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# Supply Function Equilibrium in Electricity Spot Markets with Contracts and Price Caps[1,2]

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**Abstract.** In electricity wholesale markets, generators often sign long term contracts with purchasers of power in order to hedge risks. In this paper, we consider a market where demand is uncertain, but can be represented as a function of price together with a random shock. Each generator offers a smooth supply function into the market and wishes to maximize his expected profit, allowing for his contract position. We investigate supply function equilibria in this setting, using a model introduced by Anderson and Philpott. We study first the existence of a unique monotonically increasing supply curve that maximizes the objective function under the constraint of limited generation capacity and a price cap, and discuss the influence of the generator's contract on the optimal supply curve. We then investigate the existence of a symmetric Nash supply function equilibrium, where we do not have to assume that the demand is a concave function of price. Finally, we identify the Nash supply function equilibrium which gives rise to the generators' maximal expected profit.

**Key Words.** Electricity markets, supply functions, contracts, price caps, Nash equilibrium.

## 1. Introduction

In this paper, we study the impact of contracts and price caps on the bidding behavior of electricity generators in a wholesale spot market.

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Such wholesale spot markets now operate in many different parts of the world. Though the detailed market rules vary, the majority of markets operate some sort of pool system in which generators bid different quantities of power into the market at different prices, and then an independent system operator decides how actual demand is to be met by dispatching cheaper power first. There is a single price at which the market clears and this is the price paid to each generator for all the power they supply. Actual electricity markets have to operate within the constraints of an electricity network, and moreover have a number of features designed to ensure continuity of supply, but in this paper we will consider a very simple market setup in which all the generators bid at a single node. Generators compete with each other, with each generator offering a supply schedule usually made up of tranches of power at different prices. We will abstract from this situation to suppose that generators each offer an arbitrary nondecreasing supply function. We are interested in equilibrium behavior which leads to a supply function equilibrium (SFE) Model.

The SFE concept was originally developed by Klemperer and Meyer (Ref. 1) to model a situation when each player's supply function is optimal in every realization of a random demand in an oligopoly market. Green and Newbery (Ref. 2) noted that the uncertainty of the demand is equivalent to a daily time-varying demand, when a single supply schedule is applied over a whole day and used SFE to model optimal supply functions in the electricity market in England and Wales. Since then, the SFE model has been used widely to study the bidding behavior in electricity spot market; see for instance Refs. 3–6 and the references therein. Recently, Baldick and Hogan (Ref. 7) have discussed the existence of supply function equilibria when there are price caps. A number of authors have considered restricted forms of SFE, in which the generator's bids are constrained, perhaps having price as an affine function of quantity. As pointed out by Baldick (Ref. 8), the way that supply functions are parametrized can have a significant effect on the equilibrium. The model that we use does not involve any prior parametrization, and so does not suffer from this weakness.

In electricity markets, generators often sign contracts to hedge risks. Newbery (Ref. 9) has investigated the effect of contracts on SFE and hence on market behavior in an electricity spot market. When there are constant marginal costs, limited capacities, and linear demand, Newbery derives a symmetric nonlinear SFE in the spot market and uses it to explain why generators have incentives to sell contracts, even though that makes the spot market more competitive and hence lowers the total profits.

Green (Ref. 10) discusses also how contracts can affect the optimal supply function and the resultant SFE. He shows that a generator should offer at prices which depend on the amount covered by the contracts held by the generator. When the generator's output is greater (less) than the contract amount, the generator's sale price should be greater (less) than his marginal cost. This generalizes one of the conclusions derived by Allaz and Vila (Ref. 11) in a more general context. Further, by assuming that the generators commit to a linear asymmetric SFE in the spot market, Green derives equilibria in the contract market: in a Cournot-Nash equilibrium, generators will sell no contracts, while in a Bertrand-Nash equilibrium, generators will cover all of their expected output in their contracts. Similar conclusions are also drawn by Newbery (Ref. 9).

We note that there have been other models discussing the influence of contracts on the bidding behavior in the spot markets. An early paper on this topic is by Von der Fehr and Harbord (Ref. 12) who model a spot market as a multi-unit auction and demonstrate how contracts tend to put downward pressure on spot market prices. Gans, Price, and Woods (Ref. 13) and Wolak (Ref. 14) both discuss the way that hedge contracts reduce a generator's market power. Borenstein (Ref. 15) discusses the role of long-term contracting in his recent analysis of the problems afflicting the California electricity market.

In this paper, we extend the work of Green and Newbery by studying the influence of contracts and price caps on the (Nash) SFE in a market with just two strategic generators. There may be other generators in the market, but they are assumed to be price takers: hence they influence the residual demand as a function of price, but otherwise can be left out of the analysis. We consider a spot market where the demand is uncertain, but can be represented as a function of price and a random shock. Each generator offers a smooth supply function into the spot market and may sign contracts in a contract market in order to hedge risks in the spot market. Since a generator offers a supply function into the spot market before demand is realized, he will maximize his expected profit, rather than the actual profit. Anderson and Philpott (Ref. 16) prove that the expected profit for a generator can be expressed as a line integral of a market distribution function along his offer supply curve, where the market distribution function captures the information about the probability distribution of uncertain residual demand.

Our model differs from that of Refs. 9–10 in a number of ways. First, our model is less restrictive than previous supply function treatments of the contract case. We take the market demand to be nonlinear in the price and subject to a bounded random shock. In dealing with symmetric equilibria in the spot market, we do not assume that the demand is concave, which has been a restrictive assumption in previous analyses. Second, as well as allowing for generators with capacity limits, we model also a spot market in which there is a price cap. Such price caps exist in most markets and are often the focus of considerable debate. These restrictions, together with the boundedness of the random shock in the market demand, imply that a generator's supply function is effective only on a part of the price range and hence the optimality of a supply function is considered only in such a region. This makes our analysis somewhat more complicated than those in the literature. Finally, we derive the optimal supply function of a generator by considering the maximization of the expected profit. Previous work in this area has derived the optimal continuously differentiable supply function by considering the maximization of a generator's profit on every realization of the random shock in the market demand. We will show that, in the cases that we consider, the two approaches coincide.

Our belief is that the treatment that we give is close to being as complete as possible for the symmetric duopoly case with both generators at a single node.

In practice, the case where generators supply power at different nodes in a large transmission system is of great interest. There have been a number of papers in the literature which discuss the way that generators compete over a network. Most of these papers (Refs. 17–23) consider Cournot competition, with only a few papers (Refs. 24–26) using a supply function model. Berry et al. (Ref. 24) consider a supply function model to study how generators might exercise market power by using networks. In their model, the generators submit linear supply functions to the independent system operator (ISO) at the nodes where they are located and the node prices are determined by the ISO from the solution of a maximum welfare problem under the constraints on the network flows. Strategic generators have a full knowledge of the ISO's price clearing mechanism and try to maximize their profit. Mathematically, each generator's optimization problem is a bilevel programming program: at the first level, a supply function is chosen; at the second level, the node prices are determined given the supply and fixing the other generators' supply (through the Nash conjecture).

Unfortunately, it is difficult to extend the discussion of Berry et al. to the case where the supply functions are nonlinear. In this case, the lower-level solution, which is a function of the supply function under consideration, is not a linear functional. We will no longer know that the generator's objective function is concave. Thus, it is unclear whether or not an individual generator's optimal supply function exists and whether it is unique if it does exist. It is even more difficult to obtain a Nash equilibrium in this case.

## 2. Model

We write $R_i(q, p)$ for the profit obtained by generator $i$ if this generator is dispatched an amount $q$ and the spot price is $p$, where $0 \leq q \leq q_i^M$ and $0 \leq p \leq p^M$, $q_i^M$ is the maximum generation capacity of generator $i$ and $p^M$ is the price ceiling in the spot market. We assume that the cost of generating a quantity $q$ of electricity is given by $C_i(q)$, which we assume to be an increasing convex function. Besides the money $pq$ paid to the generator and the costs incurred in that level of generation, we must consider also the contracts entered into by the generator. These are financial instruments which do not involve the actual generation of electricity, but the money paid under the contract is tied to the pool price. If generator $i$ enters into a contract at a strike price $f$ for a quantity $Q_i$ and if the actual spot price is $p$, then the generator will pay an amount $Q_i(p-f)$ to the other party in the contract. The contracts that we consider are two-way contracts for differences; so, if the spot price is lower than the contract strike price, then the generator will receive an amount $Q_i(f-p)$. Contracts of this sort are a common feature of electricity markets operating with a pool structure in which the prices for all traded electricity are determined through a combined pricing and dispatch mechanism (such as the markets operating in Australia and New Zealand and the old pool arrangements in England and Wales). This is quite a different environment than that of markets which are based on bilateral contracts, such as in the new trading arrangements in England and Wales.

Thus, we arrive at the following expression for the profit to generator $i$ as a function of the spot price $p$ and dispatched quantity $q$:

$$R_i(q, p) = pq - C_i(q) + Q_i(f - p). \tag{1}$$

Throughout this paper, we assume that the spot market demand $D(p, \epsilon)$ is dependent on the price $p$ and is subject to some random shock $\epsilon$ with known distribution. We denote by $g$ the density function of the distribution of the random shock and assume that $g$ is well defined and has the complete interval $[\epsilon_1, \epsilon_2]$ as support set. We assume also that $D''_{p\epsilon}(p, \epsilon) = 0$ [this is one of the assumptions made by Klemperer and Meyer (Ref. 1) and by others who have looked at the supply function model]. This amounts to a restriction that demand is translated horizontally by the random shock $\epsilon$. It is convenient to reparametrize $\epsilon$ so that we can write

$$D(p, \epsilon) = D(p) + \epsilon;$$

this can be done without any loss of generality once we assume that $D''_{p\epsilon}(p,\epsilon)=0$. We take the demand to be continuously differentiable and strictly decreasing in price, that is, $D'(p)<0$.

In the spot market, each generator makes an offer into the market: in many cases, this takes the form of an offer stack, being a set of quantities at increasing prices. We use a supply function $S_i$ to describe the price-quantity relationship, so that $S_i(p)$ denotes the quantity offer of generator $i$ at price $p$. The way that the market operates means that we must restrict the supply functions to be nondecreasing. The supply function is defined on $[0, p^M]$, where $p^M$ is the price ceiling that operates in the spot market. In many markets, it is possible to bid at a negative price. In the middle of the night, it can happen occasionally that the spot prices are negative. This occurs when the demand is very low and the generators have to carry the cost of turning equipment on and off: sometimes, it makes sense for a generator to pay for the privilege of being left on. However in our model, for simplicity, we normalize the prices so that the lower limit is zero. There is no loss of generality in doing this.

We will use the graph of a supply function which is defined as

$$\mathrm{s}_i=\left\{(S_i(p),p):0\le p\le p^M\right\}.$$

The graph $\mathrm{s}_i$ is a curve in the price-quantity plane, which we call the supply curve. Anderson and Philpott (Ref. 16) prove that the expected return for player $i$ by offering a supply function $S_i(p)$ can be expressed as a line integral over the supply curve $\mathrm{s}_i$,

$$E_i(\mathrm{s}_i)=\int_{\mathrm{s}_i} R_i(q,p)d\psi_i(q,p),$$

where $\psi_i$ is a continuous market distribution function for player $i$; $\psi_i(q,p)$ represents the probability that generator $i$ is not fully dispatched if it offers a quantity $q$ of power at price $p$ [equivalently, $\psi_i(q,p)$ is the probability that the residual demand left after other generators are dispatched at price $p$ is strictly less than $q$].

Let $S_j(p)$ be the aggregate supply function offered by the other generators on the market. Under our assumption on the market demand, we can write the market distribution function as

$$\begin{aligned}\psi_i(q,p)&=\Pr(D(p)+\epsilon-S_j(p)<q)\\&=G(q+S_j(p)-D(p)),\end{aligned}$$

where $G(\cdot)$ is the distribution function of the random shock $\epsilon$. Consequently,

$$(\psi_i)'_p = g(q + S_j(p) - D(p))\big(S'_j(p) - D'(p)\big),$$
$$(\psi_i)'_q = g(q + S_j(p) - D(p)).$$

Clearly, provided that $S_j$ is continuously differentiable, then $\psi_i$ will also be continuously differentiable.

Note that $E_i(\mathrm{s}_i)$ depends only on the part of the curve $\mathrm{s}_i$ where

$$d\psi_i(S_i(p), p) \neq 0,$$

or equivalently,

$$g(q + S_j(p) - D(p)) \neq 0.$$

Let $\Psi_i$ denote the set of points $(q, p)$ where

$$g(q + S_j(p) - D(p)) \neq 0;$$

thus,

$$\Psi_i = \{(q, p) : \epsilon_1 \leq q + S_j(p) - D(p) \leq \epsilon_2\}.$$

In choosing a supply curve, the generator $i$ need only consider the part of this curve located in $\Psi_i$.

## 3. Optimality Conditions

Suppose that generator $i$ holds contracts for a total quantity $Q_i$, where $0 < Q_i \leq q_i^M$, and then offers into the spot market a supply curve $\mathrm{s}_i$. Since generator $i$ cannot offer more than $q_i^M$ and since the highest offer price cannot exceed $p^M$, any supply curve from generator $i$ must be located in the region $[0, q_i^M] \times [0, p^M]$. On the other hand, as we discussed above, the expected profit $E_i(\mathrm{s}_i)$ depends only on the part of curve $\mathrm{s}_i$ located within $\Psi_i$. Therefore, our discussion of the optimality conditions for $\mathrm{s}_i$ is focused on the region $\Psi_i \cap [0, q_i^M] \times [0, p^M]$. Let $\Psi_i^o$ denote the interior of the set $\Psi_i \cap (0, p^M) \times (0, q_i^M)$. Let $\Omega_i$ denote the set of curves $\mathrm{s}_i$ such that $S_i(\cdot)$ is continuous and strictly increasing.

Suppose that $S_j$ is strictly increasing and continuously differentiable on $(0, p^M)$ and that $\mathrm{s}_i \in \Omega_i$ maximizes $E_i(\cdot)$, so the maximizing offer curve is strictly increasing. Anderson and Philpott (Ref. 16) prove that

$$Z_i(S_i(p), p) = 0,$$

for all $p \in (0, p^M)$ such that $(S_i(p), p) \in \Psi_i^o$, where

$$Z_i(q,p) \equiv (R_i)'_q(q,p)(\psi_i)'_p(q,p) - (R_i)'_p(q,p)(\psi_i)'_q(q,p).$$

Since

$$g(q + S_j(p) - D(p)) > 0,$$

we can rewrite the necessary optimality condition above as

$$\big(p - C_i'(S_i(p))\big)\big(S_j'(p) - D'(p)\big) - S_i(p) + Q_i = 0, \tag{2}$$

for $(S_i(p), p) \in \Psi_i^o$.

Green (Ref. 10) derives a similar condition to (2) for a supply function to be optimal. In his paper, the demand is a linear function of price, that is,

$$D(p) = A - bp,$$

where $A$ and $b$ are constant. He shows that, if generator $i$ supplies $S_i(p)$ at price $p$ and wishes to maximize his profit for every $p$ (market clearing price), then $S_i(p)$ must satisfy the condition

$$\big(p - C_i'(S_i(p))\big)\big(S_j'(p) + b\big) - S_i(p) + Q_i = 0.$$

In Green's framework, the supply function $S_i(p)$, if it turns out to be monotonic and hence feasible, will be an optimal offer for every demand realization in a range of demands. Essentially, each demand instance picks out one point on the $S_i(p)$ curve. Our framework is different, allowing a weaker notion of optimality (that is, optimality in expectation). However, the difference between these two approaches turns out to be minimal in this context. Observe that the defining equation (2) is independent of the distribution $g$ and so must also apply to the optimal choice of offer when the demand is deterministic. In other words, the offer curve will lead to an optimal dispatch whatever the demand. This demonstrates that, in the case that the only uncertainty relates to a demand shock involving a translation of the demand curve, the extra generality given by the expected profit framework is only of value when the optimal offer curve contains either horizontal or vertical segments (and hence is constrained by the monotonicity condition).

From (2), we see that, for an optimal supply function $S_i$, if $p$ is chosen so that $S_i(p) = Q_i$, then we must have

$$p = C_i'(S_i(p)) = C_i'(Q_i),$$

since $S_j$ is increasing and $D$ is strictly decreasing, and thus

$$S_j'(p) - D'(p) > 0.$$

Hence, an optimal supply curve $s_i$ must pass through the point $(Q_i, C_i'(Q_i))$ in the $(q, p)$ plane, so that

$$S_i\big(C_i'(Q_i)\big) = Q_i.$$

Moreover, as $S_i$ is increasing, we can see that, when $p < C_i'(Q_i)$, we must have $S_i(p) \le Q_i$. Since $S_j'(p) - D'(p) > 0$, we have from (2),

$$p \le C_i'(S_i(p)).$$

Similarly, when $p > C_i'(Q_i)$, we can show that

$$p \ge C_i'(S_i(p)).$$

This means that, if $Q_i > 0$, then in order to obtain the maximum profit, player $i$ will supply electricity up to the contract amount $Q_i$ at a price lower than its marginal cost, but will offer at higher than the marginal cost for amounts greater than the contract amount. This has been observed also by Green (Ref. 10). This result is not surprising, since from (1) if the generator is dispatched less than the contract quantity $Q_i$, then the generator makes more profit when the price is lower, and the reverse if the generator is dispatched more than the contract amount.

Though (2) can be used to find a function $S_i(p)$, it can only be a supply function if it is nondecreasing. The next result gives conditions for the function defined in this way to be monotonic and shows also what happens to the solution at the boundaries of the region $[0, q_i^M] \times [0, p^M]$. In order to prove this result, we need to use a framework which does not assume the existence of a function $S(p)$ at the outset. Thus, it is convenient to consider the following system:

$$\begin{aligned}\tilde{Z}_i(q, p) &\equiv \big(p - C_i'(q)\big)\big(S_j'(p) - D'(p)\big) - q + Q_i \\ &= 0, \end{aligned} \tag{3}$$

for $(q, p) \in \big(0, q_i^M\big) \times (0, p^M)$. Here,

$$Z_i(q, p) = g(q + S_j(p) - D(p))\tilde{Z}_i(q, p), \quad \text{for } (p, q) \in \Psi_i^o.$$

We will show that the solution to (3), when appropriately extended, will trace out the optimal offer curve in the $(q, p)$ plane. This result extends previous work which has dealt with the symmetric case (Ref. 2) or inelastic demand (Ref. 27).

**Proposition 3.1.** Suppose that $S_j$ and $D$ are twice continuously differentiable on $(0, p^M)$ and that $\big(Q_i, C_i'(Q_i)\big) \in (0, q_i^M) \times (0, p^M)$. Suppose that, for $p \in [0, p^M]$,

$$[A(p)]^2 \geq -Q_i A'(p), \tag{4}$$

$$A(p) \leq -pA'(p), \tag{5}$$

where $A(p) = D'(p) - S_j'(p)$ is the slope of the residual demand curve and is negative. Then, the locus of points satisfying (3) intersects the rectangle $(0, q_i^M) \times (0, p^M)$ in a single continuous monotonic curve. If we write $A_L = (q_L, p_L)$ and $A_H = (q_H, p_H)$ for the points through which this curve passes on the boundary of the rectangle, then we can represent the curve as a supply function $S_i(p)$ defined on $(p_L, p_H)$, or equivalently as an offer function $T_i(q) = S_i^{-1}(q)$ defined on $(q_L, q_H)$. We extend $T_i$ by defining $T_i(q) = 0$, for $0 \leq q \leq q_L$, and $T_i(q) = p_M$, for $q_H \leq q \leq q_i^M$. Then, $T_i$ is the unique optimal offer function up to changes outside $\Psi_i$.

**Proof.** By assumption, for all $(p, q) \in (0, p^M) \times (0, q_i^M)$,

$$C_i''(q) \geq 0 \quad \text{and} \quad A(p) < 0.$$

Thus,

$$(\tilde{Z}_i)_q'(q, p) = -1 + C_i''(q)A(p) < -1, \tag{6}$$

for all $(p, q) \in (0, p^M) \times (0, q^M)$. On the other hand,

$$\tilde{Z}_i\big(Q_i, C_i'(Q_i)\big) = 0$$

and by assumption

$$\big(Q_i, C_i'(Q_i)\big) \in \Psi_i^o.$$

By the implicit function theorem, there exists a unique continuous and differentiable function $S_i(p)$ in a neighborhood $\mathcal{N}$ of $C_i'(Q_i)$ such that $S_i(C_i'(Q_i)) = Q_i$ and $(S_i(p), p)$ satisfies (3) for all $p \in \mathcal{N}$.

We prove now that $S_i(p)$ is increasing over $N$ (and hence can be used as a supply function). The derivative of $S_i$ is given by

$$S_i'(p) = \big[-A'(p)\big(p - C_i'(S_i(p))\big) - A(p)\big] / \big[1 - C_i''(S_i(p))A(p)\big]. \tag{7}$$

Observe that the denominator of this expression is always positive. When $A'(p) > 0$, we have from (5)

$$\begin{aligned} -A'(p)\big(p - C_i'(S_i(p))\big) - A(p) &\geq A'(p)C_i'(S_i(p)) \\ &> 0, \end{aligned}$$

which implies $S_i'(p) > 0$.

Now, consider the case that $A'(p) \leq 0$. Using (2), we can rewrite (7) as

$$S_i'(p) = \left[-A'(p)(S_i(p) - Q_i) + [A(p)]^2\right] / \left[-A(p)\left(1 - C_i''(S_i(p))A(p)\right)\right]. \quad (8)$$

Thus, from (4),

$$\begin{aligned} -A'(p)(S_i(p) - Q_i) + A(p)^2 &\geq A'(p)Q_i + [A(p)]^2 \\ &> 0, \end{aligned}$$

which implies that $S_i'(p) > 0$.

It is easy to see that we can continue to extend the function $S_i(p)$ out to the edges of the rectangle $(0, q_i^M) \times (0, p^M)$. Essentially, all we need do is to apply the implicit function theorem again at any point $(S_i(p), p)$ which is on the boundary of $\mathcal{N}$ to keep on extending the range. The uniqueness of $S_i(p)$ follows from (6); the fact that every solution goes through $(Q_i, C'(Q_i))$ ensures that $T_i(q)$ is also unique . We can observe also that, when treated as a supply curve $s_i$, our extension amounts to joining $A_L$ to the origin and $A_H$ to the top corner of the rectangle $(q_i^M, p^M)$.

All that remains is to check the second-order sufficient conditions for a local optimum. We have already shown that $(\tilde{Z}_i)_q'(q, p) < 0$. Since the curve $\tilde{Z}_i(q, p)) = 0$ is monotonic increasing, we can deduce from the implicit function theorem that $(\tilde{Z}_i)_p'(q, p) > 0$ along that part of the supply function in $\Psi_i^o$.

Observe that, since under the conditions of the theorem we have only one line where $Z_i(q, p) = 0$, if $q_L = 0$ then $Z_i(0, p) \leq 0$ for $p < p_L$ and if $p_L = 0$ then $Z_i(q, 0) \geq 0$ for $q < q_L$. Similarly, if $q_H = q_i^M$, then $Z_i\left(q_i^M, p\right) \geq 0$ for $p > p_H$, and if $p_H = p^M$ then $Z_i\left(q, p^M\right) \leq 0$ for $q > q_H$. It is not hard to check from these observations that the sufficient conditions for a local optimum hold, as established in Ref. 28, Theorem 7. The global optimality of the solution follows, since there is an optimal solution (see Ref. 28) and no other solution can exist satisfying the necessary conditions for optimality. □

We make some comments on the assumptions (4) and (5) which are used to prove the monotonicity of $S_i$.

If the residual demand function is concave in $p$ [that is, $A'(p) < 0$], then (5) is trivially satisfied. On the other hand, when the residual demand function is convex in $p$ [that is, $A'(p) > 0$], condition (4) becomes trivial. In either case, the other constraint imposes a restriction that the absolute size of $A'(p)$is not too large in comparison with $A(p)$. Condition (5) is the generalization to this case of the restriction that $S_j(p)$ be inverse log concave that was used by Anderson and Philpott (Ref. 27). On its own, it guarantees only that $S_i(p)$ is monotonic for points to the right of $(Q_i, C_i'(Q_i))$. Note that these are general sufficient conditions which might not be necessary in specific examples: see Example 3.1.

We note also that the proposition assumes explicitly $Q_i < q_i^M$; that is, generator $i$'s contracted quantity is below the generator's generation capacity. There is no reason why this restriction has to hold in practice. Provided that the implicit function curve defined by (3) crosses the region $\Psi_i^o$, then we can still derive the same optimal supply function curve, although the starting point where we apply the implicit function theorem is no longer $(Q_i, C_i(Q_i))$. In this case the unique optimal supply function will be entirely located below the marginal cost curve.

A sufficient condition for the optimal supply curve to cross $\Psi_i^o$ is that both the points $(q_i^M, 0)$ and $(D(0) + \epsilon_2 - S_j(0), 0)$ (the highest possible demand at price 0) are located on the right-hand side of the supply function curve defined by (3). If we define

$$q_i^0 = \min\left(q_i^M, D(0) + \epsilon_2 - S_j(0)\right),$$

then we can write this condition as

$$C_i'\big(q_i^0\big)\big(S_j'(0) - D'(0)\big) + q_i^0 > Q_i.$$

Finally, we discuss the sensitivity of $S_i$ with respect to the contracted amount $Q_i$.

**Proposition 3.2.** Let $S_i(p, Q)$ be defined as in (3) for a given value of $Q$, and suppose that the conditions of Proposition 3.1 are satisfied. If $0 \le Q_i^1 < Q_i^2 \le q^M$, then $S_i(p, Q_i^1) < S_i(p, Q_i^2)$.

**Proof.** By a simple calculation, we have

$$dS_i(p, Q_i)/dQ_i = 1/[1 + \big(S_j'(p) - D'(p)\big)C_i''(S_i(p, Q_i))] > 0.$$

The result follows. □

From Proposition 3.2, we can show that, the greater the amount of contract $Q_i$, the lower the prices that the generator offers in the spot market. We write $T_i(q, Q_i)$ for the inverse function for $S_i(\cdot, Q_i)$. As $Q_i$ increases, $S_i(\cdot, Q_i)$ increases and hence $T_i(\cdot, Q_i)$ decreases. This is an observation which has been made by a number of authors (Refs. 10, 13–15) using a variety of different models.

To conclude this section, we present an example to illustrate our discussion in this section.

**Example 3.1.** Suppose that generator $i$ faces a residual demand

$$D_i(p,\epsilon)=0.5\log(1+p)-p+\epsilon$$

and that $\epsilon$ varies randomly from 0.5 to 4. Then,

$$A(p)=-1+[0.5/(1+p)],$$
$$A'(p)=-0.5/(1+p)^2<0.$$

Suppose that the generator's marginal cost for producing a quantity $q$ of electricity is $q$, i.e., $C_i'(q)=q$ and so $C_i''(q)=1$. Suppose that

$$Q_i=1,\quad q_i^M=5,\quad p^M=5.$$

From the proof of Proposition 3.2, we see that there exists a unique function $S_i$ satisfying (2). In what follows, we show $S_i$ is monotonically increasing.

Note first that, when $p<0.5\big(\sqrt{2}-1\big)$,

$$[A(p)]^2/\big(-A'(p)\big)=2(p+0.5)^2<1, \tag{9}$$

violating (4). Therefore, we cannot apply Proposition 3.2 directly and we will establish that $S_i'(p)>0$ directly. By (2),

$$S_i(p)-Q_i=[-A(p)/(1-A(P))](p-Q_i).$$

Notice that

$$-A(p)/(1-A(p))=(1+2p)/(3+4p)\in(0,0.5).$$

We want to show that

$$-A'(p)(S_i(p)-Q_i)+[A(p)]^2>0.$$

This is immediate when $p\geq 1$, since then $S_i(p)-Q_i>0$ and $A'(p)<0$. When $0<p<1$, we have

$$S_i(p)-Q_i>0.5(p-Q_i)>-0.5,$$

whereas $-[A(p)]^2/A'(p)>0.5$ from (9). This is enough to establish the inequality, and this in turn establishes that $S_i'(p)>0$, since (7) can be rewritten as (8) in this case.

## 4. Nash Supply Function Equilibrium

In this section, we consider interactions between generators in the spot market. We will consider two players each of whom maximises its expected profit through the choice of its supply functions. In order to carry the analysis through, we will need to assume that the two players are identical, and hence that we have a symmetric SFE.

First, suppose that two generators $i$ and $j$ have contracts for the same quantity $Q$ in the contract market and then compete in the spot market. We assume that the two players have identical cost function $C$ and capacity $q^M$. Each generator chooses a supply function to maximize his expected return given the other's choice. We consider a Nash equilibrium, so that no generator can increase his expected profit by unilaterally changing his supply function. One problem that we face is that we do not know if each generator's optimal response supply function is increasing in price, since our sufficient conditions (4) and (5), which guarantee the monotonicity of the optimal response supply function for generator $i$ depend on generator $j$'s supply function and vice versa. In what follows, we assume that both generators offer increasing supply functions over $[0, p^M]$. We consider symmetric equilibria, $S_i(p) = S_j(p) = S(p)$ for $[0, p^M]$; that is, after observing generator $j$'s supply function $S(p)$, generator $i$'s optimal response supply function is also $S(p)$ and vice versa.

We assume that the total demand is always positive when prices are set at the lowest marginal cost, and that the total supply is sufficient to meet demand when the price is set at the highest marginal cost. Finally, we assume that the marginal cost of production is smaller than the price cap even at the highest output levels. Notice that these last two assumptions are slightly stronger than just assuming that supply at the price cap level is sufficient to meet demand. This can be expressed algebraically as follows.

**Assumption 4.1.** We assume that

$$D\big(C'(0)\big) + \epsilon_1 > 0,$$

$$D\big(C'(q^M)\big) + \epsilon_2 < 2q^M,$$

$$C'\big(q^M\big) < p^M.$$

**Theorem 4.1.** Let Assumption 4.1 hold, let $0 < Q < q^M$, $0 < C'(Q) < p^M$, and $S'(p) > 0$ for $p \in (0, p^M)$. Then, $S(p)$ is a symmetric Nash supply function equilibrium if and only if

$$\big(S'(p)-D'(p)\big)\big(p-C'(S(p)\big)-S(p)+Q=0, \tag{10}$$

for all $p \in \Psi^*$, where

$$\Psi^* = \big\{p : \epsilon_1 \le 2S(p) - D(p) \le \epsilon_2,\ 0 \le p \le p^M\big\}.$$

**Proof.** Suppose that $S$ satisfies (10) throughout the region $\Psi^*$. We shall prove that given the supply function $S$ of generator $j$, the best response of generator $i$ is also $S$, and vice versa. Given $S_j = S$, it is clear that $S_i$ satisfies (2) and so

$$Z(S(p), p) = 0,$$

for $p \in \Psi^*$. It remains only to show the second-order conditions for optimality. Here, we can use the same argument as in the proof of Proposition 3.1 to show that $Z_q(S(p), p) < 0$ and since $S$ is monotonic increasing as a function of $p$, the implicit function theorem implies $Z_p(S(p), p) > 0$. This is enough to show from Ref. 28, Theorem 7 that $S(\cdot)$ is locally optimal. Global optimality follows, since there is no other local optimal supply curve in the considered region.

On the other hand, for any symmetric equilibrium, (10) follows from (2) provided that $(S(p), p)$ is in $\Psi_i^o = \Psi_j^o$. The only remaining case occurs when a symmetric equilibrium satisfying (10) hits the boundary of $[0, q^M] \times [0, p^M]$, within the region $\Psi^*$ and thus fails to satisfy the derivative condition of (10) at the boundary. But this is ruled out by Lemma 4.1 below. □

The lemma below deals with some issues around the boundaries of the area in which the supply function is defined. We can summarize this result by observing that it amounts to showing that a symmetric equilibrium cannot hit the boundaries of the box $[0, q_i^M] \times [0, p^M]$ before it hits the boundary of $\Psi_i = \Psi_j$. The cases where the solution to (10) hits the $p=0$ or $p=p^M$ constraints while still in the interior of the region $\Psi^*$ provides some difficulties for the modeling approach that we have adopted. For example, suppose that the solution to (10) has $S(0) = q_L$ with $D(0) + \epsilon_1 < 2q_L$. Then, for the low demand realizations, the generators will both be dispatched at price 0, with insufficient total demand for the total supply available at that price. The market mechanism will decide on an allocation of demand to the two generators, and it is natural to assume that this allocation is symmetric, with each generator being dispatched an amount $(D(0)+\epsilon)/2$ if this is less than $q_L$. However, the formal model that we have used allocates to generator $i$ an amount equal to $D(0) + \epsilon - S_j(0)$, which produces the wrong result if we take $S_j(0) = q_L$. The problem here is that the offer of a block of energy at zero price is equivalent to a vertical section in the definition of the function $S_j$; hence, it gives some indeterminacy in the definition of $S_j(0)$. Thus, we will need to treat these cases more directly with a perturbation argument.

**Lemma 4.1.** Under Assumption 4.1, if $S(p)$ is a symmetric Nash supply function equilibrium and $\Psi^* = \{p : \epsilon_1 \le 2S(p) - D(p) \le \epsilon_2,\ 0 \le p \le p^M\}$, then, $S(p) \in (0, q^M)$ for $p \in \Psi^*$ and $S(0) \le (\epsilon_1 + D(0))/2$, $S(p^M) \ge (\epsilon_2 + D(p^M))/2$.

Suppose that a solution to (10) has $S(p_L) = 0$ for some $p_L \in \Psi^*$. Thus, $D(p_L) \le -\epsilon_1 < D(C'(0))$ using Assumption 4.1. Hence, $p_L > C'(0)$. But from (10),

$$\big(S'(p_L) - D'(p_L)\big)\big(p_L - C'(0)\big) + Q = 0,$$

and this gives a contradiction since $S'$ is nonnegative and $D'$ is nonpositive.

On the other hand, suppose that a solution to (10) has $S(p_H) = q^M$ for some $p_H \in \Psi^*$. Thus,

$$D(p_H) \ge 2q^M - \epsilon_2 > D\big(C'(q^M)\big)$$

using Assumption 4.1. Hence, $p_H < C'\big(q^M\big)$. But from (10),

$$\big(S'(p_H) - D'(p_H)\big)\big(p_H - C'\big(q^M\big)\big) - q^M + Q = 0,$$

and this gives a contradiction as $Q < q^M$ and $S'(p_H) - D'(p_H)$ is nonnegative.

Now, consider the case $S(0) = q_L$ with $D(0) + \epsilon_1 < 2q_L$. We show that this cannot be an equilibrium solution. We choose a perturbation of $S_i$, by letting $S_i$ be any continuous function with $S_i(0) = (D(0) + \epsilon_1)/2$, $S_i(p) < S(p)$, for $0 < p < \delta$, and $S_i(p) = S(p)$ for $p \ge \delta$. While making a change to $S_i$, we leave $S_j = S$ unchanged. Notice that, since $C'(0) \ge 0$ and $D$ is decreasing, Assumption 4.1 implies that $S_i(0) > 0$. We can think of this perturbation as arranging for player $j$ to be given some greater priority in the dispatch when the price is zero or very near it.

If $\epsilon \in [\epsilon_1, q_L + (\epsilon_1 - D(0))/2]$, then the market still clears at price 0. For this range of $\epsilon$, the improvement in profit for player $i$ is greater than

$$C((D(0) + \epsilon)/2) - C((D(0) + \epsilon_1)/2) > 0;$$

the exact change in profit will be affected by the exact rule for sharing dispatch between the two generators. If

$$\epsilon \in (q_L + (\epsilon_1 - D(0))/2,\ 2S(\delta) - D(\delta)),$$

then the market clears at a price $p_\epsilon \leq \delta$. It is easy to see that, in this case, the perturbation will make the clearing price greater and the amount dispatched by generator $i$ less. Hence, the cost of generation is reduced and the change in profit for player $i$, which we write as $\Delta$ is bounded below by the reduction in the market payments to generator $i$ together with the loss due to the effect of increased prices on the contract payments. So,

$$\Delta \geq -\delta S(\delta) - p_\epsilon Q \geq -\delta\big(q^M + Q\big).$$

For other values of $\epsilon$, there is no change in the outcome. Since the improvement in payoff which occurs for some values of $\epsilon$ is independent of the choice of $\delta$, we will obtain an overall improvement in expected profit if $\delta$ is chosen small enough.

The argument when $S(p^M) = q_H$ with $D(p^M) + \epsilon_2 > 2q_H$ is similar. We show that this cannot be an equilibrium solution by considering a perturbation of $S_i$, with $S_i(p^M) = (D(p^M) + \epsilon_2)/2$, $S_i(p) > S(p)$, for $p^M - \delta < p < p^M$ and $S_i(p) = S(p)$ for $p \leq p^M - \delta$. Observe that Assumption 4.1 implies that $S_i(p^M) < p^M$. In this case, if

$$\epsilon \in \big(q_H + \big(\epsilon_2 - D\big(p^M\big)\big)/2, \epsilon_2\big),$$

then the perturbed solution dispatches strictly more than $S$ and the clearing price remains at $p^M$, giving an improvement which is independent of $\delta$. On the other hand, if

$$\epsilon \in \big(2S\big(p^M - \delta\big) - D\big(p^M - \delta\big)q_H + \big(\epsilon_2 - D\big(p^M\big)\big)/2\big),$$

then the perturbation will make the clearing price smaller and increase the amount dispatched by generator $i$. Suppose that a particular value of $\epsilon$ in this range is associated with price $p_\epsilon$ under $S$ and that $p'_\epsilon \leq p_\epsilon$ under the perturbed solution $S_i$, with an amount dispatched of $x$ under $S$ and $x' \geq x$ under $S_i$.

Then, using the same notation as before, this realization gives a change in profit of

$$\begin{aligned}
\Delta &= p'_\epsilon x' - C\big(x'\big) - Qp'_\epsilon - p_\epsilon x + C(x) + Qp_\epsilon \\
&\geq p^M\big(x' - x\big) - \big(C\big(x'\big) - C(x)\big) - \delta x' \\
&\geq \big(p^M - C'\big(q^M\big)\big)\big(x' - x\big) - \delta x',
\end{aligned}$$

where we have used the fact that $p'_\epsilon \geq p^M - \delta$ and that $C$ is increasing and convex on $(0, q^M)$. Now, using Assumption 4.1, we have

$$\Delta \geq -\delta q^M.$$

For other values of $\epsilon$, there is no change in the outcome. Since the improvement in payoff, which occurs for values of $\epsilon$ in the first range is independent of the choice of $\delta$, as before we will obtain an overall improvement in the expected profit if $\delta$ is chosen small enough. □

We give now a more detailed analysis of the possible symmetric supply function equilibria. We start by defining two functions. Let $S_0(p)$ be the solution of

$$S_0(p) - Q + D'(p)(p - C'(S_0(p))) = 0. \tag{11}$$

We will be interested in the solution of this equation for $p \in [0, p_M]$. To ensure a solution for every value of $p$, we choose a smooth extension of $C$ for negative arguments, with $C'(x) = C'(0) + x$ for $x < 0$. Then, the left-hand side of (11), when treated as a function of $S_0(p)$, is continuous and changes sign as $S_0(p)$ moves from $-\infty$ to $+\infty$.

From (10), we have $S'(p) = 0$ if any equilibrium solution $S(p)$ crosses $S_0(p)$ [except in the case that $p = C'(S(p)]$, i.e., at the point $(Q, C'(Q)))$. Setting $S_j(p) = 0$ in (2), we see that $S_0(p)$ is the optimal offer curve for a monopolist when faced with demand $D(p) + \varepsilon$, as has been observed by Green and Newbery in the case without contracts.

We define also $S_\infty(p)$ to be the marginal cost curve; thus, $S_\infty(p) = (C')^{-1}(p)$. Using the same extension of $C$ as defined above ensures that $S_\infty(p)$ is defined for $p \in [0, p_M]$. Observe from (10) that $S'(p) = \infty$ if any equilibrium solution $S(p)$ crosses $S_\infty(p)$ [unless this occurs at $S(p) = Q$, i.e., at the point $(Q, C'(Q))$].

These two curves intersect at $\big(Q, C'(Q)\big)$, as is illustrated in Figure 1. When $p < C'(Q)$, then $S_0(p) > S_\infty(p)$, while the reverse is true when $p > C'(Q)$. No equilibrium solution $S(p)$ can exist outside the region between these two curves, since this would imply a negative derivative at this point. There will be multiple equilibria lying in the region between the two curves. We work toward identifying which of these multiple equilibria achieves the maximum expected profit for the generators. This maximum profit equilibria will be a prime candidate for the two generators to coordinate upon. We will need to show first of all that $S_0(p)$ is increasing. To establish this we need to make one further assumption.

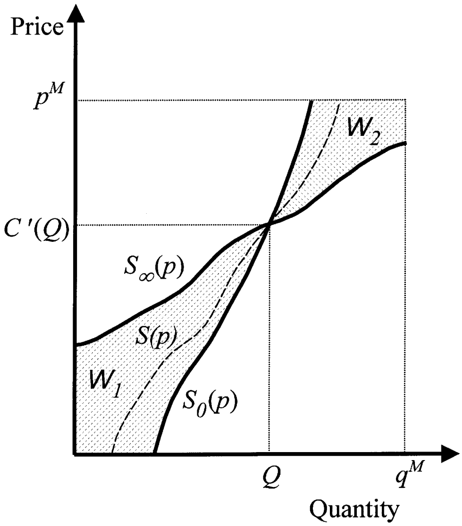


Fig. 1. Possible symmetric supply function equilibria.

**Assumption 4.2.** $\big(p - C'(Q)\big)D''(p) + D'(p) < 0$, for $p \in (0, p_M)$.

Most previous research in this area has assumed that the demand is a concave function, i.e. $D''(p) < 0$. Notice that, when the demand is concave, Assumption 4.2 will automatically be satisfied for that part of the solution lying to the right of the crossover point. But for the rest of the curve [when $p < C'(Q)$], we need instead that $D''(p)$ is positive or at least not too negative. In general, the result we give below will not hold for an arbitrary concave demand function when $Q > 0$.

**Lemma 4.2.** Under Assumption 4.2, the function $S_0(p)$ is strictly increasing on $[0, p_M]$.

**Proof.** Observe first that, from (11), $S_0'(p)$ satisfies

$$S_0'(p)\big(1 - D'(p)C''(S_0(p)) + D'(p) + D''(p)\big(p - C'(S_0(p))\big) = 0. \quad (12)$$

Suppose that $S_0(p) < Q$. Then, from (11), $p < C'(S_0(p))$. In the case that $D_p''(p) \geq 0$, we have

$$D'(p) + D''(p)\big(p - C'(S_0(p))\big) < 0 \quad (13)$$

immediately. In the case that $D''(p) < 0$, then using $C'(S_0(p)) \leq C'(Q)$, this inequality follows from Assumption 4.2.

Now, suppose that $S_0(p) \geq Q$. From (11), $p \geq C'(S_0(p))$. Hence, (13) holds immediately if $D''(p) < 0$; since $C'(S_0(p)) \geq C'(Q)$, the inequality follows from Assumption 4.2 if $D''(p) \geq 0$. In either case, once we have established (13), then $S_0'(p) > 0$ follows from (12). □

We write $\mathcal{W}$ for the region between the two curves. Specifically, we let $\mathcal{W}_1$ denote the region surrounded by the curves

$$\mathrm{s}_0^1 = \{(S_0(p), p) : 0 < p < C'(Q)\},$$
$$\mathrm{s}_\infty^1 = \{(S_\infty(p), p) : 0 < p < C'(Q)\},$$

and the lines $p = 0$ and $q = 0$. Let $\mathcal{W}_2$ denote the region surrounded by the curves

$$\mathrm{s}_0^2 = \{(S_0(p), p) : C'(Q) < p < (C')^{-1}(q^M)\},$$
$$\mathrm{s}_\infty^2 = \{(S_\infty(p), p) : C'(Q) < p < P^M\},$$

and the lines $p = p^M$ and $q = q^M$. Then, $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$.

Define

$$\tilde{f}(S, p) = [S - Q + D'(p)(p - C'(S))]/[p - C'(S)].$$

Observe that $\tilde{f}(S, p) > 0$, for any $(S, p) \in \mathrm{int}\mathcal{W}$, and that $\tilde{f}(S, p) < 0$, for $(S, p) \in [0, q^M] \times [0, p^M] \setminus \mathcal{W}$. For any point $(S, p)$ in int$\mathcal{W}$ such that $(S, p) \neq (Q, C'(Q))$, since $\tilde{f}$ is Lipschitz near the point, by a standard existence theorem in ordinary differential equation theory, there exists a unique solution $S(p)$ for (10) within a neighborhood of the point.

The lemma below shows that the possible solutions for (10) are exactly defined by just two points on the trajectory: one in $\mathcal{W}_1$ and one in $\mathcal{W}_2$. The part of the trajectory lying in $\mathcal{W}_1$ is essentially independent of that lying in $\mathcal{W}_2$, though their slopes coincide at the crossover point $(Q, C'(Q))$. In the case that there are no contracts, then only the region $\mathcal{W}_2$ occurs and this is the case considered by Green and Newbery (Ref. 2).

**Lemma 4.3.** Under Assumption 4.2, the following results hold:

(i) For any point $(S, p)$ in $\mathcal{W}$ such that $S < Q$, there exists a unique solution $S(p)$ for (10) which is increasing and remains in $\mathcal{W}$ between $(S, p)$ and $(Q, C'(Q))$. Similarly, for any point $(S, p)$ in $\mathcal{W}$ such that $S > Q$, there exists a unique solution $S(p)$ for (10) which is increasing and remains in $\mathcal{W}$ between $(Q, C'(Q))$ and $(S, p)$.

(ii) Any solution trajectory of (10) must pass through $(Q, C'(Q))$ at a slope given by (when $C''(Q) > 0$)

$$S'\big(C'(Q)\big) = 1/2\left[D'(C'(Q)) + \sqrt{D'(C'(Q))^2 - [4D'(C'(Q))/C''(Q)]}\right]. \tag{14}$$

**Proof.** Part (i) follows easily by considering tracing a curve which leaves $\mathcal{W}$ and obtaining a contradiction. For example, suppose that a curve starts at $(S, p)$ with $S < Q$ and leaves $\mathcal{W}$ on the $S_0$ boundary before reaching $(Q, C'(Q))$. Then, near to crossing the boundary, $S'$ approaches 0, but $S_0' > 0$ which gives a contradiction, since $S(p) < S_0(p)$ in this case. The other three cases can be dealt with similarly.

For part (ii), we note from (10) that, unless $p = C'(S(p))$, and hence $p = C'(Q)$, $S(p)$ has derivative

$$[S(p) - Q]/[p - C'(S(p))] + D'(p)$$

Let $\alpha$ be the left derivative of $S$ at $C'(Q)$. As $S$ approaches $Q$ and $p$ approaches $C'(Q)$ from below, we have

$$p \simeq C'(Q) - \Delta, \quad S \simeq Q - \alpha\Delta,$$

for $\Delta$ small and positive. Hence,

$$\alpha \simeq -\alpha\Delta\,/\,[C'(Q) - \Delta - C'(Q - \alpha\Delta)] + D'(C'(Q) - \Delta).$$

This gives a quadratic equation for $\alpha$. This equation has exactly one positive root and this is (14). Moreover, the argument can be repeated for $p$ approaching $C'(Q)$ from above to obtain the same limiting value. □

Note that, as $C''(Q)$ approaches zero, the value of the slope of $S$ at $C'(Q)$ goes to infinity. This corresponds to a flat offer curve which is used at $Q$ in the limit when $C''(Q) = 0$.

We have been able to give an analysis of the symmetric duopoly for the spot market equilibrium without the assumption that the demand is concave in the price. We should note that Assumption 4.2, which is new, can be dispensed with for the analysis of that part of the solution lying above the crossover point, if we assume instead that the demand is concave. This is the approach taken in the analysis of the case with fixed marginal costs carried out by Newbery (Ref. 9). He assumes that the demand is always greater than the contract amount $Q$; but this assumption is restrictive, as generators in practice are often contracted at a level close to their normal output.

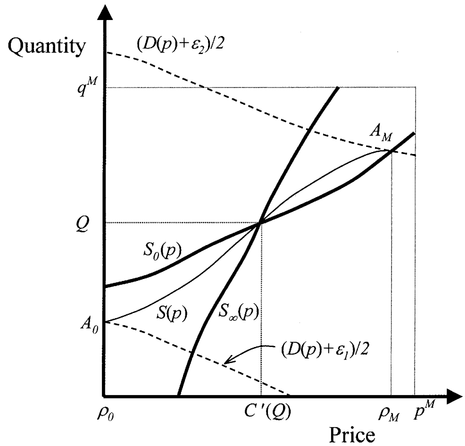


Fig. 2. SFE with maximum expected profit.

We let $\rho_0$ be the smallest nonnegative price $p$ at which the total demand could be as small as $2S_0(p)$; thus,

$$\rho_0 = \min\{p \geq 0 : D(p) + \epsilon_1 \leq 2S_0(p)\}.$$

As $D$ is decreasing and $S_0$ is increasing, there is at most one price $p$ where $D(p) + \epsilon_1 = 2S_0(p)$. This will be the value of $\rho_0$ unless $D(0) + \epsilon_1 \leq 2S_0(0)$ when $\rho_0 = 0$. We define also

$$\rho_M = \max\{p \leq p_M : D(p) + \epsilon_2 \geq 2S_0(p)\}.$$

This is the largest price $p \leq p_M$ at which the total demand could be as high as $2S_0(p)$. Figure 2 shows the definitions of $\rho_0$ and $\rho_M$, but note that the price and quantity axes have been reversed from Figure 1.

The theorem below defines the Nash supply function equilibrium in the price range $(\rho_0, \rho_M)$. Before giving the proof of this result, we note that the definition of the offer curve outside the price range defined in the theorem is irrelevant. Since the total demand will be split equally between the two generators, the offer curve outside the price range $(\rho_0, \rho_M)$ corresponds to demand values which will not occur. We write $T(q)$ for the inverse of the equilibrium supply function, so $T(q) = S^{-1}(q)$. Then, we may define a continuation of $S$, by setting

$$T(q) = \rho_0, \quad \text{for } q \in (0, D(\rho_0) + \epsilon_1)),$$
$$T(q) = \rho_M, \quad \text{for } q \in (D(\rho_M) + \epsilon_2, q_M)).$$

The values of the offer curve in these ranges are not important, since they lie outside $\Psi^o$.

**Theorem 4.2.** Under Assumptions 4.2 and 4.1, the Nash supply function equilibrium which achieves the maximum expected profit for both players is the trajectory solving (10) and passing through the points

$$A_0 = ((D(\rho_0) + \epsilon_1)/2, \rho_0), \quad A_M = (D(\rho_M) + \epsilon_2)/2, \rho_M).$$

**Proof.** We consider first that part of the equilibrium solution that lies below $(Q, C'(Q))$. The proof proceeds in two stages. First, we will show that the solution suggested is a well-defined equilibrium, and then we will show that it achieves the maximum possible profit amongst these equilibria.

We need to start by establishing that $A_0 \in \mathcal{W}_1$. First, consider the case that $\rho_0 > 0$, so $A_0$ lies on $(S_0(p), p)$. We need $S_0(\rho_0) > 0$ (or equivalently that $D(\rho_0) + \epsilon_1 > 0$). Note that

$$S_0(C'(0)) > S_\infty(C'(0)) = 0.$$

Now, if $\rho_0 > C'(0)$, then since $S_0$ is increasing, $S_0(\rho_0) > 0$. On the other hand, if $\rho_0 < C'(0)$, then since $D$ is decreasing $D(\rho_0) + \epsilon_1 > 0$ from Assumption 4.1 above. In the case that $\rho_0 = 0$, to show $A_0 \in \mathcal{W}_1$ we need to check that

$$S_\infty(0) < (D(0) + \epsilon_1)/2,$$

but this follows immediately since

$$C'((D(0) + \epsilon_1)/2) > 0.$$

Hence, we have a feasible equilibrium solution $S$ defined through the point $A_0$.

The next step is to show that any other equilibrium $\tilde{S}$ achieves a smaller profit. We do this by establishing that $\tilde{S}(p) < S(p)$ for $p \in [0, C'(Q)]$. Later, we will show that the reverse occurs when $p \in [C'(Q), p_M]$.

We know that $\tilde{S}$ is an increasing curve passing through the point $(Q, C'(Q))$. First, consider the case that $\rho_0 > 0$. If $\tilde{S}(p) > S(p)$ for some $p \in [0, C'(Q)]$, then since $\tilde{S}$ and $S$ cannot cross, $\tilde{S}(\hat{p}) = S_0(\hat{p})$ for some $\hat{p} > \rho_0$. But then, it is not possible to define a continuation of $\tilde{S}$ in $[0, \hat{p}]$, which will be an equilibrium solution. Notice that

$$D(\hat{p}) + \epsilon_1 < D(\rho_0) + \epsilon_1 = 2S_0(\rho_0),$$

and so the definition of $\tilde{S}$ in this range will affect the final profit.

In the case that $\rho_0 = 0$ (which is the case that is illustrated in Figure 2), then a curve $\tilde{S}$ may have a value between $(D(0)+\epsilon_1)/2$ and $S_0(0)$ at zero price. This is a slightly different case than that considered above: $\tilde{S}$ now has to be viewed as a function that has a vertical section at 0, corresponding to an offer of an amount $\tilde{S}(0)$ at zero price. This initial zero price bid will be used to set the price for some demands since

$$\tilde{S}(0) > (D(0)+\epsilon_1)/2.$$

The offer curve $\tilde{S}$ will not satisfy the conditions (10) and cannot be an equilibrium.

Having established that any other equilibrium offer curve $\tilde{S}$ must lie below $S$ we show that the profit is less. If demand is given by $D(p)+\varepsilon$ and we use $\tilde{S}(p)$, then the clearing price $\tilde{p}$ will be higher than that obtained with $S(p)$ [provided $(D(p)+\varepsilon)/2$ passes below the point $(Q, C'(Q))$]. Moreover the clearing price with $S(p)$ is itself higher than the clearing price $p^*$, which is obtained if both generators use $S_0(p)$.

We write $\pi$ for the profit in the event that demand is given by $D(p)+\varepsilon$ and the clearing price is $p$, so

$$\pi = p([D(p)+\varepsilon]/2) - pQ - C([D(p)+\varepsilon]/2),$$
$$d\pi/dp = (1/2)[pD'(p) + D(p) + \varepsilon - C'([D(p)+\varepsilon]/2)D'(p)] - Q.$$

Since

$$[D(p^*)+\varepsilon]/2 = S_0(p^*),$$

we can use (11) to see that, at $p^*$,

$$d\pi/dp = (1/2)[Q - S_0(p^*)] + S_0(p^*) - Q < 0.$$

We will show that $d^2\pi/dp^2 < 0$ between $p^*$ and $\tilde{p}$,

$$d^2\pi/dp^2 = (1/2)[pD''(p) + 2D'(p) - C'([D(p)+\varepsilon]/2)D''(p) - (1/2)D'(p)^2 C''([D(p)+\varepsilon]/2)].$$

Now, from Assumption 4.2,

$$\big(p - C'(q)\big)D''(p) + D'(p) < 0, \quad \text{for } q < Q,$$

using the same argument as in the proof of Lemma 4.2, and this is enough to show that $d\pi/dp$ is decreasing since the other terms are negative. Thus, $d\pi/dp < 0$ throughout the range $(p^*, \tilde{p})$. Hence, the profit under $S$ is higher than that under $\tilde{S}$. This conclusion holds independently of the value of the demand shock $\varepsilon$.

The argument for the other case (when the offer curve is above and to the right of $(Q, C'(Q)))$ is similar. □

So far, we have considered only symmetric equilibria, that is, the optimal supply strategies of both generators coincide. It is interesting to ask whether there is an asymmetric Nash supply function equilibrium. In fact, if both generators have the same marginal cost and contract, then it can be shown that no asymmetric equilibrium exists. This can be proved using a similar technique to that employed by Klemperer and Meyer (Ref. 1) to establish the equivalent result in their setting.

## 5. Conclusions

In this paper, we address the existence of an optimal monotonic supply function response in a wholesale electricity market and give conditions on the slope of the residual demand curve that are sufficient to ensure this. We give also a complete treatment of the symmetric supply function equilibrium in the case with price caps and limits on generator capacity. We are able to do this without assuming that the demand is concave. Our discussion is limited to a duopoly, but the extension to more than two identical players is not very difficult (see Newbery (Ref. 9) for a discussion of this issue in a similar context).

The fact that our detailed results apply only to the symmetric case is clearly a major restriction. An extension to the asymmetric case is problematic primarily because, in constructing nonsymmetric equilibria, the restriction to monotonic supply functions becomes very hard to ensure: we can no longer achieve this simply by giving conditions on the derivative of the residual demand. However, there are some observations that we can make on an equilibrium in the asymmetric case. If an equilibrium exists between supply the functions $S_i(p)$ for generator $i$ and $S_j(p)$ for generator $j$, then these will satisfy equation (2) and its equivalent with $i$ and $j$ reversed. It is not hard to show the equivalents of some of the observations we have made in the symmetric case. If we define $S_{i0}(p)$ to be the monopolist offer for generator $i$ [which is obtained by solving (2) with $S_j'(p) = 0$] and $S_{i\infty}(p) = (C_i')^{-1}(p)$ to be the marginal cost curve, then $S_i(p)$ will lie somewhere between these two curves, and will go through the point where they cross at $(Q_i, C_i'(Q_i))$. The equivalent holds also for $S_j(p)$. However, it is no longer the case that we can give a simple condition sufficient to ensure the existence of an appropriate monotonic equilibrium offer throughout the region between these curves.

Because our discussion is in the context of price caps and limited generator capacity, we need to be careful in considering the behavior of the supply function equilibria at points where it crosses these boundaries. Here, the use of analysis from Refs. 16 and 28 is helpful. The issue of boundaries is important also in our derivation of the supply function equilibrium which achieves the maximum profit for the generators.

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