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Nash equilibria in electricity markets with discrete prices*

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Abstract. In this paper we analyse the equilibrium structure for a particular type of electricity market. We consider a market with two generators offering electricity into a pool. Generators are centrally dispatched, with cheapest offers used first. The pool price is determined as the highest-priced offer dispatched, and both generators are paid this price for all the electricity they provide. First generators set their price points (at which bids will later be made) and these are announced. Then each generator chooses the quantities to offer at each price. This reflects the behaviour of the Australian electricity market in which prices are set for 24-hours at a time, but different quantities can be offered within each half-hour period. The demand for electricity is uncertain when offers are made (and is drawn from a probability distribution known to both players). We begin by analysing an example of this two stage game for a simple case where only one price can be chosen. The main results of the paper concern the structure of a Nash equilibrium for the quantity-setting sub-game in which each player aims to maximise their expected profit when prices have already been announced. The distribution of demand plays an important role in the existence of a Nash equilibrium. In the quantity setting game there may be Nash equilibria which are not stable. We show that, under certain circumstances, if the equilibrium offers are sufficiently close to the generators' marginal costs, then the equilibrium will be stable.

Key words: Electricity markets, Nash equilibria, Stability of equilibria, Stochastic demand

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1 Introduction

Wholesale electricity markets have been introduced in many different countries around the world. The way in which these markets are implemented varies from one country to another, but they all seek to provide electricity to consumers at a competitive price at the same time as giving appropriate signals for investment and new entry (see Chao and Huntington (1998), Stoft (2002) for more information on wholesale electricity markets). In this paper we consider a market with a central dispatch and pricing mechanism: this type of market is called an electricity pool. The first electricity market of this type was created in England and Wales, but similar markets now operate in parts of North America and Europe, in Australia and in New Zealand.

Wholesale electricity markets with a pool structure are characterized by there being a single price for electricity paid to all generators offering power at a single point of the transmission system. This price is set in a way that the market clears: generators offering power below this price have the power dispatched, and their combined output is just enough to meet the total demand. The price paid to all the generators is that for the highest-priced power which is actually used. The same idea can be applied to a network with transmission losses, and we end up with separate market clearing prices at each node. In this case an optimization problem has to be solved to find the right dispatch (the optimization problem seeks to meet the demand at least cost assuming that generators offers correctly reflect the cost of power supply). Markets of this sort are designed to produce a socially optimal outcome in the case that there are many generators and the market is competitive, with no single generator having substantial market power. In this case it is optimal for all generators to offer power at their marginal cost of generation. However real electricity markets are better modelled as oligopolies: even when the market as a whole has large numbers of generators, the transmission constraints limit the extent to which large number of generators compete for the demand at a single node. Thus we will need to use a game theory analysis to understand the likely behavior of generators operating in the market.

In this paper we discuss a two-player game that models some aspects of a wholesale electricity market. The players are generators who both offer power at the same node of the network. Each generator is concerned to maximize its own expected profit, where uncertainty arises since demand is not known in advance. The players first decide on the prices at which they will bid and these prices are announced. Then the players decide on the quantities of power to offer at their previously announced prices. Next demand occurs: the demand is random, independent of price, and is drawn from a distribution known to both players. prices. Finally a market mechanism dispatches the cheapest power to meet actual demand. The market mechanism determines the system marginal price of power and this price is paid to both generators for all of the electricity they supply.

Our focus is on the Nash equilibria that occur in this duopoly when there is a one shot game. There is a further issue that may arise in practice as a result of the repeated play that occurs in electricity markets from day to day. This enables players to learn strategies that can support high payoff outcomes that are not equilibria for the game when only played once. Though this is an important issue, we will only look at the simpler question of equilibria in a one-shot game.

The model we use is based on the wholesale electricity market that exists in Australia called the Australian National Electricity Market (NEM). At the beginning of each 24-hour period each generation unit chooses a set of ten prices that are then fixed for the whole day. Generators then choose the quantity to bid for each generation unit at each of the ten price points for each half hour of the day. Actual dispatch quantities are then determined based on system demand during a five-minute period. Within each five-minute time interval the central market mechanism will dispatch the cheapest electricity bids at that time, with all bids at the market clearing price or below being dispatched.

Much of the previous work on equilibria in electricity markets has simplified the model by assuming a continuous offer curve rather than offers at discrete prices (see for example Green and Newbery (1992), Rudkevich et al. (1998) and Anderson and Philpott (2002a)). Another approach, used by Von der Fehr and Harbord (1993), has been to look at the market from the standpoint of a multi-unit simultaneous auction. Fabra et al. (2002) argue for the importance of correctly representing the discrete nature of the generator bids, since if offer curves were truly continuous then market power can be exercised more easily. Kremer and Nyborg (2002) have also discussed this issue (their analysis is from the point of view of buyer submitting bidding schedules as occurs in treasury auctions). All of this work points to the importance of a careful treatment of the way that bidding occurs. The formulation we give has not, to our knowledge, been considered in detail before. It arises naturally from a consideration of the behaviour of the Australian market.

In this paper we make a contribution to the literature in a number of ways. First, as far as we are aware, this is the first analysis of an electricity market in which prices and quantities are set separately, as occurs in Australia. Though we can say little about the choice of prices made in this situation we are able to establish some structural properties of the quantity choices made by the generators. We also discuss the question of the stability of a Nash equilibrium for the quantity-setting sub-game when the two players use the alternating move Cournot dynamic (Fudenberg and Levine (1998)), in which each generator repeatedly adjusts its set of bids to give an optimal response to the announced bids of the other generator. Under certain circumstances we can show that stability depends on the difference between the equilibrium offers and the marginal cost offer, with stability being guaranteed if these are close enough.

The rest of this paper is organized as follows: In Section 2 we introduce the problem by discussing a small example for which we can find a (numerical) equilibrium solution for the game. In Section 3 we show that, when generators behave optimally, then it is unnecessary for either generator to offer quantities at more than one price between two adjacent prices of the other. This observation allows us to considerably simplify the analysis. In Section 4 we use the derivative of the expected return functions to give some further characterization of the prices at which positive quantities are bid in a Nash equilibrium. Finally in Section 5 we discuss the stability of a Nash equilibrium when the demand for electricity has a uniform distribution over a certain range of values.

2 A game with a single price point

In order to motivate what follows we begin by discussing the simplest example of the type we have in mind. This occurs when both players can only set one price point. We label the two players A and B. In stage 1 of the game each player chooses a price (p for player A, r for player B) and these prices are then announced. In stage 2 each player chooses a quantity (x for player A, y for player B). Finally demand occurs. Demand is inelastic and the market clears with each player being paid the clearing price for the quantity dispatched. There is a third non-strategic player C who offers a large amount at a (normalized) price of 1. Thus the demand is guaranteed to be met, and neither player A or B will offer at prices higher than 1.

We consider a specific simple symmetric example in which each player has marginal cost of zero and demand is uniformly distributed on $[0, 1]$. We further assume that each player has a total capacity of 1, which is thus an upper bound on x and y.

We are interested in a subgame perfect Nash equilibrium for the two stage game. We start by analysing the situation at the second stage, when prices have already been determined. Consider the case when the two prices are different. Without loss of generality we label A as the player with the smaller price. The situation is summarized in the following result.

Proposition 1. *If $p < r < 1$ then the only Nash equilibria are as follows. If*

$$r(3 - r - 2p + rp) < k,$$

where

$$k = \sqrt{2 - p}\sqrt{2r - p},$$

then

$$x = \frac{1}{3 - r - 2p + rp}, \quad y = \frac{1 - p}{3 - r - 2p + rp}; \tag{1}$$

otherwise there is a mixed equilibrium in which

$$x = \begin{cases} \frac{r}{k} & \text{with probability} \quad \alpha \\ \frac{r}{2r-p} & \text{with probability} \quad 1 - \alpha \end{cases}$$

where

$$\alpha = \frac{(2 - p)(3r - p - r^2 - rk(2 - r))}{r(2 - p - k)(1 - r)}, \tag{2}$$

and

$$y = \frac{2r - p - rk}{(2r - p)(1 - r)}. \tag{3}$$

Proof. We begin by looking for a pure strategy equilibrium for the choices of x and y. Consider the expected profit for player B, $E_B(y)$, given a fixed value of x, if player B chooses an amount y. If demand is less than x, player A captures all the demand and B makes no profit. If demand is between x and

$x+y$ then player B is the marginal player and the clearing price is r. If demand is greater than $x+y$ then the marginal price is 1. Thus

$$E_B(y) = \int_x^{x+y} r(u-x)\, du + \int_{x+y}^1 y\, du$$

for $y \le 1-x$. Hence $E_B(y) = ry^2/2 + (1-x-y)y$. Thus $dE_B/dy = ry - 2y + (1-x)$ and the best choice of y in the range $0 \le y \le 1-x$ is

$$y = (1-x)/(2-r). \tag{4}$$

The maximum value of $E_B(y)$ is $\frac{(1-x)^2}{2(2-r)}$. For $y > 1-x$, $E_B(y) = r(1-x)^2/2$, which is independent of y and smaller than the maximum value. Thus (4) gives the best choice of y.

Now consider the choice of x, given a fixed value for y. The profit for player A is given by

$$E_A(x) = \int_0^x up\, du + \int_x^{x+y} xr\, du + \int_{x+y}^1 x\, du,$$

with the obvious changes to this expression when $x > 1-y$. Hence

$$E_A(x) = \begin{cases} px^2/2 + xyr + (1-x-y)x & \text{if} \quad x < 1-y, \\ px^2/2 + x(1-x)r & \text{if} \quad x \ge 1-y. \end{cases}$$

In finding the maximum value of $E_A(x)$ we need to treat the two sections separately. When $x < 1-y$, $dE_A/dx = px + ry + 1 - y - 2x$, and the maximum value in the range $[0, 1-y]$ is achieved when

$$x = x_0(y) = \begin{cases} (1-y(1-r))/(2-p) & \text{if} \quad y < (1-p)/(1-p+r), \\ 1-y & \text{otherwise.} \end{cases} \tag{5}$$

At this point

$$E_A(x_0) = \begin{cases} \frac{(yr+1-y)^2}{2(2-p)} & \text{if} \quad y < (1-p)/(1-p+r), \\ p(1-y)^2/2 + y(1-y)r & \text{otherwise.} \end{cases}$$

When $x \ge 1-y$, $dE_A/dx = px + r - 2rx$, and the maximum value in the range $[1-y, 1]$ is achieved when

$$x = x_1(y) = \begin{cases} r/(2r-p) & \text{if} \quad y > (r-p)/(2r-p), \\ 1-y & \text{otherwise.} \end{cases} \tag{6}$$

At this point

$$E_A(x_1) = \begin{cases} \frac{r^2}{2(2r-p)} & \text{if} \quad y > (r-p)/(2r-p), \\ p(1-y)^2/2 + y(1-y)r & \text{otherwise.} \end{cases}$$

It is not hard to show, using $r^2 < r$, that $(r-p)/(2r-p) < (1-p)/(1-p+r)$, hence we cannot have both the maximum over $[0, 1-y]$, and the maximum over $[1-y, 1]$ occurring at $1-y$, so $E_A(x)$ is never maximised at the point $x = 1-y$ between the two regions.

First suppose that, at the equilibrium, y is such that $E_A(x_1) > E_A(x_0)$ so that the maximum for generator A occurs at the point $x_0(y)$ (rather than

$x_1(y)$). Then, after some algebra, we can show that the values (1) solve the simultaneous equations (4) and (5). Since $3 - r - 2p + rp = 2 - p + (1 - p)(1 - r)$ it is easy to see that these values do satisfy the condition $x + y \leq 1$.

We write $E_A^*(p, r)$ and $E_B^*(p, r)$ for the values of E_A and E_B at the equilibrium values; these can be calculated as

$$E_A^*(p,r) = \frac{2-p}{2(3-r-2p+rp)^2},$$

$$E_B^*(p,r) = \frac{(2-r)(1-p)^2}{2(3-r-2p+rp)^2}.$$

Now consider the case that at the equilibrium $E_A(x_1) > E_A(x_0)$, and the maximum for generator A occurs at $x_1(y)$. Then we have an immediate contradiction since from (6) $x \geq 1 - y$, but from (4) $y < 1 - x$. So the only pure strategy equilibrium has the form (1).

However this equilibrium will only exist when $E_A^*(p, r) \geq E_A(x_1)$ and the maximum for generator A occurs at x_0. In the case that $E_A^*(p, r) < E_A(x_1)$, i.e. when

$$\frac{r^2}{2(2r-p)} > \frac{2-p}{2(3-r-2p+rp)^2}, \tag{7}$$

there cannot be a pure strategy equilibrium. In this situation the only possibility is a mixed strategy equilibrium in which player A chooses the two maximizing values x_0 and x_1 with some probabilities, α and $(1 - \alpha)$ say. In this case, because of linearity, the profit for player B for a particular value of y is given by the same formula (4), but replacing x with the expected value for x, which is $\alpha x_0 + (1 - \alpha)x_1$. Hence B will still choose a pure strategy, since there is a single optimal value for y . For a mixed strategy equilibrium to exist the two choices of x must give the same payoff for player A. i.e.

$$\frac{(1-y(1-r))^2}{2(2-p)} = \frac{r^2}{2(2r-p)}.$$

This implies the value of y given by (3). From this we can calculate

$$x_0(y) = \frac{rk}{(2r-p)(2-p)} = \frac{r}{k}.$$

The value of x_1 is independent of y. This establishes most of what we require for the Proposition. It only remains to check that, with the value of α given, we have $y = (1 - \alpha x_0 - (1 - \alpha)x_1)/(2 - r)$ and $0 \leq \alpha \leq 1$.

To establish that y satisfies this equation we have to show that α is given by

$$\frac{(2-r)y - 1 + x_1}{x_1 - x_0} = \frac{(2-p)((2-r)(2r-p)y - (r-p))}{r(2-p-k)}$$

which, after substitution for y, is easily seen to be the same as the expression (2). To show the inequalities for α it is enough to show that $1 - (2 - r)y$ lies between r/k and $r/(2r - p)$. Now

$$1 - (2-r)y = \frac{(2-r)rk - (2r-p)}{(2r-p)(1-r)},$$

so one of the inequalities we require for this mixed equilibrium is (after multiplying through by k)

$$\frac{r(2-r)(2-p)(2r-p)-(2r-p)k}{(2r-p)(1-r)} \geq r.$$

But this simplifies to $k \leq r((2-r)(2-p)-(1-r))$. After squaring both sides this becomes precisely the inequality (7). So that $\alpha \geq 0$ precisely when the pure strategy equilibrium does not occur.

The other inequality we require is

$$\frac{(2-r)rk-(2r-p)}{(2r-p)(1-r)} \leq \frac{r}{(2r-p)}.$$

After simplification this inequality becomes

$$(3r-p-r^2)^2 - r^2(2-p)(2r-p)(2-r)^2 \geq 0.$$

But the left hand side of this inequality can be re written as

$$(r-1)^2\left((r-p)(5r-p-2pr)+4r^2(1-r)+pr^3+pr^2(r-p)\right),$$

and as each bracketed term in this expression is positive (using $1 > r > p$), the inequality is established.

Hence in all cases there is just a single equilibrium possible.

Our discussion in the proof above shows that, when we allow for the possibility of a mixed strategy equilibrium, the expected profit at equilibrium for player A is given by

$$E_A^*(p,r) = \max\left(\frac{r^2}{2(2r-p)}, \frac{2-p}{2(3-r-2p+rp)^2}\right).$$

It turns out, after some algebra, that we can express the equivalent value for player B as

$$E_B^*(p,r) = \min\left(\frac{(2-r)((2r-p)^{1/2}-r(2-p)^{1/2})^2}{2(1-r)^2(2r-p)}, \frac{(2-r)(1-p)^2}{2(3-r-2p+rp)^2}\right).$$

Notice that when p approaches r, the values of E_A^* and E_B^* do not coincide, unless $p = 0$. In fact E_B^* is smaller by a factor of at least $(1-p)^2$ (since this is the difference between the second of the terms in the two expressions). This difference occurs because the limit of the profit to player A as his announced price p approaches r from below is greater than the profit for player B.

We need to treat the case when $p = r$ separately. First we note there is the need to define the amount dispatched by the two players (a sharing rule). We shall assume that demand is shared between the players in proportion to the amount offered. Thus provided $x + y \leq 1$,

$$E_A(x) = \int_0^{x+y} u\frac{x}{x+y}p\, du + \int_{x+y}^1 x\, du = \frac{(x+y)xp}{2} + (1-x-y)x.$$

So $dE_A/dx = xp + yp/2 + 1 - y - 2x$. Similarly $dE_B/dy = yp + xp/2 + 1 - x - 2y$. At the equilibrium x and y are chosen so that these are both zero, i.e.

$$[1+\frac{p}{2}(x+y)-x-y]+\frac{px}{2}-x=[1+\frac{p}{2}(x+y)-x-y]+\frac{py}{2}-y=0.$$

From this we can deduce that (provided $p \leq 2/3$)

$$x=y=\frac{2}{3(2-p)},$$

giving

$$E_A^*(p)=E_B^*(p)=\frac{2}{9(2-p)}. \quad (8)$$

The other case we have to consider is that $x+y>1$. In this case $E_A(x)=(1/2)px/(x+y)$ and both players will maximise their profit by bidding their entire capacity. Hence we have an alternative equilibrium when $x=y=1$ and $E_A^*(p)=E_B^*(p)=p/4$. This is the only equilibrium possible when $p>2/3$.

Now we consider the form of an equilibrium for the first stage of the game (in which p and r are chosen). The game is symmetric and we look for a symmetric mixed equilibrium. Suppose first of all that there is some price p_0 in $(0,2/3)$ such that both players have a positive probability of choosing this price in stage 1. If both players choose this price and then coordinate on the equilibrium which does not bid their entire capacity, then the expected profit is given by (8). In most cases there cannot be an equilibrium with these properties since it would then be possible for one of the players to improve their overall profit by choosing to replace the price p_0 with a price just undercutting p_0, and then do better after the stage 2 quantity game is finished. To see this observe that for p_0 in $(0,1)$, $2p_0^2<3p_0$ and hence $3(2-p_0)>2(3-3p_0+3p_0^2)$. Squaring both sides and rearranging gives

$$\frac{2}{9(2-p_0)}<\frac{2-p_0}{2(3-3p_0+p_0^2)^2},$$

provided $p_0>0$. Moreover the alternative equilibrium (with $x=y=1$) gives a payoff which is also smaller than $(2-p_0)/(2(3-3p_0+p_0^2)^2)$.

Thus we are left with a price of zero as the only one which can be given (with a non-zero probability) by both players in an equilibrium solution to the game. However we can show that a pure strategy equilibrium in which both players announce a price of zero does not occur. Either player can improve their payoff (from 1/9 to 1/8) by announcing a price of 1 instead.

Thus for the first stage game we must seek a mixed strategy equilibrium in which each player has a certain probability γ of offering at price zero and a certain probability, $(1-\gamma)$ of choosing a price from a continuous distribution having density function g over $[0,1]$. Because of symmetry the equilibrium solution has a single distribution g to be played by both players. Suppose now that player B selects r using the density function g, we can calculate the expected profit for A if he chooses a price p, which we write as $\Pi_A(p)$. This has three terms depending on whether r is zero, between zero and p, or more than p:

$$\Pi_A(p)=\gamma E_B^*(0,p)+(1-\gamma)\int_0^p E_B^*(r,p)g(r)\,dr+(1-\gamma)\int_p^1 E_A^*(p,r)g(r)\,dr.$$

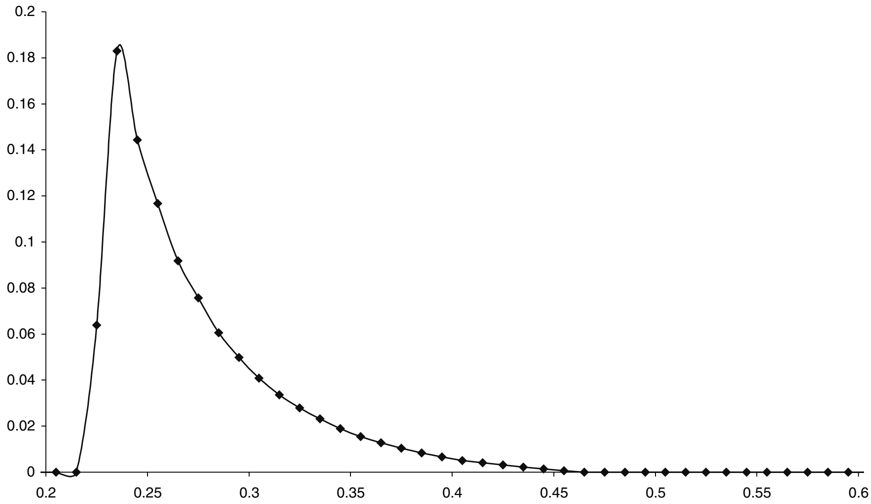

Fig. 1. The equilibrium density function g

In order for there to be a mixed strategy equilibrium, $\Pi_A(p)$ must be maximised at all the values of p, which may be chosen, i.e. at all the values in the support of g, together with 0 if $\gamma > 0$. The next step is to find values of γ and the function g which achieve this. We will need $\Pi_A(p)$ stationary as p varies in the support of g. We solve this problem numerically, by discretising the interval $[0, 1]$ into 100 discrete points and estimating the integrals by assuming the density function is concentrated on these points. An optimization procedure is used to find values of γ and $g(1/100)$, $g(2/100)$, etc. which minimize the variation in the values of $\Pi_A(p)$ at points with non-zero values. We obtain the approximate density function g shown in Figure 1 and a γ value of zero. The support of g is the interval $(0.22, 0.48)$ and $\Pi_A(p) = 0.1165$ in this range.

Thus we can summarise the subgame perfect Nash equilibrium strategy as follows. In the first stage of the game each player chooses a price randomly in the range $(0.22, 0.48)$ using a continuous probability distribution g. Then prices are announced and players decide their quantities using the mechanism of Proposition 2. There is only a vanishingly small probability of both players announcing the same price, p (in which case they may as well choose a quantity $\min(1, \frac{2}{3(2-p)})$).

3 Fundamentals of the market model

We will embark on our analysis of the more general case in which each generator can select a number of different prices. We continue to assume that there are just two generators, both located at the same node. In this paper we will not consider any issues arising as a result of the transmission network. We suppose that prices are set by the generators in stage 1 and during stage 2 quantities are set at each of the price points. The set of price-quantity pairs are then fixed for the bidding period (half an hour in the Australian context). Demand is uncertain and the generator wishes to choose the quantities to bid in order to maximise profit.

The market demand during the bidding period is random but with a known distribution. We denote the demand by D and the density function of its distribution by f. We assume that the support set of f is $[d_1, d_2]$, where $0 \leq d_1 < d_2$. Notice that this model can also be used to represent the circumstance when demand varies during the course of the bidding period, but bids cannot be changed. In this case the various values of demand which are used in determining dispatch (every 5 minutes in the Australian market) can be thought of as realisations of the demand process D. This is similar to the approach that Green and Newbery suggest in their analysis of the England and Wales market over a 24-hour period using the supply function equilibrium of Klemperer and Meyer (1989). Essentially the load-duration curves (giving the number of hours that demand is above a certain level) correspond to a cumulative distribution function for f if taken over the period for which a single set of quantity bids are valid.

We have chosen to assume that demand is unaffected by price. This is a more reasonable assumption for the short term market behaviour considered here than it would be for a longer time horizon. The structure of the market, both in Australia and elsewhere does allow demand-side bids, however these are not yet common in practice. There will nevertheless be some large industrial users of power who are participants in the market and are likely to reduce their load during periods of very high price. However for much of the time the degree of price elasticity in the half hour time frame considered here is close to zero. The assumption of zero price elasticity is also made by Anderson and Philpott and Rudkevich et al.

We write $R_A(X, p)$ for the profit obtained by generator A if this generator is dispatched an amount X and the market clearing price is p. Besides the money pX paid to the generator, there are the costs incurred in that level of generation, which we denote by $C_A(X)$. Thus

$$R_A(X, p) = pX - C_A(X).$$

Similarly

$$R_B(Y, p) = pY - C_B(Y)$$

is the profit for generator B if it is dispatched an amount Y with a clearing price of p. There are interesting questions that arise in practice when generators hold hedging contracts, but we will not address these issues here.

We now focus on the second stage of the game. Suppose that generator A has already chosen prices $p_1 < p_2 < \cdots < p_n$ and generator B has already chosen prices $r_1 < r_2 < \cdots < r_m$, which then remain fixed. We will assume that none of the prices p_i, $i = 1, 2, \ldots n$ and r_i , $i = 1, 2, \ldots m$ coincide. In cases where there is a price which is chosen by both players and the market clears at this price, then the two generators will have to share dispatch and the market rules will determine how this sharing takes place. However our analysis of the simple game in the previous section suggests that the situation where each player chooses prices from a continuous distribution is important; in this case we can safely assume that prices do not coincide.

For a given bidding period, generator A offers a quantity x_i at price p_i, $i = 1, 2, \ldots, n$, and generator B offers a quantity y_i at price r_i, $i = 1, 2, \ldots, n$. Let T_A and T_B denote respectively the capacities of generator A and generator B. Then the total quantities offered by each generator at each bidding period will not exceed its maximum generating capacity.

Let

$$X_i = \sum_{k=1}^{i} x_k,\ i = 1, \ldots, n$$

and

$$Y_i = \sum_{k=1}^{i} y_k,\ i = 1, \ldots, m.$$

Since the market demand is uncertain, the total supply of the electricity to the market must not be less than the maximum possible demand d_2, that is, $X_n + Y_m \geq d_2$. This is an awkward constraint since it makes the possible choices of one generator dependent on the choices of the other. One way to deal with this is to introduce a third non-strategic generator as we did in the previous section (this is also equivalent to having a given "Value of Lost Load" which becomes the market clearing price if demand cannot be met and load has to be shed). However we will instead introduce the constraint that all the capacity of a generator has to be bid in at some price. Hence we obtain the following set of possible strategies for the two generators:

$$\mathcal{S}_A = \{(X_1, \ldots, X_n) : 0 \leq X_1 \leq \ldots \leq X_n = T_A\}$$

and

$$\mathcal{S}_B = \{(Y_1, \ldots, Y_m) : 0 \leq Y_1 \leq \cdots \leq Y_m = T_B\}.$$

Throughout, we will frequently use X to denote the vector $(X_1, \ldots, X_n)$ and Y the vector $(Y_1, \ldots, Y_m)$.

The market clearing price if the generators use strategies (X, Y) and the demand realisation is d will be:

$$p_c(X, Y, d) = \inf\{p : \sum_{p_j \leq p} x_j + \sum_{r_j \leq p} y_j \geq d\}.$$

Since we assume that prices offered by different generators do not coincide, there is exactly one generator offering at the clearing price, p_c. We call this the marginal generator. The system will dispatch from the marginal generator just enough electricity to meet the demand. Any electricity offered at prices strictly below the clearing price is dispatched completely.

The proposition below shows that if there are a whole set of prices for one generator lying between two adjacent prices for the other, then we can find an optimal solution that only uses the highest price in this set. Thus bidding at any other prices in such a case is unnecessary.

Proposition 2. *If $r_i < p_j < \cdots < p_{j+k} < r_{i+1}$ for some $k > 1$ then there is an optimal policy for generator A that offers no quantity at any of the prices $p_j, p_{j+1}, \ldots, p_{j+k-1}$. Similarly if $p_j < r_i < \cdots < r_{i+h} < p_{j+1}$ for $h > 1$ then there is an optimal policy for generator B that offers no quantity at any of the prices $r_i, r_{i+1}, \ldots r_{i+h-1}$.*

Proof. Suppose that X is a strategy for generator A, and that $r_i < p_j < p_j$ $+1 < \cdots < p_{j+k} < r_{i+1}$. We define a new strategy $\tilde{X}$ for player A by first setting $\tilde{x}_s = x_s$, for $s \leq j - 1$ or $s \geq j + k + 1$ and then defining the other values as follows:

$$\tilde{x}_j = \tilde{x}_{j+1} = \cdots = \tilde{x}_{j+k-1} = 0, \tilde{x}_{j+k} = \sum_{s=j}^{j+k} x_s.$$

Observe that under any given demand realisation d, the quantity dispatched from A is unaltered by changing from X to $\tilde{X}$ since the total quantity offered between the prices r_i and r_{i+1} is unchanged. Let X_d be the quantity dispatched by player A, let c_d be the market clearing price under strategy X. Let c'_d be the market clearing price under strategy $\tilde{X}$. If $c_d \le r_i$ or $c_d \ge r_{i+1}$, then $c_d = c'_d$ and hence $R_A(X_d, c_d) = R_A(X_d, c'_d)$.

If $p_j \le c_d \le p_{j+k}$, then $c'_d = p_{j+k} \ge c_d$ and $X_d \ge X_{j-1}$. Thus $R_A(X_d, c_d) \le R_A(X_d, c'_d)$. So whatever the demand, the profit for player A under strategy X is less than that under strategy $\hat{X}$, and so $\tilde{X}$ is a better strategy for player A. The second half of the proposition, related to generator B, is proved in exactly the same way.

This proposition implies that, once prices have been announced, both players can ignore all but the highest price between adjacent prices of the other generator. Consequently we only need to consider an interleaved scenario in which each player offers at most one price between two adjacent prices for the other. Here we will concentrate on the case when $n = m$ and

$$r_1 < p_1 < r_2 < p_2 < \cdots < r_n < p_n.$$

We call this the *standard price arrangement*. The other possibility (where n and m differ by 1 and the same player has both the highest and lowest price) is not significantly different.

Now we are in a position to give explicit expressions for the expected profits for the two generators. We suppose that generator A uses policy $X \in \mathcal{S}_A$ and generator B uses policy $Y \in \mathcal{S}_B$. We first consider $E_A(X, Y)$ the expected profit for generator A. If demand is in the range $X_{i-1} + Y_{i-1}$ to $X_{i-1} + Y_i$, then the quantity dispatched from A will be X_{i-1} and the price will be r_i. On the other hand if demand is in the range $X_{i-1} + Y_i$ to $X_i + Y_i$, say the demand is $q + Y_i$, then the quantity dispatched from A will be q and the price will be p_i. Thus we arrive at the following expression for the expected profit for A:

$$E_A(X,Y) = \sum_{i=1}^{n} \left(\int_{X_{i-1}+Y_i}^{X_i+Y_i} R_A(q - Y_i, p_i) f(q)dq + R_A(X_{i-1}, r_i) \int_{Y_{i-1}+X_{i-1}}^{Y_i+X_{i-1}} f(q)dq \right).$$

Here we take $X_0 = Y_0 = 0$. Similarly the expected profit for generator B is given by

$$E_B(X,Y) = \sum_{i=1}^{n} \left(\int_{Y_{i-1}+X_{i-1}}^{Y_i+X_{i-1}} R_B(q - X_{i-1}, r_i) f(q)dq + R_B(Y_i, p_i) \int_{X_{i-1}+Y_i}^{X_i+Y_i} f(q)dq \right).$$

4 Characterization of a Nash equilibrium

Our analysis will be based on the derivatives of the expected return functions E_A and E_B. The existence of these derivatives will depend on the behaviour of the density function f. We will assume that f is continuous except at the ends of the interval $[d_1, d_2]$. This will imply that there may be a difficulty in defining the derivatives of E_A and E_B when for example some $X_i + Y_i = d_1$. We will assume that the cost functions C_A and C_B are increasing and differentiable.

Let Ω denote the set of all pairs of policies for the two players, so that generator A chooses a strategy X from $\mathcal{S}_A$ and generator B chooses a strategy Y from $\mathcal{S}_B$, that is,

$$\Omega = \{(X, Y) : X \in \mathcal{S}_A, Y \in \mathcal{S}_B\}.$$

E_A and E_B are piecewise continuously differentiable. For $(X, Y) \in \Omega$ and $i = 1, \dots, n-1$,

$$\begin{aligned}\frac{\partial E_A(X,Y)}{\partial X_i} = &\ [f(X_i + Y_i)(p_i - r_{i+1}) + f(X_i + Y_{i+1})(r_{i+1} - p_{i+1})]X_i \\ &+ (r_{i+1} - C_A'(X_i))(F(X_i + Y_{i+1}) - F(X_i + Y_i)),\end{aligned} \tag{9}$$

If $X_i + Y_i = d_1$ or d_2, or $X_i + Y_{i+1} = d_1$ or d_2, then we must interpret these partial derivatives as one-sided and evaluate f accordingly. Similarly

$$\begin{aligned}\frac{\partial E_B(X,Y)}{\partial Y_i} = &\ [f(Y_i + X_{i-1})(r_i - p_i) + f(Y_i + X_i)(p_i - r_{i+1})]Y_i \\ &+ (p_i - C_B'(Y_i))(F(X_i + Y_i) - F(X_{i-1} + Y_i)),\ i = 1, \dots, n-1.\end{aligned} \tag{10}$$

If $X_i + Y_i = d_1$ or d_2, or $X_{i-1} + Y_i = d_1$ or d_2, then we must interpret these partial derivatives as one-sided and evaluate f accordingly.

The next proposition establishes straightforward general conditions at a Nash equilibrium. We use the notation $\frac{\partial E(X,Y)}{\partial X^-}$, $\frac{\partial E(X,Y)}{\partial X^+}$ for the left and right partial derivatives of a function E at (X, Y).

Proposition 3. *Suppose that (X^*, Y^*) is a Nash equilibrium. Then*

(i) for $i = 1, \dots, n-1$, if $x_i^* > 0$, then $\frac{\partial E_A(X^*,Y^*)}{\partial X_i^-} \geq 0$, if $x_{i+1}^* > 0$, then $\frac{\partial E_A(X^*,Y^*)}{\partial X_i^+} \leq 0$; moreover if both $x_i^* > 0$ and $x_{i+1}^* > 0$ and $\frac{\partial E_A(X^*,Y^*)}{\partial X_i}$ exists, then $\frac{\partial E_A(X^*,Y^*)}{\partial X_i} = 0$;

(ii) for $i = 1, \dots, n-1$, if $y_i^* > 0$, then $\frac{\partial E_B(X^*,Y^*)}{\partial Y_i^-} \geq 0$, if $y_{i+1}^* > 0$, then $\frac{\partial E_B(X^*,Y^*)}{\partial Y_i^+} \leq 0$; moreover, if both $y_i^* > 0$ and $y_{i+1}^* > 0$ and $\frac{\partial E_B(X^*,Y^*)}{\partial Y_i}$ exists, then $\frac{\partial E_B(X^*,Y^*)}{\partial Y_i} = 0$.

Proof. We prove part (i); part (ii) can be proved in a similar way. Suppose for a contradiction that $\frac{\partial E_A(X^*,Y^*)}{\partial X_i^-} < 0$ and $x_i^* > 0$. Let $\tilde{X}$ be such that $\tilde{x}_k = x_k^*$ for $k \neq i, i+1$, $\tilde{x}_i = x_i^* - \varepsilon$, and $\tilde{x}_{i+1} = x_{i+1}^* + \varepsilon$, with ε chosen small enough so that $\tilde{x}_i > 0$. This has the effect of reducing X_i^* while leaving all the other components of X^* unchanged. Then for ε small enough the sign of $\frac{\partial E_A}{\partial X_i^-}$ will imply $E_A(\tilde{X}, Y^*) > E_A(X^*, Y^*)$, which contradicts the optimality of X^*. In a similar way, when $x_{i+1}^* > 0$ and $\frac{\partial E_A(X^*,Y^*)}{\partial X_i^+} > 0$, we can increase E_A by reducing x_{i+1}^* by a small amount and increasing x_i^* by the same amount. Again this contradicts the optimality of X^*.

It is clear that $p_c(X, Y, d_1)$ is the lowest possible market clearing price, which we write as $p^L(X, Y)$. Similarly $p^H(X, Y) = p_c(X, Y, d_2)$ is the highest possible clearing price. We need to treat the special case $d_1 = 0$ separately: we instead define $p^L(X^*, Y^*)$ as $\lim_{\varepsilon\to 0} p_c(X^*, Y^*, \varepsilon)$. This means that when $d_1 = 0$, $p^L(X^*, Y^*)$ will be the lowest price at which a non-zero quantity is offered. When it will not cause confusion we will drop the (X, Y) from our notation and just write p^L and p^H for the lowest and highest clearing prices possible.

Now we are ready to give a result showing that the prices between the highest and lowest clearing price will all have non-zero quantities offered at an equilibrium. The idea of the proof is to observe that if one player (say generator A) offers a zero quantity at a price, say $p_i > p^L$, then the other player will be better off by moving a small amount from r_i to r_{i+1} and this will undermine the Nash equilibrium. In essence the argument is the same as that of Proposition 2 which rules out having two prices r_i and r_{i+1} with no p_j between them.

Proposition 4. *Suppose that* (X^*, Y^*) *is a Nash equilibrium under the standard price arrangement and* $d_1 > 0$. Then

(i) $x_i^* > 0$ *for each* i *with* $p^L(X^*, Y^*) \le p_i \le p^H(X^*, Y^*)$;
(ii) $y_i^* > 0$ *for each* i *with* $p^L(X^*, Y^*) \le r_i \le p^H(X^*, Y^*)$.

Proof. We define $h_A = \min\{i : p^L \le p_i \le p^H,\ x_i^* = 0\}$, and $h_B = \min\{i : p^L \le r_i \le p^H,\ y_i^* = 0\}$. If the proposition statement is true, then both of these sets are empty and the values h_A and h_B undefined. We suppose on the contrary that one or both of h_A and h_B is defined (if either is not defined we let its value be ∞). We suppose without any real loss of generality that $h_A \ge h_B$ so that $p_{h_A} > r_{h_B}$ (the case that $h_A < h_B$ can be dealt with similarly).

Now consider the value of $x_{h_B-1}^*$. Since $y_{h_B}^* = 0$, the market cannot clear at r_{h_B} and $p^L < r_{h_B} < p^H$. If $x_{h_B-1}^* = 0$, then $h_B - 1$ satisfies all the conditions from the definition of h_A, and so $h_A \le h_B - 1$ which contradicts our assumption. So we have established that $x_{h_B-1}^* > 0$.

Since $y_{h_B}^* = 0$, we have $X_{h_B-1}^* + Y_{h_B-1}^* = X_{h_B-1}^* + Y_{h_B}^*$. Since $r_{h_B} < p^H$ we know that $X_{h_B-1}^* + Y_{h_B}^* < d_2$. So by (9),

$$\begin{aligned}\frac{\partial E_A(X^*, Y^*)}{\partial X_{h_B-1}} &= f(X_{h_B-1}^* + Y_{h_B-1}^*)(p_{h_B-1} - p_{h_B})X_{h_B-1}^* \\ &< 0.\end{aligned}$$

Thus we have a contradiction from part (i) of Proposition 3 and the fact that $x_{h_B-1}^* > 0$.

We might expect that, at an equilibrium, where a generator makes a positive offer, then it will do so at a price above its marginal cost of generation. In fact we have a slightly weaker result: the marginal cost of generation will be less than the next higher price from the other generator. We let i_A be the index of the lowest p_i which can occur as a clearing price, and similarly for i_B. Thus $i_A = \min\{i : p^L \le p_i\}$ and $i_B = \min\{i : p^L \le r_i\}$.

Proposition 5. *Suppose that* (X^*, Y^*) *is a Nash equilibrium. Then under the standard price arrangement, we have for* $i_A \le i < n$, $C_A'(X_i^*) \le r_{i+1}$ *and for* $i_B \le i < n$, $C_B'(Y_i^*) \le p_i$.

Proof. Suppose that there exists an index i' such that $i_A \leq i' < n$ and $r_{i'+1} < C'_A(X^*_{i'})$, and in particular we define i' to be the first index satisfying these conditions. Then $x^*_{i'} > 0$, since if $x^*_{i'} = 0$, we have $C'_A(X^*_{i'}) = C'_A(X^*_{i'-1}) \leq r_i < r_{i'+1}$, which contradicts the definition of i'. Using (9), we have

$$\frac{\partial E_A(X^*, Y^*)}{\partial X_{i'}} < 0,$$

which, however, is impossible according to part (i) of Proposition 3. The relationship between p_i and $C'_B(Y^*_i)$ can be proved similarly.

At first sight it seems counter-intuitive that offers may be made and the market clear at a price below the marginal cost of generation: why would a generator deliberately bid in a way which will on some occasions lead to a loss? A simple example may clarify what is happening. Suppose that demand is uniformly distributed between 0 and 2, generator A has marginal cost of 2 , a capacity of 1 and can choose quantities to offer at prices 1 and 4. If it is known that generator B will bid its entire capacity of 1 at price 3 what should generator A do? Suppose that generator A bids an amount x at price 1, and the remainder at price 4. Then the profit that A makes is given by

$$E_A = \int_0^x u(1-2)(1/2)du + \int_x^{x+1} x(3-2)(1/2)du$$
$$+ \int_{x+1}^2 (u-1)(4-2)(1/2)du = 1/2 + x/2 - 3x^2/4.$$

This achieves a maximum value of 7/12 when $x = 1/3$, which is therefore the optimal bidding policy. So in this case there will be a probability of 1/6 that the market clears at price 1 and generator A makes a loss. However a smaller value of x leads to less profit when the other player sets the price.

Our next result is related to the highest clearing price that can occur. We show that, under certain conditions on the distribution of demand, offers in an equilibrium will be organized to allow the market to clear at the highest possible price when demand is at its upper limit. The condition on the demand distribution f is not restrictive.

Proposition 6. *Suppose that $(X^*, Y^*) \in \Omega$ is a Nash equilibrium under the standard price arrangement. Suppose also either that the density function f is bounded away from zero on $(d_2 - \delta, d_2]$ or the density function is non-increasing on the interval $(d_2 - \delta, d_2]$ for some small positive number δ. Then $p^H(X^*, Y^*) = p_n$.*

Proof. We suppose that for a contradiction that the highest clearing price $p_c(X^*, Y^*)$ is p_i for $i < n$. The other case, where the highest clearing price is some r_i, can be dealt with similarly. Thus $X^*_i + Y^*_i \geq d_2$ and $X^*_{i-1} + Y^*_i < d_2$.

We will show that there is a solution with a higher expected profit for A, to establish the contradiction. We do this in two stages. First we define a solution $\tilde{X}$ with $\tilde{x}_k = x^*_k$, for $k = 1, \ldots, i-1, i+1, \ldots, n-1$, and

$$\tilde{x}_i = d_2 - (X^*_{i-1} + Y^*_i).$$

Now as the greatest amount that A is dispatched under (X^*, Y^*) is $\tilde{X}_i$, it is easy to see that $\tilde{X}$ will have the same behaviour as X^* when played against Y^*. So,

$$E_A(\tilde{X}, Y^*) = E_A(X^*, Y^*). \tag{11}$$

Now observe that $\tilde{X}_i + Y_i^* = d_2$, and hence by assumption, $\tilde{x}_i > 0$. For $\epsilon > 0$ we define $\tilde{X}(\epsilon)$ as follows: $\tilde{x}_k(\epsilon) = \tilde{x}_k$, for $k = 1, \ldots, i-1, i+2, \ldots, n$, and

$$\tilde{x}_i(\epsilon) = \tilde{x}_i - \epsilon,\ \tilde{x}_{i+1}(\epsilon) = \tilde{x}_{i+1} + \epsilon.$$

By considering the derivative of $E_A(\tilde{X}(\epsilon), Y^*)$ with respect to changes in the quantity X_i we will show that the expected profit is greater if generator A uses $\tilde{X}(\epsilon)$ than if it uses $\tilde{X}$.

Notice that $\tilde{X}_i(\epsilon) + Y_i^* = d_2 - \epsilon$, $\tilde{X}_i(\epsilon) + Y_{i+1}^* = d_2 + y_{i+1}^* - \epsilon$. First we suppose that $y_{i+1}^* > 0$. Then we can choose $\epsilon > 0$ small enough so $d_1 < \tilde{X}_i(\epsilon) + Y_i^* < d_2$ and $\tilde{X}_i(\epsilon) + Y_{i+1}^* > d_2$. Hence,

$$\begin{aligned}\frac{\partial E_A(\tilde{X}(\epsilon), Y^*)}{\partial X_i} &= [f(\tilde{X}_i(\epsilon)+Y_i^*)(p_i - r_{i+1}) + f(\tilde{X}_i(\epsilon)+Y_{i+1}^*)(r_{i+1} - p_{i+1})]\tilde{X}_i(\epsilon) \\ &\quad + (r_{i+1} - C_A'(\tilde{X}_i(\epsilon)))(F(\tilde{X}_i(\epsilon)+Y_{i+1}^*) - F(\tilde{X}_i(\epsilon)+Y_i^*)) \\ &= f(d_2-\epsilon)(p_i - r_{i+1})(\tilde{X}_i - \epsilon) + (r_{i+1} - C_A'(\tilde{X}_i - \epsilon))(1 - F(d_2 - \epsilon)).\end{aligned}$$

Consider the case that the density function f is bounded away from zero on $(d_2 - \delta, d_2]$. Then for ϵ small enough the first term dominates the second and $\partial E_A(\tilde{X}(\epsilon), Y^*)/\partial X_i < 0$.

In the other case when the density function is non-increasing on the interval $(d_2 - \delta, d_2]$ then

$$0 \le 1 - F(d_2 - \epsilon) \le f(d_2 - \epsilon)\epsilon,$$

for $\epsilon \le \delta$. Thus

$$\begin{aligned}\frac{\partial E_A(\tilde{X}(\epsilon), Y^*)}{\partial X_i} &\le f(d_2 - \epsilon)[(p_i - r_{i+1})(\tilde{X}_i - \epsilon) + (r_{i+1} - C_A'(\tilde{X}_i - \epsilon))\epsilon]. \\ &< 0,\end{aligned}$$

for ϵ small enough (where the last inequality uses the fact that $\tilde{X}_i - \epsilon > 0$ for ϵ small enough).

In the case that $y_{i+1}^* = 0$ we have

$$\begin{aligned}\frac{\partial E_A(\tilde{X}(\epsilon), Y^*)}{\partial X_i} &= f(\tilde{X}_i(\epsilon) + Y_i^*)(p_i - p_{i+1})\tilde{X}_i(\epsilon) \\ &< 0,\end{aligned}$$

for ϵ small enough.

Hence in all cases we have established that

$$E_A(\tilde{X}(\epsilon), Y^*) > E_A(\tilde{X}, Y^*)$$

which, with our earlier observation (11), contradicts the assumption that (X^*, Y^*) is a Nash equilibrium.

It is interesting to compare this result with some of the results available for supply function equilibria. Here we have shown that, under mild conditions, an equilibrium solution will have the property of achieving the highest possible price for at least some demand realisations. We might expect that the continuous supply function equilibrium model would be obtained in the limit of a very large number of prices. In that case the equivalent result in a continuous setting would have the effect of ruling out equilibria which do not reach a price cap. However the usual formulations of supply function equilibrium do not have this property: there may be equilibria which reach the price cap, but there are often other more competitive equilibria, i.e. equilibria with lower prices at high demand (Baldick and Hogan(2001)).

5 Stability of a Nash equilibrium

We now turn to the stability of a Nash equilibrium in this setting. Any discussion of this topic presupposes a dynamic component in the behaviour of the market participants. A variety of models have been proposed as frameworks to capture this dynamic behaviour. Within an electricity market adjustments of the offers made can occur on a daily basis as market participants adjust their behaviour on the basis of the observed behaviour of other participants on the previous day. In the Australian market, which has served as motivation for the model we consider, bid adjustment can occur at any time during the day up to dispatch, and generators may choose to make a number of adjustments of their offer quantities in response to sensitivity information which derives from the offers made by other participants.

We have chosen to use the simplest definition of stability. We say that a Nash equilibrium (X^*, Y^*) is *stable* if one generator, say generator A, makes an initial offer X^0 sufficiently close to X^* and generator B responds optimally to A's offer, and then generator A responds optimally to B's offer, and so on, then the strategies played will approach X^* and Y^*. Many people have observed the unsatisfactory nature of this Cournot dynamic in which players go to the effort of making an optimal response to another player, when they know in advance that the other player will immediately change their play. More satisfactory models often incorporate some continuous adjustment method, in which players adjust their strategies in a direction which improves their payoff, without going as far as the optimal response (see Fudenberg and Levine(1998)). We have not attempted to establish any kind of stability result within a continuous adjustment framework, but it seems likely that similar results would hold.

To find conditions under which a Nash equilibrium is stable requires some stringent assumptions on the density function f. But rather than working through a further complex set of assumptions, we consider a straightforward case where the market demand has a uniform distribution, that is, the density function f is constant over $[d_1, d_2]$. It may appear that such an assumption is unrealistic given the fact that demand is by nature less likely to take the extreme values d_1 and d_2 than it is to take values in the middle of the possible range. Nevertheless, it can be a sensible model. Recall that as we mentioned in section 3, demand may change over a bidding period in a predictable way. If we consider a demand which changes linearly over the bidding period, and model this using "samples" of demand being taken every five minutes, then it may well be appropriate to think of the samples as being taken from a uniform distribution.

The theorem below is interesting because of the form of the conditions necessary for stability. It turns out, as we shall show, that when demand is uniform and prices become more spread out as they become higher, then stability is guaranteed when offers are close enough to marginal cost. This result suggests the possibility that an equilibrium may not be stable if it involves generators offering power at prices very much higher than their marginal costs. This conjecture is consistent with the work of Baldick and Hogan (2001) who have shown that in the supply function case a restriction to stable equilibria will rule out many possible equilibria at which generators offer at high prices.

Theorem 7 *Suppose that* (X^*, Y^*) *is a Nash equilibrium with* $Y_1^* > d_1$. *Suppose that*

$$f(t) = \begin{cases} \frac{1}{d_2 - d_1} & t \in [d_1, d_2] \\ 0 & \text{otherwise} \end{cases}$$

Suppose also that for $i = 1, \ldots, n-1$,

$$p_i - C_A'(X_i^*) < \frac{1}{2}(p_i + p_{i+1}) - r_{i+1},$$

and

$$r_i - C_B'(Y_i^*) < \frac{1}{2}(r_i + r_{i+1}) - p_i,$$

then (X^*, Y^*) *is stable.*

Proof. First observe that we can rewrite the conditions of the theorem as

$$r_{i+1} - C_A'(X_i^*) < \frac{1}{2}(p_{i+1} - p_i), \quad i = 1, \ldots, n-1. \tag{12}$$

$$p_i - C_B'(Y_i^*) < \frac{1}{2}(r_{i+1} - r_i), \quad i = 1, \ldots, n-1, \tag{13}$$

For simplicity of notation, we let

$$H_i(X, Y) = \frac{\partial E_A(X, Y)}{\partial X_i}, \; i = 1, \ldots, n-1,$$

$$H_n(X, Y) = X_n^* + Y_n^* - X_n - Y_n,$$

and

$$G_i(X, Y) = \frac{\partial E_B(X, Y)}{\partial Y_i}, \; i = 1, \ldots, n.$$

Since (X^*, Y^*) is a Nash equilibrium, it follows from Proposition 6 that $X_{n-1}^* + Y_n^* < d_2$ and $p_H(X^*, Y^*) = p_n$. Note that by assumption $Y_1^* > d_1$ and $X_n^* + Y_n^* > d_2$. Therefore $H_i(\cdot, \cdot)$, $i = 1, \ldots, n-1$ and $G_i(\cdot, \cdot)$, $i = 1, \ldots, n$ are well defined in a small neighborhood of (X^*, Y^*). Moreover, since $Y_1^* > d_1$, we have $p_L(X^*, Y^*) = r_1$ and hence from Proposition 4 we know that there is a non-zero quantity offered at every price. Thus the equilibrium point (X^*, Y^*) is a solution of the following system of equations:

$$\begin{aligned} H(X, Y) &= 0, \\ G(X, Y) &= 0, \end{aligned}$$

this follows from using Proposition 3, or just observing that the solution is not on any of the constraints in (X, Y) space.

Now, for $i = 1, \ldots, n-1$, we have

$$\frac{\partial H_i(X^*, Y^*)}{\partial X_i} = [(p_i - p_{i+1}) - C''_A(X_i^*)(Y_{i+1}^* - Y_i^*)]/(d_2 - d_1),$$

$$\frac{\partial H_i(X^*, Y^*)}{\partial Y_i} = -(r_{i+1} - C'_A(X_i^*))/(d_2 - d_1),$$

$$\frac{\partial H_n(X^*, Y^*)}{\partial X_n} = -1,$$

and

$$\frac{\partial H_i(X^*, Y^*)}{\partial Y_{i+1}} = (r_{i+1} - C'_A(X_i^*))/(d_2 - d_1),$$

$$\frac{\partial H_n(X^*, Y^*)}{\partial Y_n} = -1.$$

Since $\frac{\partial H_i(X^*,Y^*)}{\partial X_i} < 0$ for $i = 1, \ldots, n-1$, and $\frac{\partial H_i(X^*,Y^*)}{\partial X_j} = 0$ for $j \neq i$, the Jacobi matrix $\nabla_X H(X^*, Y^*)$ is nonsingular. By the classical implicit function theorem, there exists an open ball $\mathcal{U}(Y^*)$ with center Y^* and an implicit function $u : \mathcal{U} \to \mathbb{R}^n$ such that $X^* = u(Y^*)$ and $u(\cdot)$ is player A's best response function satisfying

$$H(u(Y), Y) = 0$$

for $Y \in \mathcal{U}(Y^*)$. Moreover

$$\frac{\partial u_i(Y^*)}{\partial Y_i} = -\frac{r_{i+1} - C'_A(X_i^*)}{p_{i+1} - p_i + C''_A(X_i^*)(Y_{i+1}^* - Y_i^*)},$$

$$\frac{\partial u_i(Y^*)}{\partial Y_{i+1}} = \frac{r_{i+1} - C'_A(X_i^*)}{p_{i+1} - p_i + C''_A(X_i^*)(Y_{i+1}^* - Y_i^*)},$$

$$\frac{\partial u_n(Y^*)}{\partial Y_n} = -1,$$

and

$$\frac{\partial u_i(Y^*)}{\partial Y_j} = 0, \ j \neq i, i+1.$$

Let $\|M\|$ denote the 2-norm of a matrix M. Then by (12), we have an estimate for the 2-norm of the Jacobi matrix $\nabla u(Y^*)$,

$$\|\nabla u(Y^*)\| \le \max\left(1, \ \max_{i=1}^{n-1} 2\frac{r_{i+1} - C'_A(X_i^*)}{p_{i+1} - p_i}\right) = 1.$$

In a similar way we can define v as a best response function for player B on an open ball $\mathcal{V}(X^*)$ centered at X^* with similar formula except for $i = n$, when

$$\frac{\partial G_n(X^*,Y^*)}{\partial Y_n} = [(r_n - 2p_n + C_B'(Y_n^*)) - C_B''(Y_n^*)(d_2 - Y_n^* - X_{n-1}^*)]/(d_2 - d_1),$$

$$\frac{\partial G_n(X^*,Y^*)}{\partial X_{n-1}} = -(p_n - C_B'(Y_n^*))/(d_2 - d_1).$$

Thus

$$\frac{\partial v_n(X^*)}{\partial X_{n-1}} = \frac{-(p_n - C_B'(Y_n^*))}{r_n + C_B'(Y_n^*) - 2p_n - C_B''(Y_n^*)(d_2 - Y_n^* - X_{n-1}^*)}$$

and

$$\frac{\partial v_n(X^*)}{\partial X_j} = 0,\ j \neq n-1.$$

Hence we can show, using (13), that

$$\|\nabla v(X^*)\| \leq \max\left\{\frac{p_n - C_B'(Y_n^*)}{2p_n - r_n - C_B'(Y_n^*)},\ \max_{i=1}^{n-1} 2\frac{p_i - C_B'(Y_i^*)}{r_{i+1} - r_i}\right\}$$
$$<1.$$

Let $\gamma > 0$ be a small positive number with $\gamma < 1 - \|\nabla v(X^*)\|$ and $\mathcal{V}(X^*)$ be small enough so that

$$\|v(X)) - v(X^*)\| \leq (\gamma + \|\nabla v(X^*)\|)\|X - X^*\|$$

for every $X \in \mathcal{V}(X^*)$. Let $\delta > 0$ be such that

$$\delta < \frac{1}{\gamma + \|\nabla v(X^*)\|} - \|\nabla u(Y^*)\|$$

and $\mathcal{U}(Y^*)$ be sufficiently small so that

$$\|u(Y) - u(Y^*)\| \leq (\|\nabla u(Y^*)\| + \delta)\|Y - Y^*\|$$

for every $Y \in \mathcal{U}(Y^*)$.

Let $\mathcal{U}(Y^*)$ and $\mathcal{V}(X^*)$ be sufficiently small so that

$$u(\mathcal{U}(Y^*)) \subset \mathcal{V}(X^*),\ v(\mathcal{V}(X^*)) \subset \mathcal{U}(Y^*).$$

Now suppose that Y^1 is a small perturbation of Y^* in $\mathcal{U}(Y^*)$. Then the best response from player A is $X^1 = u(Y^1)$ and

$$\|X^1 - X^*\| = \|u(Y^1) - u(Y^*)\| \leq (\|\nabla u(Y^*)\| + \delta)\|Y^1 - Y^*\|.$$

Given X^1, the best response from player B is $Y^2 = v(X^1)$ and

$$\|Y^2 - Y^*\| = \|v(X^1) - v(X^*)\| \leq (\|\nabla v(X^*)\| + \gamma)\|X^1 - X^*\|$$
$$\leq (\|\nabla v(X^*)\| + \gamma)(\|\nabla u(Y^*)\| + \delta)\|Y^1 - Y^*\|.$$

Let $\alpha = (\|\nabla v(X^*)\| + \gamma)(\|\nabla u(Y^*)\| + \delta)$. By the choice of δ, $\alpha < 1$ and consequently $\|Y^2 - Y^*\| < \|Y^1 - Y^*\|$. Thus $Y^2 \in \mathcal{U}(Y^*)$. For Y^2, the best response from player A is $X^2 = u(Y^2)$ and

$$\|X^2 - X^*\| \leq (\|\nabla u(Y^*)\| + \delta)\|Y^2 - Y^*\|$$
$$\leq \alpha(\|\nabla u(Y^*)\| + \delta)\|Y^1 - Y^*\| < (\|\nabla u(Y^*)\| + \delta)\|Y^1 - Y^*\|.$$

Continuing with this process, we obtain two sequences $\{X^k\}$ and $\{Y^k\}$ such that $\{X^k\} \subset \mathcal{V}$ and $\{Y^k\} \subset \mathcal{U}$ with

$$\|Y^k - Y^*\| < \alpha^{k-1}\|Y^1 - Y^*\|,$$

and

$$\|X^k - X^*\| < \alpha^{k-1}(\|\nabla u(Y^*)\| + \delta)\|Y^1 - Y^*\|.$$

Since $\alpha < 1$, we have shown that $\{X^k\} \to X^*$ and $\{Y^k\} \to Y^*$ as $k \to \infty$. In the case that player A makes a first perturbation, we can present a similar analysis and draw the same conclusion. □

Example 8. Suppose that demand is uniformly distributed between zero and one, $C_A(X) = X$ and $C_B(Y) = Y$, and $r_1 = 2$, $p_1 = 3$, $r_2 = 4$, $p_2 = 5$, $r_3 = 6$, $p_3 = 7$, $r_4 = 8$, $p_4 = 9$, $r_5 = 10$, $p_5 = 11$. Let

$$X^*(t) = (0.1294, 0.2373, 0.3409, 0.4429, t),$$
$$Y^* = (0.1294, 0.2157, 0.3107, 0.4081, 0.5065),$$

where $0.4935 < t \leq 1$. Then we can show that $(X^*(t), Y^*)$ is a Nash equilibrium.

Note that the equilibria we obtain in this example do not satisfy the conditions of the theorem. Indeed, none of the set of equilibria $(X^*(t), Y^*)$ is stable. We have coded a Matlab program and tested the Nash equilibrium with $t_0 = 0.4935$ (note that any value of t greater than this obviously gives the same result). We start with $(X^*(t_0), Y^*)$ and then player A makes its best response to the given strategy Y^* by player B: this is X^1. Then player B reacts to X^1 optimally by playing Y^1 and so on. Now note that $(X^*(t_0), Y^*)$ can only be an approximate Nash equilibrium, since the equilibrium is obtained numerically. Thus we will not have (X^1, Y^1) being exactly the same as $(X^*(t), Y^*)$. In the case that the equilibrium is stable then we would find that multiple iterations of this process would leave the solution very close to its starting point. However if the equilibrium is not stable, then starting with an approximate solution will, over a number of iterations of the process, lead away from the Nash equilibrium. This latter pattern is exactly what we observe. After 1 iteration, we have

$$X^1 = (0.1295, 0.2375, 0.3409, 0.4426, 0.4935),$$

$$Y^1 = (0.1295, 0.2161, 0.3102, 0.4076, 0.5065).$$

After 5 iterations, the sequences (X^k, Y^k) enters cycling with

(a) $X^{6+5k} = (0.0015, 0.8315, 0.8315, 0.8315, 0.8315),$

$Y^{6+5k} = (0.0015, 0.7988, 0.7988, 0.7988, 0.7988),$

(b) $X^{7+5k} = (0.7489, 0.7489, 0.7489, 0.7489, 0.7489),$

$Y^{7+5k} = (0.6667, 0.6667, 0.6667, 0.6667, 0.6667),$

(c) $X^{8+5k} = (0, 0.001, 0.002, 0.003, 0.3333),$
$Y^{8+5k} = (0, 0.002, 0.003, 0.004, 0.9064),$

(d) $X^{9+5k} = (0.002, 0.003, 0.004, 0.8964, 0.8965),$
$Y^{9+5k} = (0.0013, 0.0023, 0.0033, 0.8853, 0.8853),$

and

(e) $X^{10+5k} = (0.0015, 0.0025, 0.8721, 0.8721, 0.8721),$
$Y^{10+5k} = (0.0013, 0.0023, 0.855, 0.855, 0.855),$

for $k = 0, 1, 2, \ldots$.

We should note that the fact that cycling can occur for some initial conditions does not in itself make a Nash equilibrium unstable.

6 Discussion

Other than our discussion of the single price example in section 2 we have said nothing about the equilibrium solution in stage 1 of the game, when prices are set. The question of which price points should be selected by the generators is interesting, but very complex when multiple prices are involved. The underlying structure of the problem makes it probable that an equilibrium solution will involve a continuous distribution over prices. Not only is this the behaviour we observe in the one price case, but it also occurs in the price-setting game considered by von der Fehr and Harbord (1993). However any type of exact analysis seems out of reach.

The usual market models involve the choice of prices and the choice of quantities at the same time. However participants in actual markets may choose to limit the variation in prices that they offer. This is because a market in which prices are freely adjusted, and in which power is offered in blocks, has the characteristics of a Bertrand equilibria in which there is a tendency to push towards a competitive outcome even with only two players. This arises because generators have the capacity to observe a competitor's bid and then to undercut that bid by a small amount. Though this is a result which might be welcomed by market regulators, there is no long term incentive for participants in the market to adopt this strategy. In fact in most actual electricity market contexts this would be perceived as a very aggressive pricing strategy. It may be that a type of collusive outcome in which there is a voluntary decision to limit the extent to which different prices are utilised makes it more likely that an equilibrium with relatively higher prices will be reached. There have been suggestions that in the early years of the England and Wales electricity pool the two dominant players bid prices that were interleaved in exactly the fashion indicated by Proposition 2.

If we consider our fixed price point model as the number of price points increases, then the restrictions imposed by this form of bid structure lessen. As an extreme example we might simply restrict prices offered to be at a whole number of cents. However, as the number of possible price points multiplies, the assumption we have made that there are no prices in common between the two players becomes harder to justify.

One important observation from the work discussed here is the key role played by the distribution of demand. The equilibria we study do not occur when there is no uncertainty in demand. This approach supports previous work by the authors (Anderson and Philpott (2002b), Anderson and Xu (2002)) in which stochastic demand is an important ingredient in the analysis.

Our final result, in which we show a link between the stability of a Nash equilibrium and the extent to which the generator bids differ from their marginal costs, is of interest, but is much weaker than we might wish: first because of the very restrictive conditions under which it is proved (uniform demand, expanding price gaps) and second because we have not established the reverse implication. Thus we have found a case where stability is ensured by sufficiently competitive bidding in an equilibrium, but we have not shown that an equilibrium that has bids far above marginal costs is necessarily unstable. Nevertheless this result is interesting from a policy perspective because it suggests that generators in an oligopoly may find it hard to support a non-competitive equilibrium because of stability issues. However to verify whether or not this is true would require much more work, perhaps including considerable simulation testing. For the continuous supply function equilibrium approach Baldick and Hogan (2001) give an extensive treatment of the issue of stability. Their results go beyond ours in demonstrating both theoretically and empirically that many non-competitive equilibria will not be stable.

One limitation of this paper is the restriction to the case with just two generators. Unfortunately the situation with three or more generators is very much harder to analyse. The tools we have used are very closely tied to the two player case. Equally it is very hard to generalise our results to a situation in which demand is price sensitive. Finally, we note that it is possible to introduce in our discussion hedging contracts that generators may sign prior to entering spot markets. Anderson and Xu (2001) consider this situation and derive different forms of some of the results here.

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