 3)/(4p + 3), & p \in [0, 2.6236), \\ 2.0201, & p \in [2.6236, 5]. \end{cases}$$

This is illustrated in Figure 1, where the two optimal solutions are shown. The dashed line shows the marginal cost curve which is below the optimal offer without contracts (at least over the range of possible dispatch), but intersects the optimal offer for the $Q = 1.5$ case at the contract quantity. It is important to note that the strike price $f = 1$ is not reflected in $S_2(p)$. In other words, if $f = 2$, then we still have the same optimal supply curve so long as Q is unchanged. We will use this result in later examples in which we consider one way contracts (see Examples 4.1 and 5.1).

Note also that the optimal curve of $S_0(p)$ is above that of $S_2(p)$ in the q, p-plane, which illustrates Lemma 3.2.

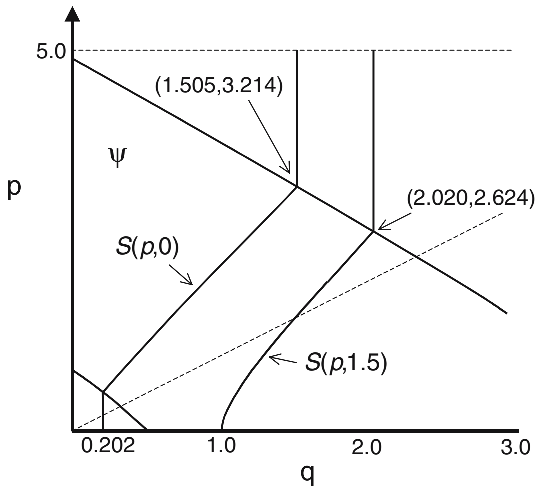

Fig. 1 Optimal supply curves for Example 3.1

4 One-way contract: call option

As we discussed above, the generator's profit function when it has sold a call option is given by R^1 defined as in (3). The optimization problem for the generator becomes

$$\max_{\mathbf{s}} v(\mathbf{s}) \equiv \int_{\mathbf{s}} R^1(q, p) d\psi(q, p), \tag{8}$$

where **s** is a non-decreasing curve. Observe that R^1 is a combination of R_0 and R_Q, that is, when the market clears below the strike price f, $R^1(q, p) = R_0(q, p)$ and when the market clears above the strike price f, $R^1(q, p) = R_Q(q, p)$. This indicates that the solution of (8) may be some type of combination of $S(p, Q)$ and $S(p, 0)$. Note that neither $S(p, Q)$ nor $S(p, 0)$ is a solution of (8). In fact, there is no monotonically strictly increasing and continuously differentiable optimal supply curve for (8). We need to consider a monotonic non-decreasing supply function. The following theorem shows how a unique global optimal supply function of (8) can be constructed.

Theorem 4.1 *Suppose that Assumption 3.1 is satisfied for both contract quantities Q and 0. Let $S(p, 0)$, $q_L(0)$, $p_L(0)$, $S(p, Q)$, $q_U(Q)$ and $p_U(Q)$ be defined as in Lemma 3.1. Define*

$$S(p) \equiv \begin{cases} q_L(0), & p \in [0, p_L(0)], \\ S(p, 0), & p \in [p_L(0), f), \\ S(p, Q), & p \in [f, p_U(Q)], \\ q_U(Q), & p \in [p_U(Q), p_M]. \end{cases} \tag{9}$$

Then the graph of S is the unique optimal solution of (8).

Theorem 4.1 shows that the optimal supply curve contains a horizontal segment in the q, p-plane. In other words, the optimal supply function S jumps at the strike price $p = f$. The following lemma concerns the size of this jump, which is the amount of power offered at the strike price f.

Proposition 4.1 *Under the conditions of Theorem 4.1, if the optimal supply function S offers a quantity W at the strike price f, then $W \leq Q$ with equality if and only if the generator's marginal cost is constant over the range of quantities q for which the supply function offers at the strike price f.*

Proof From Theorem 4.1 we know that $W = S(f, Q) - S(f, 0)$. From the proof of Lemma 3.2 we have

$$W = (C'(S(f, 0)) - C'(S(f, Q)))(-D'_{\mathrm{A}}(f)) + Q.$$

The result follows since $-D'_{\mathrm{A}}(f) > 0$ and C' is increasing. For equality we require $C'(S(f, 0)) = C'(S(f, Q))$, which is equivalent to the condition of the lemma. □

A well-known observation is that, when a generator signs a two-way contract, it should sell in the spot market at the marginal cost for $q = Q$, below its marginal cost for $q < Q$, and above the marginal cost for $q > Q$. Thus the curve $S(p, Q)$ passes through the point $(Q, C'(Q))$ in the q, p -plane. From a market perspective, the generator needs to sell the uncontracted part at a price above the marginal cost to make a profit, while it does not need to do so for the contracted part. See Green (1999, Proposition 1) and also a discussion in Anderson and Xu (2005). The following result shows that this is not going to be true when the generator has sold a call option, in that the generator's offer price may be greater than its marginal cost at any level of its offer.

Proposition 4.2 *Suppose that $f \geq C'(Q)$ and the conditions of Theorem 4.1 hold. Then the optimal supply curve is above the marginal cost curve, that is, for $p \in [p_{\mathrm{L}}(0), p_{\mathrm{U}}(Q)]$,*

$$S(p) \leq (C')^{-1}(p).$$

Proof By definition for $p \leq f$, $S(p) = S(p, 0)$ and for $p > f$, $S(p) = S(p, Q)$. From (6), since $D'_{\mathrm{A}}(p) < 0$ for all p, and $S(p, 0) \geq 0$, it follows that $S(p, 0) \leq (C')^{-1}(p)$. On the other hand, for $p > f > C'(Q)$, if $S(p, Q) \leq Q$, then $p > C'(S(p, Q))$ and we have a contradiction from (6). Hence $S(p, Q) > Q$ and so $p > C'(S(p, Q))$, again using (6), i.e. $S(p, Q) < (C')^{-1}(p)$ for any $p > f$. The conclusion follows. □

The lemma shows that if a generator has sold a call option contract, and the strike price is greater than the marginal cost for generating the contracted quantity of electricity, then the generator's sale price will be greater than his marginal cost at any level of output.

In some cases a generator may sell one-way call option contracts at a number of different prices. In this case we may use a construction like that of Theorem 4.1 to obtain an optimal supply function which jumps at each strike price.

Example 4.1 We give an example to illustrate Theorem 4.1. The example is a variation of Example 3.1. We suppose that a generator has sold a call option of quantity 1.5 at strike price $f = 1$.

The effective response region is the same, that is,

$$\Psi = \{(q, p) : q \geq 0, 0 \leq p \leq 5 : 0.5 - p + 0.5 \log(1 + p) \leq q \leq 4 - p + 0.5 \log(1 + p)\}.$$

We use $S(p, 0)$ and $S(p, 1.5)$ which are obtained in Example 3.1 to construct an optimal supply function following Theorem 4.1.

Since the equations for $S(p, 0)$ and $S(p, 1.5)$ have unique solutions in increasing curves, Assumption 3.1 holds for both $Q = 0$ and $Q = 1.5$.

The optimal supply curve is shown as the solid line in Figure 2. It starts from $(0.2019, 0)$ and goes vertically to the point $(0.2019, 0.50247)$ to enter the Ψ region. Then it follows the $S(p, 0)$ curve until it reaches the call option strike price $p = 1$ at which point $q = 0.42857$. Using Theorem 4.1, the optimal curve then goes horizontally to the point $(1.2857, 1)$ which is on the $S(p, 1.5)$ curve. The optimal curve then follows $S(p, 1.5)$ until it crosses the upper boundary of Ψ at point $(2.0201, 2.6236)$. Finally it goes vertically to the price cap at $(2.0201, 5)$. The optimal supply function can be written as

$$S(p) = \begin{cases} 0.20109, & p \in [0, 0.50247), \\ (2p^2 + p)/(4p + 3), & p \in [0.50427, 1), \\ (2p^2 + 4p + 3)/(4p + 3), & p \in [1, 2.6236), \\ 2.0201, & p \in [2.6236, 5]. \end{cases}$$

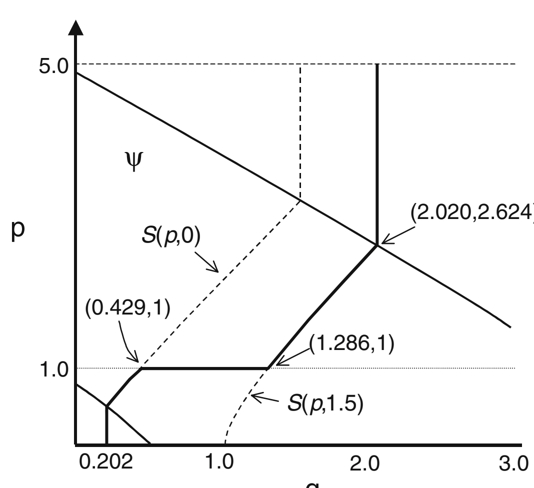

Fig. 2 Optimal supply curves for Example 4.1

5 One way contract: put option

Now we discuss the case that generator A's profit function is given by (4). The generator needs to solve the following maximization problem:

$$\max_{\mathbf{s}} v(\mathbf{s}) \equiv \int_{\mathbf{s}} R^2(q, p) d\psi(q, p). \tag{10}$$

Observe that R^2 is a combination of R_0 and R_Q. As in the R^1 case, we would like to use $S(p, Q)$ and $S(p, 0)$ to construct the optimal solution of (10). The construction will be more complex in this case than it was for a call option. Instead of switching from $\mathbf{s}_0$ (the graph of $S(p, 0)$) to $\mathbf{s}_Q$ (the graph of $S(p, Q)$), we need to make the switch the other way round. Since $S(f, Q)$ is greater than $S(f, 0)$, moving from supply curve $\mathbf{s}_Q$ to $\mathbf{s}_0$ at price f will result in non-monotonicity of the supply function as a whole. So we need to connect the two supply curves with a vertical line segment in the (q, p)-plane to ensure the optimal supply function is monotonic increasing.

Let

$$Z(q, p) = \begin{cases} Z_Q(q, p), & p \leq f, \\ Z_0(q, p), & p > f. \end{cases}$$

Note that the value of $Z(q, f)$ will not be important in the following discussion.

The lemma below is derived from Theorem 3.1 in Anderson and Xu (2002), which was in turn derived from an earlier result of Anderson and Philpott (2002).

Lemma 5.1 *Suppose that* $\mathbf{s} = \{x(t), y(t), 0 \leq t \leq T\}$ *is an increasing continuous offer curve which is optimal. Suppose that there exist m numbers* $0 \leq t_1 < t_2 < \cdots < t_m \leq T$ *with* $0 < x(t) < q_M$ *and* $0 < y(t) < p_M$ *for* $t_1 < t < t_m$ *and such that on each section* (t_{i-1}, t_i), $i = 2, \ldots, m$, $\mathbf{s}$ *is either strictly increasing in both components, or horizontal, or vertical, with different characteristics in successive segments. On segments for which* $\mathbf{s}$ *is strictly increasing in both components,* $Z(x(t), y(t)) = 0$. *On horizontal or vertical segments,*

$$\int_{t_{i-1}}^{t_i} Z(x(\tau), y(\tau))(x'(\tau) + y'(\tau))d\tau = 0, \quad i = 2, \ldots, m,$$

unless the segment is horizontal and at the strike price, f.

Proof The proof is by perturbations of the offer curve $\mathbf{s}$. All the arguments given by Anderson and Philpott carry through provided that when an area integral of Z uses an area crossing the strike price, f, the integral is carried out in two parts (above and below the strike price) to allow the application of Green's theorem. We omit the details here. □

Now we are in a position to provide a characterization of the optimal supply curve around the strike price f.

Theorem 5.1 *If Assumption 3.1 is satisfied for both contract quantities Q and* 0*, then there is an optimal supply function, and if $S(p)$ is optimal then there are constants ζ, $\delta_1 > 0$, $\delta_2 > 0$, such that $S(p) = \zeta$ for $p \in \{f - \delta_1, f + \delta_2\}$.*

Note that a vertical section like this in the (q, p) plane implies that none of the generator's capacity is bid at prices close to the strike price of the contract.

For the call option we established a stronger result, with a unique global optimal solution under the same assumption. Unfortunately to obtain a similar result for the put option case requires stronger conditions, which are technically complicated and almost impossible to verify.

The theorem shows that an optimal solution when there is a put option involves a vertical segment. Lemma 5.1 shows that the line integral of Z along a vertical segment is zero, and this can often be used to determine the position of this vertical segment.

Example 5.1 We give an example to illustrate Theorem 5.1, which is a variation of Example 3.1. Suppose that the generator has bought a put option for quantity 1.5 at strike price $f = 2$. We have the same $S(p, 0)$ and $S(p, 1.5)$ as before, that is,

$$S(p, 1.5) = \frac{2p^2 + 4p + 3}{4p + 3},$$

and

$$S(p, 0) = \frac{2p^2 + p}{4p + 3}.$$

The difference is that here the two curves are connected by a vertical segment rather than a horizontal segment in the (q, p)-plane.

The optimal supply curve is shown in Figure 3. Since $S(p, 1.5)$ equals 0 at $p = 0$, the curve $S(p, 1.5)$ does not cross the lower boundary of Ψ. The optimal

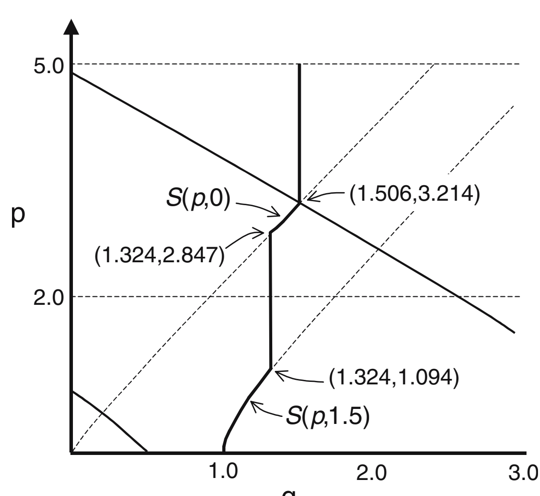

Fig. 3 Optimal supply curve for Example 5.1

supply curve starts from $(1, 0)$ and follows the $S(p, 1.5)$ curve until it reaches the point $(1.3245, 1.0939)$. Then it moves vertically to the point $(1.3245, 2.8469)$ on the $S(p, 0)$ curve. These points are chosen so that

$$\int_{1.0939}^{2.8469} Z(1.3245, y)dy = 0.$$

The optimal supply curve then follows the $S(p, 0)$ curve until it crosses the upper boundary of Ψ at $(1.5055, 3.2137)$. Finally it goes vertically up to the price cap at $(1.5055, 5)$. The optimal supply function can be written as

$$S(p) = \begin{cases} (2p^2 + 4p + 3)/(4p + 3), & p \in [0, 1.0939), \\ 1.3245, & p \in [1.0939, 2.8469), \\ (2p^2 + p)/(4p + 3), & p \in [2.8469, 3.2137), \\ 1.5055, & p \in [3.2137, 5]. \end{cases}$$

6 Piecewise continuously differentiable cost functions

In this section we consider a case where the cost function is non-smooth. This happens when a single generator owns several generation units with different marginal costs and its overall marginal cost function is discontinuous from one generation unit to another. Here we assume that a generator's marginal cost is piecewise continuous, leading to a piecewise smooth cost function.

In most discussions in the literature (Anderson and Philpott 2002; Anderson and Xu 2005; Baldick et al. 2004; Green 1999; Green and Newbery 1992; Borenstein 2001; Newbery 1998; Rudkevich 1999) the generators' cost functions are assumed to be smooth. In some cases, the marginal cost function is even assumed to be affine. Baldick and Hogan (2001) pointed out that this assumption does not capture jumps in marginal cost from, say, coal, to gas technology, and there may also be jumps simply because of the different ages of the generation units.

In order to focus on the impact of non-smoothness in the cost function, we ignore the possible non-smoothness in the profit function caused by other factors such as one way contracts. Therefore, we take generator A's profit function to be

$$R(q, p) = qp - C(q),$$

where

$$C(q) = \begin{cases} C_1(q) & 0 \le q \le q_1, \\ C_2(q) & q_1 < q \le q_2, \\ \vdots & \\ C_n(q) & q_{n-1} < q \le q_n, \\ C_{n+1}(q) & q_n < q \le q^M. \end{cases} \tag{11}$$

We assume that for $i = 1, \ldots, n$

$$C'_{i+1}(q_i^+) > C'_i(q_i), \tag{12}$$

where

$$C'_{i+1}(q_i^+) = \lim_{\delta \downarrow 0} C'_{i+1}(q_i + \delta).$$

The inequality (12) implies that there is a jump in marginal cost in moving from generation unit i to generation unit $i + 1$.

As before, our problem is

$$\max_{\mathbf{s}} v(\mathbf{s}) \equiv \int_{\mathbf{s}} R(q, p) d\psi(q, p), \tag{13}$$

where the function R now incorporates jumps in marginal cost. Let $S_i(p)$ denote the optimal supply function which corresponds to C_i. We will use a similar approach to that we used in the previous section; that is we use the optimal supply functions S_i, corresponding to different cost functions C_i, to construct a global optimal supply function.

We let p_i^l and p_i^u be the prices at which the curve defined by S_i crosses the boundaries of the region between q_{i-1} and q_i . Thus $S_i(p_i^l) = q_{i-1}$ and $S_i(p_i^u) = q_i$. We can use (6) to show that p_i^l and p_i^u satisfy the following equations:

$$\left(p_i^l - C'(q_{i-1}^+)\right) D'_{\mathrm{A}}\left(p_i^l\right) + q_{i-1} = 0,$$
$$\left(p_i^u - C'(q_i)\right) D'_{\mathrm{A}}\left(p_i^u\right) + q_i = 0.$$

Now we are in a position to use S_i to construct an optimal supply function over $[0, p^M]$ for the nonsmooth cost function $C(\cdot)$.

For simplicity, assume the supply curve of S_1 and S_{n+1} intercept the boundary of Ψ respectively at $(p_{\mathrm{L}}, q_{\mathrm{L}})$ and $(p_{\mathrm{U}}, q_{\mathrm{U}})$. We define the following supply function (see Fig. 4):

$$S(p) \equiv \begin{cases} q_{\mathrm{L}}, & p \in [0, p_{\mathrm{L}}], \\ S_1(p) & p \in [p_{\mathrm{L}}, p_1^u] \\ q_i, & p \in [p_i^u, p_{i+1}^l],\ i = 1, \ldots, n, \\ S_i(p), & p \in [p_i^l, p_i^u],\ i = 2, \ldots, n, \\ S_{n+1}(p), & p \in [p_{n+1}^l, p_{\mathrm{U}}], \\ q_{\mathrm{U}}, & p \in [p_{\mathrm{U}}, p^M]. \end{cases} \tag{14}$$

The following theorem shows that this supply function is globally optimal. First we define $Z_i(q, p)$ to be the function Z when costs are given by C_i, $i = 1, \ldots, n+1$. Thus

$$Z_i(q, p) = g(q - D_{\mathrm{A}}(p)) \left[(p - C'_i(q))(-D'_{\mathrm{A}}(p)) - q\right].$$

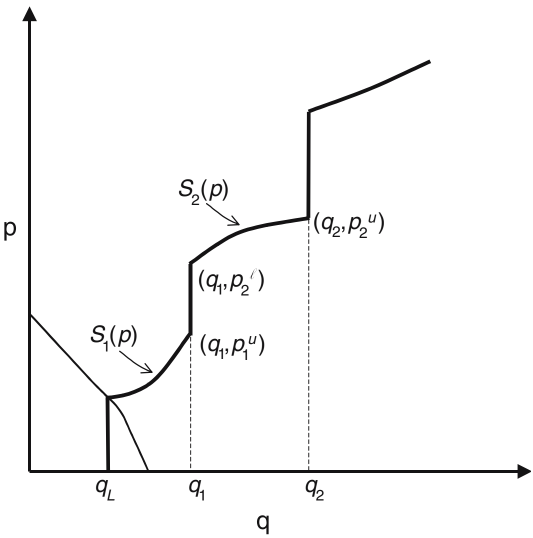

Fig. 4 Optimal supply curve when marginal cost has discontinuities

Theorem 6.1 *Suppose that, for a given contract quantity Q, Assumption 3.1 is satisfied in each region* $q_{i-1} < q \leq q_i$, $i = 1, 2, \ldots n+1$, *so that the supply functions* $S_i(p)$ *are uniquely defined. Let* $S(p)$ *be defined as in (14). Then* $S(p)$ *is the unique optimal solution of (13).*

We can express the optimal supply curve as a function of q, which we write $T(q)$. Then $T(q)$ is strictly increasing and is discontinuous at q_i, $i = 1, 2, \ldots, n$, where the marginal cost function C' jumps. This is consistent with what we would expect: when the marginal cost jumps upwards, the generator should increase its offer price.

Example 6.1 We give an example to illustrate Theorem 6.1. The example is a variation of Example 3.1. We suppose that a generator has a piecewise continuously differentiable cost function as follows

$$C(q) = \begin{cases} \frac{1}{20}q^2, & 0 \leq q < 1, \\ \frac{1}{2}q^2 - 0.45, & 1 \leq q. \end{cases}$$

Obviously $C(q)$ is piecewise smooth and

$$C'(q) = \begin{cases} \frac{1}{10}q, & 0 \leq q < 1, \\ q, & 1 \leq q. \end{cases}$$

is discontinuous at $q = 1$.

Suppose that the generator faces a residual demand $D_A(p)+\epsilon$, where $D_A(p) = 0.5\log(1+p) - p$, and ϵ has a uniform distribution over the interval [0.5, 4]. There is a price cap of $p^M = 5$. We use Theorem 6.1 to work out the optimal supply function for the generator.

The effective response region is

$$\Psi = \{(q,p) : q \geq 0, 0 \leq p \leq 5 : 0.5 - p + 0.5\log(1+p) \leq q \leq 4 - p + 0.5\log(1+p)\}.$$

The first step is to calculate optimal supply functions $S_1(p)$ and $S_2(p)$ which corresponds to $C(q) = \frac{1}{20}q^2$ and $C(q) = \frac{1}{2}q^2 - 0.45$ respectively. We have,

$$D'_{\mathrm{A}}(p) = \frac{1}{2(1+p)} - 1.$$

Substituting $D'_{\mathrm{A}}(p)$, $C'(q) = \frac{1}{20}q$ into (6), we have

$$(p - 0.1q)\left(1 - \frac{1}{2(1+p)}\right) - q = 0.$$

Solving for q in terms of p in this equation, we obtain

$$S_1(p) = 10p\frac{2p+1}{22p+21}.$$

Similarly, we can obtain

$$S_2(p) = p\frac{2p+1}{4p+3}.$$

Since the equations for $S_1(p)$ and $S_2(p)$ have unique solutions in increasing curves, Assumption 3.1 holds (for $Q = 0$).

Next we identify optimal supply curves. We start with the point $(q^1_{\mathrm{L}}(0), p^1_{\mathrm{L}}(0))$ at which the optimal supply curve enters the Ψ region. This is the point where the $S_1(p)$ curve crosses the lowest residual demand curve at the boundary of the effective region Ψ . To calculate $p^1_{\mathrm{L}}(0)$ and $q^1_{\mathrm{L}}(0)$ we solve

$$0.5 - p + 0.5\log(1+p) = 10p\frac{2p+1}{22p+21},$$

and obtain $p^1_{\mathrm{L}}(0) = 0.420$; hence $q^1_{\mathrm{L}}(0) = S_1(p^1_{\mathrm{L}}(0)) = 0.255$. Similarly we can identify a point $(1.584, 3.214)$ where the $S_2(p)$ curve crosses the highest residual demand curve at the boundary of the effective region Ψ.

Now we can describe the optimal supply curve in the q, p-plane (shown in Figure 5). It starts from $(0.255, 0)$ and goes vertically to the point $(0.255, 0.420)$ to enter the Ψ region. Then it follows the $S_1(p)$ curve until it crosses $q = 1$ line at $(1, 1.368)$. It jumps up to the $S_2(p)$ curve and then follows this until it crosses the upper boundary of Ψ at $(1.584, 3.214)$. Finally it goes vertically up to the price cap at $(1.584, 5)$. We can express this as follows

$$S(p) = \begin{cases} 0.255, & p \in [0, 0.420), \\ 10p\frac{2p+1}{22p+21}, & p \in [0.420, 1.368), \\ 1, & p \in [1.368, 2.186], \\ p\frac{2.0p+1.0}{4.0p+3.0} & p \in (2.186, 3.214] \\ 1.584, & p \in (3.214, 5]. \end{cases}$$

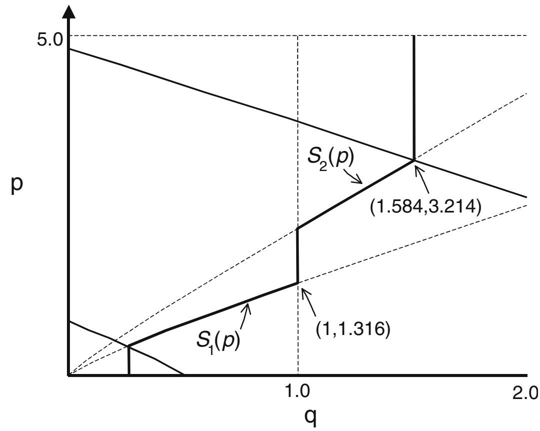

Fig. 5 Optimal supply curve for Example 6.1

7 Discussion

We have shown how to construct optimal supply functions in the presence of put or call options, or when generators have generation units with different cost characteristics. It is important not to miss the fundamental characteristics of the optimal supply functions. Whenever a generator has sold a call option at a strike price f, some of the generator's capacity should be bid at the price f (unless all the capacity is bid below f, or all capacity is bid above f.) On the other hand when a generator has bought a put option at strike price f, then there is a band around f of prices such that none of the generator's capacity is bid within this band. It is not surprising that a call option should lead to bidding at the strike price, it is less intuitive that a put option deters bidding near the strike price.

The same methods can be used to analyse the situation with jumps in the marginal cost of generation, corresponding to bringing more expensive units on line. As might be expected, the optimal offer curve has jumps at the quantities which correspond to the (cumulative) capacities of the different generation units, i.e. the points at which the marginal cost of generation has discontinuities.

The focus of this paper is on the impact of non-smoothness of the profit function on the optimal supply function of a single generator without considering the reaction of other generators. Nevertheless an important question is the existence of Nash supply function equilibrium when generators hold option contracts or have non-smooth costs. As we mentioned earlier a significant problem is that when one generator's supply curve contains a horizontal segment or a vertical segment, then the residual demand curve for another generator is no longer strictly decreasing and continuously differentiable. In this case ψ is not continuously differentiable.

Anderson and Xu (2005) investigated this situation and showed that there may be no optimal supply function, at least when generators are able to bid at any price. This is because a generator can benefit by undercutting the other generator's price by ϵ, which can be arbitrarily small. Anderson and Xu also showed that an ϵ-optimal supply function may exist under such a circumstance (for a detailed discussion, see Anderson and Xu 2005). The problem occurs when one of the generators uses a supply function with a jump in quantity at a particular price. This analysis suggests that there may be a difficulty in obtaining any equilibrium when generators sell call options. In practice the situation is complicated by restrictions on prices (e.g. to whole number of cents) and other market rules (in the case of Australia a set of 10 prices at which a generator can bid needs to be made in advance, and then remains fixed for the day, see Anderson and Xu 2004).

In the case when there is jump in price at a particular quantity (as happens with changes in marginal cost, or a generator buying a put option) then it might be possible that an equilibrium exists. Baldick et al. 2004 give some discussion of piecewise affine supply function equilibrium. A fuller investigation of equilibrium models in these circumstances is a topic for further research.

An alternative viewpoint is to consider approximate Nash equilibria in markets with many generators. In this case the impact of any single generator's offer of quantity or price is negligible. Hence it may be reasonable to take D_A as a smooth approximation of the actual residual demand curve.

Appendix

Proof of Lemma 3.2 Since $S(p, Q)$ and $S(p, 0)$ are strictly increasing functions, we have

$$(p - C'(S(p, Q)))(-D_A'(p)) - S(p, Q) + Q = 0,\quad \text{for all } p \in [p_L(Q), p_U(Q)], \tag{15}$$

and

$$(p - C'(S(p, 0)))(-D_A'(p)) - S(p, 0) = 0, \quad \text{for all } p \in [p_L(0), p_U(0)]. \tag{16}$$

Subtracting (16) from (15), we have, for all $p \in [p_L(Q), p_U(Q)] \bigcap [p_L(0), p_U(0)]$,

$$(C'(S(p, 0)) - C'(S(p, Q)))(-D_A'(p)) + (S(p, 0) - S(p, Q)) + Q = 0. \tag{17}$$

Suppose that $S(p, 0) \geq S(p, Q)$ for some $p \in [p_L(Q), p_U(Q)] \bigcap [p_L(0), p_U(0)]$. Then as C is convex and differentiable and $D_A' \leq 0$, the first term in (17) is non-negative. Since $Q > 0$, this gives a contradiction and the conclusion follows. □

Proof of Theorem 4.1 Let **s** denote the graph of supply function S defined as in (9) (see Fig. 6) and $\widetilde{\mathbf{s}}$ be an arbitrary non-decreasing supply curve defined over $[0, q^M] \times [0, p^M]$. We need to compare the expected return based on **s** with that based on $\widetilde{\mathbf{s}}$. We want to achieve this by applying the classical Green's theorem (which shows that the line integral of R_Q with respect to $d\psi$ along the boundary of a region is exactly the area integral of Z_Q in that region).

Let L and U denote respectively the points where **s** enters and exits Ψ. Let A and B denote the points where the lowest residual demand curve intercepts the p-axis and the q-axis, and E and F denote the points where the highest residual curve hits the $p = p^M$ line and the $q = q^M$ line.

Observe that we can always regard point L as being located on the lowest residual demand curve between A and B since if $q_L(0) = 0$ and $p_L(0) > 0$, then we can regard the curve as entering Φ at A and hence set L = A; on the other hand

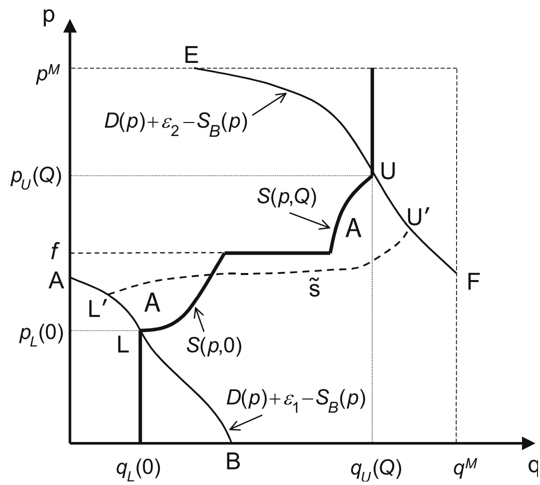

Fig. 6 Optimal supply curve with a call option

if $q_{\mathrm{L}}(0) > 0$ and $p_{\mathrm{L}}(0) = 0$, then we can regard the curve **s** as entering Φ at B and hence set L = B. Using a similar argument, we can regard point U as being located on the highest residual demand curve between points E and F. Similarly, we assume that $\widetilde{\mathbf{s}}$ enters Φ at L′ which is located on the lowest residual demand curve between points A and B and exits at U′ which is located on the highest residual demand curve between points E and F.

We write $\mathcal{A}$ for the area surrounded by **s**, $\widetilde{\mathbf{s}}$ and the boundary of Ψ (notice that this may consist of two or more different regions if **s** and $\widetilde{\mathbf{s}}$ cross). The area $\mathcal{A}$ may lie entirely on one side of the $p = f$ line, then we can apply Green's theorem directly since R^1 is continuously differentiable (coinciding either with R_0 or with R_Q over the entire area).

Suppose that $\mathcal{A}$ lies on both sides of the $p = f$ line as in the figure. We cannot apply Green's formula straightforwardly since in this case R^1 is not differentiable at $p = f$, so we split the area into two, above and below the $p = f$ line.

In the area below the $p = f$ line, $R^1 = R_0(q, p)$. We can apply Green's theorem to the area by extending Z_0 values upwards to the $p = f$ line. We can deal with the area above the $p = f$ line in a similar way (by extending Z_Q values downwards to the $p = f$ line). Note that in applying the theorem, we need to add or subtract a line integral of R^1 with respect to $\psi(q, p)$ along the segment of the $p = f$ line located in the area. These items do not appear as they cancel each other.

Thus we obtain

$$\begin{aligned} v(\mathbf{s}) - v(\widetilde{\mathbf{s}}) = &\int_{\mathrm{LL}'} R_Q(q, p) d\psi(q, p) + \int_{\mathrm{U}'\mathrm{U}} R_Q(q, p) d\psi(q, p) \\ &+ \int\int_{\mathcal{A}, p \le f} \mathrm{sign}(q, p) Z_0(q, p) dq dp \\ &+ \int\int_{\mathcal{A}, p \ge f} \mathrm{sign}(q, p) Z_Q(q, p) dq dp, \end{aligned} \tag{18}$$

where sign(q, p) equals 1, 0 and -1 respectively if (q, p) is located above, on or below the curve $\mathbf{s}$.

Since $Z_0(q, p) > 0$ for (q, p) located above $\mathbf{s}$ and $Z_0(q, p) < 0$ for (q, p) located below $\mathbf{s}$, we have

$$\int\int_{\mathcal{A}, p \le f} \text{sign}(q, p) Z_0(q, p) dq dp \ge 0.$$

On the other hand, since L and L′ are on the lowest residual demand curve, and ψ is constant on the curve, then

$$\int_{\mathrm{LL}'} R_Q(q, p) d\psi(q, p) = 0.$$

Similarly

$$\int\int_{\mathcal{A}, p \ge f} \text{sign}(q, p) Z_Q(q, p) dq dp \ge 0,$$

and

$$\int_{\mathrm{U}'\mathrm{U}} R_Q(q, p) d\psi(q, p) = 0.$$

This shows $v(\mathbf{s}) - v(\widetilde{\mathbf{s}}) \ge 0$. Thus $\mathbf{s}$ is globally optimal. □

Proof of Theorem 5.1 We begin by showing the existence of an optimal solution. This follows from a similar argument to the result of Anderson and Xu who show that the supply functions, when treated as continuous curves in the (q, p) plane, are compact under the Hausdorff metric (Anderson and Xu, 2002, Lemma 2). In order to show the existence of an optimal solution, we need the continuity of the objective function (under the Hausdorff metric). Anderson and Xu prove this under the additional restriction that the profit function $R(q, p)$ is continuously differentiable. See (Anderson and Xu, 2002, Theorem 3). However in our case we only have a piecewise continuously differentiable profit function R. Observe, however, that the result we require is that a small perturbation of the supply curve under the Hausdorff metric results in a small change in objective.

The difference between the two objectives is given by

$$\int_{\mathcal{C}} R(q, p) d\psi(q, p) = \int\int_{\mathcal{A}} Z(q, p) dp dq$$

where $\mathcal{C}$ is the closed curve formed by the difference between the perturbed and original supply curves, and $\mathcal{A}$ is the region between them. Here we have extended the application of Green's theorem (Kaplan 1962) by splitting the region $\mathcal{A}$ into the parts above and below the strike price f and applying Green's theorem separately to the two parts. Since Z is bounded the right hand side approaches zero as the

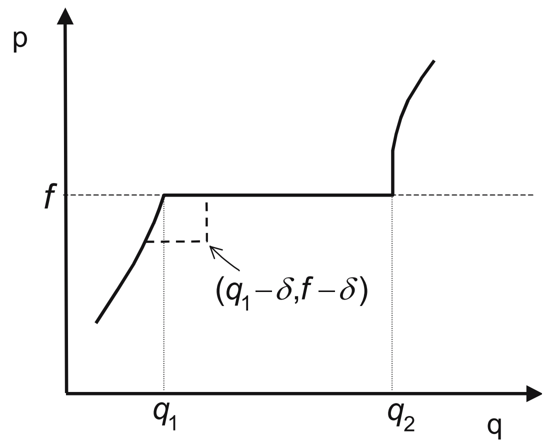

Fig. 7 A perturbation of the supply curve

perturbation gets smaller, which is enough to establish the existence of an optimal solution.

When treated as a curve in the (q, p) plane the supply curve must cross the line $p = f$ either (a) in a vertical section, or (b) in a strictly monotonically increasing segment (which is not vertical), or (c) with a section that is horizontal. To prove the result we need to show that options (b) and (c) cannot occur.

Suppose (b) occurs. Then the supply curve has the characteristic of being strictly increasing, but not vertical, on either side of the $p = f$ line. Hence, from Lemma 5.1 the supply curve must follow the $Z = 0$ curve on both sides of the $p = f$ line. Now from Assumption 3.1 there is just a single $Z = 0$ curve both above and below $p = f$ corresponding to $S(p, 0)$ and $S(p, Q)$. But from Lemma 3.2 $S(f, Q) > S(f, 0)$ giving a contradiction.

Now suppose that (c) occurs. Suppose that the horizontal section of the optimal supply curve is from (q_1, f) to (q_2, f), $q_1 < q_2$ as shown in Figure 7. Let $S_0(f) = \alpha$ and $S_Q(f) = \beta$ where $\alpha < \beta$. Since the supply curve is not horizontal immediately below and to the left of (q_1, f) we can apply a perturbation at this point as shown in the figure (with the perturbation parameterized by $\delta > 0$). Since the supply curve is optimal, using Green's theorem shows that the area integral of Z_Q over the region between the curve and its perturbation is non-positive. Letting the size of the perturbation, δ, go to zero and using the continuity of Z_Q shows that $Z_Q(q_1, f) \leq 0$. Hence, from Assumption 3.1, $q_1 \geq \beta$.

Similarly we can apply a perturbation at (q_2, f) to show that $q_2 \leq \alpha$ which is a contradiction. □

Proof of Theorem 6.1 We begin by establishing that S is monotonic and well-defined. To do this we have to establish that $p_i^u < p_{i+1}^l$, $i = 1, \ldots, n-1$. This will show that the supply curve jumps up at the break points q_i when considered in the q, p -plane. Notice that, since (q_i, p_i^u) is on the optimal supply curve S_i, $Z_i(q_i, p_i^u) = 0$. Moving across the q_i boundary, from Z_i to Z_{i+1}, involves replacing $C_i'(q_i)$ with $C_{i+1}'(q_i^+)$ which decreases the value of Z. Hence $Z_{i+1}(q_i^+, p_i^u) < 0$. Thus the point (q_i, p_i^u) (or strictly points (q, p_i^u) with q approaching q_i from the right) is in the region below the $Z_{i+1}(q, p) = 0$ curve, and hence $p_i^u < p_{i+1}^l$ as required.

The graph of S divides the effective response region Ψ into two in the q, p-plane: Ψ_1 is located above and to the left of the curve and Ψ_2 is located below and to the right. We show that $Z(q, p) > 0$, for $(q, p) \in \Psi_1$ (and not on the graph of S) and $Z(q, p) < 0$ for $(q, p) \in \Psi_2$ (and not on the graph of S). Now $Z(q, p) = Z_i(q, p)$, for $q_{i-1} < q \leq q_i$. So the result we require is immediate, since it holds for each region $q \in (q_{i-1}, q_i)$ separately.

The unique optimality of $S(p)$ now follows from similar arguments to those given earlier using Green's theorem. We omit the details. □

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