Coping with Risk Aversion in the Newsvendor Model with a Backorder Case

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Abstract

In this paper, we study the optimal order quantity decisions for a risk-averse newsvendor with a backorder case, where it is assumed that all or part of the excess demands of the customers can be backlogged. The optimal decisions are obtained under the popular Conditional Value-at-Risk (CVaR) criterion, which is to control the risk of the profit due to uncertain market demands. We study two basic models. The first is the pure CVaR model. It is found that the optimal order quantity for a risk-averse newsvendor is less than that for a risk-neutral newsvendor, who is to maximize the expected profit. It also shows that low risk means low expected profit while high expected profit comes with high risk. The second is the mixed model that balances the CVaR and the expected profit model. Important monotone properties are obtained for the two models and their relationships with the existing results are revealed. Finally, a numerical example is given to demonstrate the obtained results companied by interesting interpretation of some management insights.

Key words: Inventory, Conditional Value-at-Risk, Backorder, Newsvendor model, Optimal order quantity

1 Introduction

The newsvendor model has long been applied to many settings such as production planning and supply chain management. In this model, for a perishable product with a short selling

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season and a stochastic demand of the market, a newsvendor needs to decide the optimal order quantity of the product to maximize his profit before the selling season. If the order quantity is greater than the realized demand of the market, the newsvendor has to dispose the excess order as a loss. In contrast, if the order quantity is lower than the realized demand of the market, there would result in lost sales with/without shortage penalties. Thus, the newsvendor must find a balance between over ordering and under ordering their products in order to maximize its expected profit. With the growing emphasis on globalization and competition, some basic assumptions of this model seem to be unrealistic and many researchers thus concentrate on relaxing the premise of the classical newsvendor model. It is therefore not surprising to see that many extensions of the classical newsvendor model emerged in the recent years, see Khouja [10] and Qin et al. [17] for more details.

One of the important extensions is to allow backlogging part or all of the excess orders. The classical newsvendor model does not allow further replenishment when stockout occurs and this is not a valid assumption for most of the inventory systems nowadays. In reality, many newsvendor-type products can be replenished or backlogged when there exists unsatisfied demand in the selling season (see Lau and Lau [11], Gurnani and Tang [8]). In the classical model, all the unsatisfied demands turn to lost sales, which ranges from profit loss on the scale to some unspecific loss of goodwill of the customers. Such lost sales in a highly competitive environment can be devastating and can, for example, disrupt the production process or even shut down an entire production line. Close scrutiny of such lost sales reveals that all (or part) of the unsatisfied customers can actually wait for their demands to be met soon after they place their orders, which is often referred to as the backorder case. The benefit from the backorder being realized is obvious: the newsvendor can gain more profits from the otherwise lost sales. Moreover, it is believed that backlogging the unsatisfied demands of the customers can also promote the relationships between the newsvendor and the customers. Therefore, a promising subject about the newsvendor model is to relax the premise of complete lost sales when a stockout occurs, and to study the case that part or even all of the excess demands can be backlogged (i.e., the backorder case).

Although the backorder offers a great mechanism to rescue the lost sales for a newsvendor, there associate certain risks that should be factored into the optimal decision process. For example, there exist certain replenishment costs, the ordering cost, and/or the customer waiting cost (see [12, 13]). It is also seen as a risky strategy if a newsvendor heavily relies on backlogging to meet the excess demands, which would in no doubt result in the loss of some customers’
goodwill. In other words, there would exist some potential risks unaccounted for if a newsvendor only intends to maximize the profit for the backorder case. However, to our best knowledge, most of the existing studies about the backorder case mainly focus on the maximization of the expected profit or minimization of the expected cost, while risk analysis and control are ignored. It is the purpose of this paper to study models that take into account of risk control for a risk-averse newsvendor with the backorder case.

Similar arguments for the need of controlling risks also appear in other scenarios of newsvendor models. Those have led to models that aim to strike a balance between maximizing its profit and controlling the associated risk. For example, the mean-variance model [5, 25] and the Conditional Value-at-Risk (CVaR) model [3, 7] are devised to cope with the risk aversion of a newsvendor’s decision problem. However, those models do not cover the backorder case, which has begun to attract more attention of many researchers.

This paper thus considers the optimal order quantity decision for a risk-averse newsvendor with a backorder case. We propose two different models. One is the pure CVaR model, where the CVaR criterion is applied to measure and control potential risks that originate from the fluctuation in market demand. An optimal order quantity is guaranteed to exist. The second is the mixed CVaR model that aims to strike a balance between the expected profit and the amount of risks that the newsvendor may face in their optimal decision making process. The pure CVaR model is an extreme case of the mixed model. We establish some important monotone properties for both models and reveal their relationships with the existing results from other models.

Our paper contributes to the growing study on the newsvendor model in the following aspects.

(i) This research integrates the backorder case and the CVaR criterion in a newsvendor setting and obtains the optimal order quantity decisions for a newsvendor with different objectives about the profit function in a backorder case. In particular, this study shows that, in a backorder case, the risk-averse newsvendor with the CVaR objective will choose a lower order quantity to reduce the risk that originates from the fluctuation in the market demand. The more risk-averse the newsvendor is, the lower order quantity he would make.

(ii) This study also shows that the expected profit of the risk-averse newsvendor under the optimal order quantity in maximizing the CVaR objective is decreasing in the confidence level $\alpha$, which means that low risk means low expected profit while high expected profit comes with high risk.

The rest of this paper is organized as follows. In the following section, we review some related works. The main Section 3 contains the two models and their favourable properties. All
the proofs for this section can be found in Appendix. Section 4 gives a numerical example and includes its sensitivity analysis to demonstrate the obtained results. Section 5 concludes the paper.

2 Literature Review

It is almost an impossible task to review the latest advances in various newsvendor models. Therefore, our short review has to be selective and is only for the most relevant studies about the backorder case and the risk aversion.

The classical newsvendor model assumes that all the unsatisfied demand of customers when stockout occurs turn to lost sales. In an effort to represent the diversity of customer responses to unsatisfied demands and rescue the excess demands, several researchers have generalized the classical newsvendor model to the case that all or part of the excess demands of the customers can be backlogged. For example, Montgomery et al. [15] first introduced an exact solution procedure to determine the optimal policy of newsvendor model with fixed partial backorder. Weng [24] considered a backorder case where all excess demand is backlogged but some of them may not be satisfied. San José et al. [20] introduced an inventory model with partial backlogging, where unsatisfied demand is partially backlogged according to an exponential function and developed a general approach for finding the optimal policy to this model. Lodree [12] studied a two-level supply chain within a newsvendor framework, incorporating with shortages a combination of backorders and lost sales, and also allowing the retailer to start emergency replenishment. Lodree et al. [13] investigated the case in which all the excess demand are backlogged through an emergency procurement process, where it is assumed that costs incurred during the emergency procurement process include a variable ordering cost and a customer waiting cost. Zhou and Wang [26] extended the model proposed by Weng [24] to the case where excess demand is partially backlogged and showed that the decentralized system would perform best if the manufacturer covers utterly the second production setup cost, which is opposite to that obtained in [24]. It appears that those models mainly aim to maximize the profit of a newsvendor by backlogging the unsatisfied demands or to coordinate the optimal decisions of the supplier and the newsvendor in a two-echelon newsvendor model, while little attention has been paid to the risk control for such a problem.

The newsvendor literature addressing risk aversion includes (but not limited to) the following several papers. Eeckhoudt et al. [6] studied the effects of risk and risk aversion in the single-period newsvendor model and concluded that comparative-static effects of changes in price and
cost parameters are determined by and related to the risk aversion of the newsvendor. Lau and Lau [11] explored the two-product newsvendor problem and developed a method to find the optimal production quantities of each product that will maximize the probability of achieving a profit target. Agrawal and Seshadri [1] studied the optimal order quantity and optimal retail price when the risk-averse newsvendor faces uncertain customer demand. Choi et al. [21] considered a multi-product risk-averse newsvendor with Law-Invariant coherent measures of risk. In 2000, CVaR measure was introduced (see Rockafellar and Uryasev [18, 19]) to financial management problems and proved to be effective for risk management. CVaR has subsequently been used for the risk management of the newsvendor problem (see Chen et al. [3], Gotoh and Takano [7], Jammernegg and Kischka [9], Chen et al. [4]). In general, these papers show that CVaR measure is efficient in coping with the risk control in their newsvendor models. But, they did not include the backorder case that is considered in this paper.

Instead of separately considering backorder and CVaR in the newsvendor model appeared in the above literature, we will examine the optimal order quantity decisions of the newsvendor model when these two parts are integrated. Our research thus contributes to the study of risk aversion in a newsvendor setting, and therefore complements both the backorder and CVaR research in the newsvendor model.

3 The Two Models and Their Properties

This is the major part of the paper. We first describe the basic assumptions in Subsection 3.1, followed by the two models in Subsection 3.2. We then study their properties in the rest of the section. All proofs can be found in the Appendix.

3.1 Basic Assumptions

We consider the common setting for a newsvendor model. Suppose the market demand $\xi$ is a random variable, and let $f(\cdot)$ and $F(\cdot)$ be its probability density function and cumulative distribution function respectively. Without loss of generality, it is assumed that $F(0) = 0$, $F(+\infty) = 1$, $F(\cdot)$ is continuously differentiable and increasing, and the inverse of $F(\cdot)$ exists. For a given order quantity $q$ and a realized market demand $\xi$, if it satisfies $q \geq \xi$, then there exists some excess order which can be salvaged at a salvage price $r$. Otherwise (i.e., $q < \xi$), there exists some excess demands that can not be satisfied immediately. For this case, it is assumed that all or part of the unsatisfied demands can be backlogged, and the backorder rate is represented by $w \in [0, 1]$. 

Furthermore, in order to encourage the unsatisfied customers to accept this backorder, the newsvendor will offer a discount $s$ in the retail price for unit backlogged product to compensate these loyal customers for the inconvenience of such a backorder. Here, we consider a very basic backorder case for the newsvendor model with most other complicating factors such as the lead time for the backlogged products removed. It is easy to see that the realized profit of the newsvendor in this basic backorder case is given by

$$P(q, \xi) = p \min\{q, \xi\} - cq + r(q - \xi)^+ + w(p - c - s)(\xi - q)^+, \quad (1)$$

where $X^+ = \max\{X, 0\}$. Here, $p$ is the retail price per unit of product, $c$ is the wholesale price of unit product from the supplier, $s$ is the discount in the retail price for unit backlogged product. Without loss of generality, it is assumed that $p \geq c + s > c \geq r \geq 0$. In the right hand of (1), the first item represents the sales income of the newsvendor, the second the order cost, the third the salvage income from excess order (which may not exist), and the last the profit from the backlogged products of the excess demands (which may not exist).

We end this subsection with the following remarks.

(R1) When $w = 0$, the profit function $P(q, \xi)$ in (1) is reduced to the profit function with no penalty for lost sales, studied in Chen et al. [3]. When $w = -1$ and $s = 0$, the profit function $P(q, \xi)$ is reduced to the profit function with a positive penalty $(p - c)$ for each lost sale, studied in Gotoh and Takano [7].

(R2) Thus, the proposed profit function $P(q, \xi)$ can be seen as a generalized form of the profit function in the classical newsvendor model. Our models are to make the optimal order quantity decisions with different performance measures about the above profit function $P(q, \xi)$.

3.2 The Two Models

The performance measure that we are going to use for the profit function $P(q, \xi)$ is the well-known CVaR, whose various definitions can be found in Rockafellar and Uryasev [18] and Pfug [16]. See also Artzbet et al. [2] for general axioms in defining risk measures. Let $\ell(x, \xi)$ be a loss function associated with the decision variable $x$ and the random variable $\xi$. The corresponding CVaR is defined as

$$\text{CVaR}_\alpha(\ell(x, \xi)) = \min_{v \in \mathbb{R}} \left\{ v + \frac{1}{1 - \alpha} \mathbb{E} \left[ \ell(x, \xi) - v \right]^+ \right\},$$

where $\mathbb{E}$ is the expectation operator and $\alpha \in [0, 1)$ reflects the degree of risk aversion for the newsvendor (the bigger $\alpha$ is, the more risk averse the newsvendor is).
A natural loss function for the newsvendor is \((-P(q, \xi))\). We therefore have

\[
\text{CVaR}_\alpha(-P(q, \xi)) = \min_{v \in \mathbb{R}} \left\{ v + \frac{1}{1 - \alpha} \mathbb{E} \left[ -P(q, \xi) - v \right]^+ \right\} = \min_{v \in \mathbb{R}} \left\{ -v + \frac{1}{1 - \alpha} \mathbb{E} \left[ v - P(q, \xi) \right]^+ \right\} = -\max_{v \in \mathbb{R}} \left\{ v - \frac{1}{1 - \alpha} \mathbb{E} \left[ v - P(q, \xi) \right]^+ \right\}.
\]

It follows that

\[
\min_{q \geq 0} \text{CVaR}_\alpha(-P(q, \xi)) = -\max_{q \geq 0} \max_{v \in \mathbb{R}} \left\{ v - \frac{1}{1 - \alpha} \mathbb{E} \left[ v - P(q, \xi) \right]^+ \right\}.
\]  

(2)

Let

\[
f_p(q) = \max_{v \in \mathbb{R}} \left\{ v - \frac{1}{1 - \alpha} \mathbb{E} \left[ v - P(q, \xi) \right]^+ \right\}.
\]  

(3)

Then, problem (2) is equivalent to the following problem

\[
\max_{q \geq 0} f_p(q).
\]  

(4)

We refer to (4) as the pure CVaR model with the given confidence level \(\alpha\). We note that a similar function as \(f_p(q)\) has also been used in Chen et al. [4].

When \(\alpha = 0\), \(f_p(q) = \mathbb{E} \left( P(q, \xi) \right)\), the expected profit that is central to the classical (risk-neutral) newsvendor problems. However, this expected measure may lead to a great loss since the risk arising from this criterion is not considered, which can not be accepted by some risk-averse newsvendors who are sensitive to profit variations. In other words, the research about expected profit maximization excludes the fact that different newsvendors may have different preferences about risk. The CVaR is known as a risk measure which possesses preferable properties such as coherence, and meaningful also to a decision maker who faces an uncertain situation.

Although the pure CVaR model (4) can recover the expected profit \(\mathbb{E} \left[ P(q, \xi) \right]\) in the extreme case \(\alpha = 0\), more explicit involvement of \(\mathbb{E} \left[ P(q, \xi) \right]\) can be derived through the following convex combination of \(\mathbb{E} \left[ P(q, \xi) \right]\) and \(f_p(q)\).

\[
f_m(q) = \lambda \mathbb{E} \left[ P(q, \xi) \right] + (1 - \lambda) f_p(q)
\]  

(5)

\[
= \lambda \mathbb{E} \left[ P(q, \xi) \right] - (1 - \lambda) \text{CVaR}_\alpha(-P(q, \xi)),
\]

where \(\lambda \in [0, 1]\) is the parameter that balances the expected profit and the associated risk in terms of CVaR. Our interest is to study the following problem.

\[
\max_{q \geq 0} f_m(q).
\]  

(6)

We call (6) the mixed CVaR model in this paper. In the rest of the paper, we study the properties of the two models.
3.3 Optimality Properties of the Pure CVaR Model

Regarding to the pure CVaR model (4), we have the following major result.

**Theorem 3.1** Under the basic assumptions in Subsection 3.1, the optimal order quantity \( q^*_1 \) that solves the problem (4) is given by

\[
q^*_1 = F^{-1} \left( \frac{(1-\alpha)(w(p-c)+ws)}{p-r-w(p-c-s)} \right).
\]

In the sequel, we discuss some major implications of this result.

(a) **Implications to the risk-neutral newsvendor.** As noted above, when \( \alpha = 0 \) the pure CVaR model (4) becomes the model that maximizes the expected profit for a risk-neutral newsvendor. In this case, \( q^*_1 \) becomes

\[
q^*_0 = F^{-1} \left( \frac{w(p-c)+ws}{p-r-w(p-c-s)} \right).
\]

It is obtained by Chen et al. [3] that, if there is no shortage penalty for lost sales, the optimal order quantity to maximize the expected profit for a risk-neutral newsvendor is given as \( q^*_n \). It is also obtained by Gotoh and Takano [7] that, if there exists a positive shortage penalty \( b \) for unit lost product, then the optimal order quantity to maximize the expected profit for a risk-neutral newsvendor is given as \( q^*_p \). The quantities \( q^*_n \) and \( q^*_p \) are given respectively by

\[
q^*_n = F^{-1} \left( \frac{p-c}{p-r} \right) \quad \text{and} \quad q^*_p = F^{-1} \left( \frac{p-c+b}{p-r+b} \right).
\]

It can be easily checked that

\[
\frac{p-c+b}{p-r+b} \geq \frac{p-c}{p-r} \geq \frac{p-c-w(p-c-s)}{p-r-w(p-c-s)} = \frac{(1-w)(p-c)+ws}{p-r-w(p-c-s)}.
\]

Since \( F^{-1}(\cdot) \) is monotonically increasing, it follows from the above inequalities that

\[
q^*_p \geq q^*_n \geq q^*_0.
\]

The result declares that for maximizing the expected profit, compared to the optimal order quantity \( q^*_n \) with no shortage penalty, a newsvendor will order more products when there exists shortage penalty for lost sales, and order less products if all or part of the excess demand can be backlogged. This conclusion shows that the shortage penalty for lost sales forces the newsvendor to order more products to prevent the loss from excess demand, while backorder induces the newsvendor to order less products to avoid the loss from excess order.
(b) Existence of decision bias. It is obvious that \( q_1^* \leq q_0^* \). This implies that the optimal order quantity from the pure CVaR model for a risk-averse newsvendor is smaller than the optimal order quantity from maximizing the expected profit for a risk-neutral newvendor. In terms of the terminology of Wang and Webster [23], there exists a decision bias (i.e., the quantity \( q_0^* - q_1^* \) is positive). We note that there also exists decision bias in the newsvendor model when all the excess demands turn to lost sales and there is no penalty for lost sales [3]. An interpretation of the decision bias is as follows. Since the excess demands can be all or partially satisfied by a backorder, then the risk that may lead to a loss would mainly come from the excess order. Thus it is better to order less products for a risk-averse newsvendor to lower the potential risk.

(c) Monotone properties of the optimal order quantity. The optimal order quantity \( q_1^* \) can be regarded as a function of the salvage price \( r \), the price discount \( s \), the backorder rate \( w \), and the confidence level \( \alpha \). The following results are about the monotone properties of \( q_1^* \) with respect to those variables.

**Corollary 3.2** We have the following monotone properties of \( q_1^* \).

(i) For \( r, s \geq 0 \), \( q_1^* \) is an increasing function of \( r \) and \( s \) respectively.

(ii) For \( \alpha \in [0,1) \) and \( w \in [0,1] \), \( q_1^* \) is a decreasing function of \( \alpha \) and \( w \) respectively.

More precisely, we have

\[
\frac{\partial q_1^*}{\partial r} > 0, \quad \frac{\partial q_1^*}{\partial s} > 0, \quad \text{and} \quad \frac{\partial q_1^*}{\partial \alpha} < 0, \quad \frac{\partial q_1^*}{\partial w} < 0.
\]

We have the following remarks.

(R3) Cor. 3.2(i) means that if the salvage price \( r \) improves (meaning that the loss from the excess order will decrease), then the newsvendor should order more product in order to avoid the potential loss of excess demands. Cor. 3.2(i) also means that if the price discount \( s \) increases (meaning that the newsvendor will pay a higher cost for each backlogged product), then the newsvendor should order more products in order to avoid the potential loss of excess demands.

(R4) Cor. 3.2(ii) means that if the backorder rate \( w \) grows (meaning that more excess demands can be backlogged), then the newsvendor had better order less product (i.e., smaller order quantity). The confidence level \( \alpha \) indicates the risk preference of a risk-averse newsvendor. If \( \alpha \) grows bigger, the newsvendor becomes more risk-averse. Cor. 3.2(ii) shows that, the
more risk-averse the newsvendor is, the less products he orders. In the extreme case \( \alpha = 1 \), we have \( q_1^* = 0 \) from (8). That is to say, if the newsvendor expects there is no risk for his order quantity, he had better not order any products and wait to backlog any demands of the customers.

An important question to ask is how the expected profit would change under the optimal order quantity \( q_1^* \) when \( \alpha \) changes. We have the following result to address this issue.

**Corollary 3.3** For \( \alpha \in [0, 1) \), \( E[P(q_1^*, \xi)] \) is decreasing in the confidence level \( \alpha \). That is

\[
\frac{\partial E[P(q_1^*, \xi)]}{\partial \alpha} < 0.
\]

Cor. 3.3 implies that, if the confidence level \( \alpha \) grows bigger (implying that the newsvendor becomes more risk-averse and thus orders less products to reduce risk), then he will expect a lower profit. This result shows that low risk means low expected profit, while high profit always comes with high risk. This is also consistent with the common sense among risk-averse newsvendors.

### 3.4 Optimality Properties of the Mixed CVaR Model

Similar properties can also be obtained for the mixed CVaR model (6), but with lightly more involved calculations. First we define the quantity:

\[
\tau_0 = \frac{c - r}{\alpha(p - r - w(p - c - s))}.
\]

Note that \( (p - r - w(p - c - s)) = (1 - w)(p - r) + w(c + s - r) > 0 \) because \( c + s > r \) according to our basic assumption. Hence, \( \tau_0 \) is well defined.

**Theorem 3.4** Under the basic assumptions in Subsection 3.1, the optimal order quantity \( q_2^* \) that solves the problem (6) is given by

\[
q_2^* = \begin{cases} 
F^{-1}\left(\frac{\lambda((1-w)(p-c)+ws)-(1-\lambda)(c-r)}{\lambda(p-r-w(p-c-s))}\right) & \text{if } \lambda \geq \tau_0 \\
F^{-1}\left(\frac{(1-\alpha)((1-w)(p-c)+ws)}{(1-\lambda\alpha)(p-r-w(p-c-s))}\right) & \text{otherwise}
\end{cases}
\]

(8)

It is noted that the optimal order quantity \( q_0^* \) from maximizing the expected profit (for a risk-neutral newsvendor) and the the optimal order quantity \( q_1^* \) from the pure CVaR model (for a risk-averse newsvendor) can be treated as extreme cases of this result. In particular, \( q_2^* = q_0^* \) when \( \lambda = 1 \) and \( q_2^* = q_1^* \) when \( \lambda = 0 \). It is also important to point out that

\[
q_1^* \leq q_2^* \leq q_0^*.
\]
That is, the decision bias \((q_0^* - q_2^*)\) is less than that from the pure CVaR model.

Patterned as in the previous subsection, we have the following monotone properties of \(q_2^*\) when it is treated as a function of its parameters \(r, s, w, \alpha, \) and \(\lambda\).

**Corollary 3.5** We have the following monotone properties of \(q_2^*\).

(i) For \(r, s \geq 0\), \(q_2^*\) is a increasing function of \(r\) and \(s\) respectively.

(ii) For \(\alpha \in [0, 1)\) and \(w \in [0, 1]\), \(q_2^*\) is a decreasing function of \(\alpha\) and \(w\) respectively.

(iii) For \(\lambda \in [0, 1]\), \(q_2\) is increasing in the weight \(\lambda\).

More precisely, we have

\[
\frac{\partial q_2^*}{\partial r} > 0, \quad \frac{\partial q_2^*}{\partial s} > 0, \quad \text{and} \quad \frac{\partial q_2^*}{\partial \alpha} < 0, \quad \frac{\partial q_2^*}{\partial w} < 0, \quad \text{and} \quad \frac{\partial q_2^*}{\partial \lambda} > 0.
\]

Similar comments can be made for the first two results in the above corollary as for Cor. 3.2 (see Remarks (R3) and (R4)). The last result in Cor. 3.5 (iii) indicates that \(q_2^*\) strictly increases from \(q_1^*\) to \(q_0^*\) as \(\lambda\) increases from 0 to 1. This well reflects the role that \(\lambda\) plays in the mixed CVaR model, which is to balance the expected profit \(E(P(q, \xi))\) and the CVaR objective \(f_p(q)\).

We finished this section by stating one more result concerning the expected profit function under the optimal order quantity \(q_2^*\).

**Corollary 3.6** For \(\alpha \in [0, 1]\), \(E[P(q_2^*, \xi)]\) is decreasing in the confidence level \(\alpha\). That is,

\[
\frac{\partial E[P(q_2^*, \xi)]}{\partial \alpha} < 0.
\]

For \(\lambda \in [0, 1]\), \(E[P(q_2^*, \xi)]\) is increasing in weight \(\lambda\). That is,

\[
\frac{\partial E[P(q_2^*, \xi)]}{\partial \lambda} > 0.
\]

The first statement in Cor. 3.6 means that if the newsvendor becomes more risk-averse, then he should expect a lower profit from the mixed CVaR model. This remark is similar to that for Cor. 3.3. The second statement in Cor. 3.6 shows that if the newsvendor pays more attention to expected profit over the risk control, he should expect a higher profit return.

### 4 Numerical Results

In this section, we present some numerical experiments on a randomly generated problem so as to demonstrate the main results obtained in Sec. 3. Along the way, we will draw some management insights.

We first present our example.
Example 4.1 For the newsvendor problem with a backorder case, suppose the market demand $\xi$ is subject to the uniform distribution $U(0,1000)$ and the parameters take the following basic values:

$$p = 8, \quad c = 3, \quad r = 1, \quad \text{and} \quad s = 1.$$  

These values will vary when we conduct the sensitivity analysis of the optimal order quantities.

We will conduct three groups of sensitivity analysis. The first group is on the basic parameters $p$, $c$, $r$, and $s$. The second analysis is on the model parameters $w$ and $\alpha$. The last group of analysis is on the expected profit function under the optimal order quantities.

(a) Sensitivity analysis on the basic parameters $p$, $c$, $r$, and $s$. Throughout this group of test, we set $w = 0.5$, $\alpha = 0.5$, and $\lambda = 0.5$. We compute the optimal order quantities $q^*_0$, $q^*_1$ and $q^*_2$ with different basic parameters $p$, $c$, $r$ and $s$. We plot those quantities in Figure 1. It can be seen from Figure 1 that all the quantities $q^*_0$, $q^*_1$ and $q^*_2$ are increasing in the retail price $p$, the salvage price $r$ and the price discount $s$, and decreasing in the wholesale price $c$, respectively. Moreover, the decision bias from the mixed CVaR model is below the decision bias from the pure CVaR model, which confirms our theoretical result: $q^*_0 \geq q^*_2 \geq q^*_1$.

(b) Sensitivity analysis on the model parameters $w$ and $\alpha$. In this group of test, we let $p = 8$, $c = 3$, $r = 1$ and $s = 1$ and compute the optimal order quantities $q^*_0$, $q^*_1$ and $q^*_2$ with different parameters $w$, $\alpha$ and $\lambda$. We plot those quantities in Figures 2, 3, and 4. By Figure 2, it is easily checked that $q^*_0$, $q^*_1$ and $q^*_2$ all are decreasing in the backorder rate $w$. By Figure 3, $q^*_1$ and $q^*_2$ both are decreasing in the confidence level $\alpha$, while for a risk-neutral newsvendor, i.e., $\alpha = 0$, the optimal order quantity $q^*_0$ stays the same. By Figure 4, the optimal order quantity $q^*_2$ for the combined objective is increasing in the weight $\lambda$. That is to say, if a newsvendor pays more attention to the maximization of his expected profit rather than risk control and assigns a big value to the weight $\lambda$, he had better improve his order quantity accordingly. Besides, by Figure 2 and Figure 3, $q^*_0 \geq q^*_2 \geq q^*_1$ always holds for different backorder rate $w$ and confidence level $\alpha$.

(c) Sensitivity analysis on the expected profit function under the optimal order quantities. In this group of test, we let $p = 8$, $c = 3$, $r = 1$, $s = 1$ and $w = 0.5$ and compute the expected profit $E[P(q^*_0, \xi)], E[P(q^*_1, \xi)]$ and $E[P(q^*_2, \xi)]$ with different parameters $\alpha$ and $\lambda$. We plot those quantities in Figure 5 and Figure 6 respectively. By Figure 5, for
Figure 1: Behavior of the optimal order quantities $q_0^*$, $q_1^*$ and $q_2^*$ in terms of the retail price $p$, the wholesale price $c$, the salvage price $r$ and the price discount $s$.

Figure 2: Behavior of the optimal order quantities $q_0^*$, $q_1^*$ and $q_2^*$ in terms of the backorder rate $w$. 

\[ \text{Figure 1: Behavior of the optimal order quantities } q_0^*, q_1^* \text{ and } q_2^* \text{ in terms of the retail price } p, \]
\[ \text{the wholesale price } c, \text{ the salvage price } r \text{ and the price discount } s. \]

\[ \text{Figure 2: Behavior of the optimal order quantities } q_0^*, q_1^* \text{ and } q_2^* \text{ in terms of the backorder rate } w. \]
Figure 3: Behavior of the optimal order quantities $q_0^*$, $q_1^*$ and $q_2^*$ in terms of the confidence level $\alpha$.

![Graph showing behavior of optimal order quantities](image)

Figure 4: Behavior of the optimal order quantity $q_2^*$ in terms of the weight $\lambda$.

![Graph showing behavior of optimal order quantity $q_2^*$](image)

A risk-neutral newsvendor, i.e. $\alpha = 0$, the expected profit $E[P(q_0^*, \xi)]$ stays the same, while the expected profits $E[P(q_1^*, \xi)]$ and $E[P(q_2^*, \xi)]$ of the newsvendor are both decreasing in the confidence level $\alpha$, which implies if the newsvendor wants to reduce the potential risk, he will expect a lower profit. Moreover, Figure 5 also illustrates that the expected profit $E[P(q_1^*, \xi)]$ is smaller than the expected profit $E[P(q_2^*, \xi)]$, while $E[P(q_0^*, \xi)]$ is the biggest of all. By Figure 6, the expected profit $E[P(q_2^*, \xi)]$ of the newsvendor is increasing in the parameter $\lambda$, which implies if the newsvendor pays more attention to the maximization of his expected profit rather than
risk control and assigns a big value to the weight $\lambda$, he then will expect a higher profit.

Figure 5: Expected profits $E[P(q_0^*, \xi)]$, $E[P(q_1^*, \xi)]$ and $E[P(q_2^*, \xi)]$ with confidence level $\alpha$

To summarize this section, the numerical experiment and sensitivity analysis confirm that the results obtained in the above section are qualitatively robust. Here, for the newsvendor model with a backorder case, there are following suggestions for the newsvendor to decide an order quantity. If the newsvendor pays more attention to the maximization of the expected profit rather than risk control, then he had better give a larger order quantity; otherwise, if the newsvendor pays more attention to reducing the risk originated from the fluctuation in the
market demand, then he had better give a smaller order quantity.

5 Conclusions

In the newsvendor model, to obtain profit that is saved from the excess demands of the unsatisfied customers, a newsvendor always does his best to backlog the excess demands as many as possible. Nowadays, with the rapid development of logistics industry, the goods can be delivered to the newsvendor quickly and it makes backorder easier to implement. Thus, a recent extension of the classical newsvendor model to backlog the excess demands of loyal customers has received more and more attentions from researchers. However, the existing literature pays more attention to the maximization of profit or minimization of cost for the newsvendor model with a back order case, while the risk control for such a problem is often ignored.

In order to control the risk originated from the fluctuation in the market demand for the newsvendor model with a backorder case, we introduce the CVaR measure to quantify the above risks, which leads to our pure CVaR model (4). Based on the obtained results, it is easily checked that the optimal order quantity of a risk-averse newsvendor in maximizing the CVaR objective is less than that of a risk-neutral newsvendor in maximizing the expected profit in a backorder case. It has the following rationale: Since the excess demands can be all or partially satisfied by a backorder, the risk originates mainly from the possible excess order, thus it is better to order less products to lessen the risk from the excess order for a risk-averse newsvendor. Moreover, this study also shows that, if the confidence level $\alpha$ improves, which implies the newsvendor becomes more risk-averse and thus lays more importance on reducing the risks, then the expected profit under the optimal order quantity in maximizing the CVaR objective will decrease. In other words, if the newsvendor intends to reduce the potential risk, then he will expect a lower profit.

As a coherent risk measure, there still exists some limitations about the CVaR criterion. For example, this criterion pays more attention to the profit below the certain quantile level while the profit above this quantile level is always neglected. This is very conservative for some newsvendors since only the worst outcomes are considered. To balance the profit maximization and risk control, this paper subsequently introduces another objective which is a combination of the expected profit and the CVaR objective, leads to our mixed CVaR model (6). This objective satisfies the newsvendor’s need to maximize his expected profit on the one hand, and minimize the downside risk of his profit on the other hand. Then, a newsvendor can attach different importance on expected profit and CVaR by changing the parameter $\lambda$. We think this objective is helpful to strike a balance between profit maximization and risk control for the newsvendor.
in a backorder setting.

Several extensions of this study are possible. First, it is assumed in this paper that the backorder rate $w$ is exogenously determined. But, in some literature, it is assumed that the backorder rate $w$ is associated with the lead-time of the excess demand. Extending our study to incorporate a lead time-dependent backorder rate is appealing. Second, another interesting issue to be investigated is the profit function $P(q, \xi)$, in which the replenishment cost for the backlogged products may be considered. For such a case, the newsvendor needs to consider not only the cost for compensating the loyal customers but also the replenishment cost for the backlogged products.
Appendix

**Proof of Theorem 3.1.** Define

\[ h(q, v) = v - \frac{1}{1 - \alpha} E [v - P(q, \xi)]^+. \]

It can be easily checked that

\[ h(q, v) = v - \frac{1}{1 - \alpha} \int_0^q [v - (p - r)t + (c - r)q]^+ dF(t) \]

\[ = -\frac{1}{1 - \alpha} \int_q^\infty [v - ((1 - w)(p - c) + ws)q - w(p - c - s)t]^+ dF(t) \]  

It follows from [19, Cor. 11] that \( h(q, v) \) is jointly concave in \((q, v)\) since \( P(q, \xi) \) is concave in \( q \).

We note that the pure model (4) is equivalent to

\[ \max_{q \geq 0} \left[ \max_{v \in \mathbb{R}} h(q, v) \right]. \]

The proof below aims to find the optimal solution of the inner problem when a particular \( q \) is given. We consider the following three cases.

**Case 1.** \( v < (r - c)q \). For this case, it follows from (10) that

\[ h(q, v) = v, \]

and hence

\[ \frac{\partial h(q, v)}{\partial v} = 1 > 0. \]

Due to the concavity of \( h(q, \cdot) \), the maximum of \( h(q, \cdot) \) cannot be attained for this case.

**Case 2.** \((r - c)q \leq v < (p - c)q\). For this case, it follows from (10) that

\[ h(q, v) = v - \frac{1}{1 - \alpha} \int_0^{v+(c-r)q/p-r} [v - (p - r)t + (c - r)q] dF(t) \]

and

\[ \frac{\partial h(q, v)}{\partial v} = 1 - \frac{1}{1 - \alpha} F \left( \frac{v + (c - r)q}{p - r} \right). \]

Obviously, it satisfies \( \left. \frac{\partial h(q, v)}{\partial v} \right|_{v=r-c} = 1 > 0 \). If \( q \geq F^{-1}(1 - \alpha) \), we must have

\[ \left. \frac{\partial h(q, v)}{\partial v} \right|_{v=(p-c)q} = 1 - \frac{1}{1 - \alpha} F(q) \leq 0. \]

By the continuity of \( \frac{\partial h(q, v)}{\partial v} \), there exists \( v^* \in [(r - c)q, (p - c)q] \) such that

\[ 0 = \left. \frac{\partial h(q, v)}{\partial v} \right|_{v=v^*} = 1 - \frac{1}{1 - \alpha} F \left( \frac{v^* + (c - r)q}{p - r} \right) = 0, \]

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which implies

\[ v^* = (p - r)F^{-1}(1 - \alpha) - (c - r)q. \]

Due to the concavity of \( h(q, \cdot) \), the maximum of \( h(q, \cdot) \) is attained at \( v^* \) when \( q \geq F^{-1}(1 - \alpha) \).

**Case 3.** \( v \geq (p - c)q \). For this case, it follows from (10) that

\[
h(q, v) = v - \frac{1}{1 - \alpha} \int_0^q [v - (p - r)t + (c - r)q]dF(t) - \frac{1}{1 - \alpha} \int_q^{v - ((1 - w)(p - c) + ws)q/w(p - c - s)} [v - ((1 - w)(p - c) + ws)q - w(p - c - s)t]dF(t),
\]

and

\[
\frac{\partial h(q, v)}{\partial v} = 1 - \frac{1}{1 - \alpha} F \left(\frac{v - ((1 - w)(p - c) + ws)q}{w(p - c - s)}\right).
\]

Note that \( \alpha > 0 \), we have

\[
\lim_{v \to \infty} \frac{\partial h(q, v)}{\partial v} < 0.
\]

If \( q \leq F^{-1}(1 - \alpha) \), we have

\[
\left. \frac{\partial h(q, v)}{\partial v} \right|_{v = (p - c)q} = 1 - 1 - \frac{1}{1 - \alpha} F(q) \geq 0.
\]

Due to the continuous differentiability of \( h(q, \cdot) \), there must exist \( v^* \) such that

\[
\left. \frac{\partial h(q, v)}{\partial v} \right|_{v = v^*} = 0,
\]

which implies

\[ v^* = w(p - c - s)F^{-1}(1 - \alpha) + ((1 - w)(p - c) + ws)q. \]

Due to the concavity of \( h(q, \cdot) \), the maximum of \( h(q, \cdot) \) is attained at \( v^* \) when \( q \leq F^{-1}(1 - \alpha) \).

Putting together the three cases, we have proved that the optimal solution \( v^* \) to the inner problem of (10) is given by

\[
v^* = \begin{cases} (p - r)F^{-1}(1 - \alpha) - (c - r)q & q \geq F^{-1}(1 - \alpha), \\ w(p - c - s)F^{-1}(1 - \alpha) + ((1 - w)(p - c) + ws)q & q \leq F^{-1}(1 - \alpha). \end{cases}
\]

Therefore, the problem (10) is reduced to

\[
\max_{q \geq 0} h(q, v^*).
\]

Since \( v^* \) is a linear function of \( q \), \( h(q, v^*) \) is concave in \( q \) due to the fact that \( h(q, v) \) is jointly concave in \((q, v)\). The rest of the proof is to show that the maximum of \( h(q, v^*) \) is attained.

We consider two situations. One is \( q \geq F^{-1}(1 - \alpha) \). It follows from (11) and (10) that

\[
h(q, v^*) = (p - r)F^{-1}(1 - \alpha) - (c - r)q - \frac{1}{1 - \alpha} \int_0^{F^{-1}(1 - \alpha)} (p - r)[F^{-1}(1 - \alpha) - t]dF(t)
\]
We calculate $E$. It follows from (1) that

\[ \frac{\partial h(q, v^*)}{\partial q} = -(c - r) < 0. \]

By the concavity of $h(q, v^*)$ in $q$, the maximum cannot be achieved for this situation.

The remaining situation is $q \leq F^{-1}(1 - \alpha)$. It follows from (11) and (10) that

\[
\begin{align*}
    h(q, v^*) &= w(p - c - s)F^{-1}(1 - \alpha) + ((1 - w)(p - c) + ws)q \\
    &\quad - \frac{1}{1 - \alpha} \int_0^q [w(p - c - s)F^{-1}(1 - \alpha) - (p - r)t + ((p - r) - w(p - c - s))q]dF(t) \\
    &\quad - \frac{1}{1 - \alpha} \int_q^{F^{-1}(1 - \alpha)} w(p - c - s)(F^{-1}(1 - \alpha) - t)dF(t)
\end{align*}
\]

and

\[
\frac{\partial h(q, v^*)}{\partial q} = (1 - w)(p - c) + ws - \frac{p - r - w(p - c - s)}{1 - \alpha} F(q).
\]

It is easy to check that

\[
\frac{\partial h(q, v^*)}{\partial q} |_{q=0} = (1 - w)(p - c) + ws > 0 \quad \text{and} \quad \frac{\partial h(q, v^*)}{\partial q} |_{q=F^{-1}(1-\alpha)} = -(c - r) < 0.
\]

There must exist $q_1^*$ such that

\[
\frac{\partial h(q_1^*, v^*)}{\partial q} = 0,
\]

which gives

\[
q_1^* = F^{-1}\left(\frac{(1-\alpha)((1-w)(p-c)+ws)}{p-r-w(p-c-s)}\right).
\]

Due to the concavity of $h(q, v^*)$, $q_1^*$ is the optimal solution of (12), and hence is also the optimal solution of (10). This completes the proof.

\[\square\]

**Proof of Cor. 3.2.** Since $F(\cdot)$ is monotonically increasing, it follows from the formula of $q_1^*$ that $q_1^*$, as a function of $\alpha$, $r$, and $s$, has the corresponding increasing or decreasing property as stated in the corollary. When $q_1^*$ is viewed as a function of $w$, we have

\[
\frac{\partial q_1^*}{\partial w} = -\frac{1}{f\left(F^{-1}\left(\frac{(1-\alpha)((1-w)(p-c)+ws)}{p-r-w(p-c-s)}\right)\right)} \frac{(1-\alpha)(c-r)(p-c-s)}{(p-r-w(p-c-s))^2} < 0.
\]

Hence, $q_1^*$ is decreasing in $w$. The complete the proof.

\[\square\]

**Proof of Cor. 3.3.** It follows from (1) that

\[
E[P(q, \xi)] = [(1 - w)(p - c) + ws]q + w(p - c - s)E(D) - [p - r - w(p - c - s)] \int_0^q (q - t)dF(t).
\]

We calculate

\[
\frac{\partial E[P(q_1^*)]}{\partial \alpha} = [(1 - w)(p - c) + ws - (p - r - w(p - c - s))F(q_1^*)] \frac{\partial q_1^*}{\partial \alpha}.
\]

(13)
We recall that
\[ q_0^* = F^{-1}\left(\frac{(1-w)(p-c) + ws}{p-r-w(p-c-s)}\right), \]
which gives
\[ (1-w)(p-c) + ws - (p-r-w(p-c-s))F(q_0^*) = 0. \]
We also recall \( q_1^* \leq q_0^* \). The monotonicity of \( F \) implies
\[ (1-w)(p-c) + ws - (p-r-w(p-c-s))F(q_1^*) \geq (1-w)(p-c) + ws - (p-r-w(p-c-s))F(q_0^*) = 0. \]
Equation (13) and Cor. 3.2(ii) yield
\[ \frac{\partial E[P(q_1^*, \xi)]}{\partial \alpha} \leq 0, \]
which proves that \( E[P(q_1^*, \xi)] \) is decreasing in the confidence level \( \alpha \).

**Proof of Theorem 3.4.** The proof is similar to that for Thm. 3.1. Define
\[ k(q, v) = \lambda E[P(q, \xi)] + (1 - \lambda)[v - \frac{1}{1 - \alpha} E[v - P(q, \xi)]]^+. \]
Then the problem (6) is equivalent to
\[ \max_{q \geq 0} \max_{v \in \mathbb{R}} k(q, v). \quad (14) \]
Note that the first item in the right hand of \( k(q, v) \) has nothing to do with \( v \). For any fixed \( q \), it is concluded from the proof of Theorem 4.2 that the optimal solution to inner problem \( \max_{v \in \mathbb{R}} k(q, v) \) of (14) is given by (11). Therefore, the problem (14) is reduced to
\[ \max_{q \geq 0} k(q, v^*). \quad (15) \]
Since the function \( k(q, v) \) is jointly concave due to [19, Cor. 11] and \( v^* \) is linear in \( q \), \( k(q, v^*) \) is concave in \( q \). We consider the following two cases.

**Case 1.** \( q \geq F^{-1}(1 - \alpha) \). It follows from (11) and the definition of \( k(q, v) \) that (we omit the detailed computation)
\[ \frac{\partial k(q, v^*)}{\partial q} = \lambda[(1-w)(p-c) + ws - (p-r-w(p-c-s))F(q)] - (1-\lambda)(c-r). \quad (16) \]
If \( \lambda \geq \tau_0 \), it is straightforward to see that
\[ \frac{\partial k(q, v^*)}{\partial q} |_{q=F^{-1}(1-\alpha)} \geq 0. \]
Equation (16) also means
\[ \lim_{q \to \infty} \frac{\partial k(q,v^*)}{\partial q} < 0. \]

By the continuous differentiability of \( k(\cdot,v^*) \) in \( q \), there exists \( q_2^* \) such that
\[ 0 = \frac{\partial k(q_2^*,v^*)}{\partial q} = \lambda((1-w)(p-c) + ws - (p-r-w(p-c-s))F(q_2^*)) - (1-\lambda)(c-r), \]
which gives
\[ q_2^* = F^{-1}\left(\frac{\lambda((1-w)(p-c) + ws) - (1-\lambda)(c-r)}{\lambda(p-r-w(p-c-s))}\right). \]

By the concavity of \( k(q,v^*) \) in \( q \), \( q_2^* \) solves the problem (15).

**Case 2.** \( q \leq F^{-1}(1-\alpha) \). It follows from (11) and the definition of \( k(q,v) \) that (again we omit the detailed computation)
\[ \frac{\partial k(q,v^*)}{\partial q} = (1-w)(p-c) + ws - \frac{1-\lambda\alpha}{1-\alpha}(p-r-w(p-c-s))F(q). \]

If \( \lambda \leq \tau_0 \), it is straightforward to see that
\[ \frac{\partial k(q,v^*)}{\partial q} |_{q=F^{-1}(1-\alpha)} \leq 0. \]

It is also easy to verify that
\[ \frac{\partial k(q,v^*)}{\partial q} |_{q=0} = (1-w)(p-c) + ws > 0. \]

By the continuous differentiability of \( k(\cdot,v^*) \) in \( q \), there exists \( q_2^* \) such that
\[ 0 = \frac{\partial k(q_2^*,v^*)}{\partial q} = (1-w)(p-c) + ws - \frac{1-\lambda\alpha}{1-\alpha}(p-r-w(p-c-s))F(q_2^*), \]
which gives
\[ q_2^* = F^{-1}\left(\frac{(1-\alpha)((1-w)(p-c) + ws)}{(1-\lambda\alpha)(p-r-w(p-c-s))}\right). \]

By the concavity of \( k(q,v^*) \) in \( q \), \( q_2^* \) solves the problem (15).

Putting the two cases together, we proved that the quantity \( q_2^* \) actually solves the problem (14) and hence the original problem (6). This completes the proof. \( \square \)

**Proof of Cor. 3.5.** The proof is straightforward and only involves simple differentiation operations. We omit the details for the benefit of shortening the length of the paper. \( \square \)
Proof of Cor. 3.6. For the first result, we consider two cases. \( \lambda \geq \tau_0 \) and \( \lambda \leq \tau_0 \). For the first case, we compute

\[
\frac{\partial q_2^*}{\partial \lambda} = \frac{1}{f \left( F^{-1} \left( \frac{\lambda (1-w)(p-c)+ws}{\lambda(p-r-w(p-c-s))} \right) \right)} \frac{c-r}{\lambda^2(p-r-w(p-c-s))} > 0.
\]

For the second case, we compute

\[
\frac{\partial q_2^*}{\partial \lambda} = \frac{1}{f \left( F^{-1} \left( \frac{(1-\alpha)(1-w)(p-c)+ws}{(1-\lambda\alpha)(p-r-w(p-c-s))} \right) \right)} \frac{\alpha(1-\alpha)(1-w)(p-c) + ws}{(1-\lambda\alpha)^2(p-r-w(p-c-s))} > 0.
\]

For both cases, we proved that \( q_2^* \) is increasing in \( \lambda \). Furthermore, at the boundary \( \lambda = \tau_0 \), the derivatives are equal. Hence, \( q_2^* \) is increasing in \( \lambda \) for the whole range \( \lambda \in [0,1] \).

The second part can be similarly proved as for Cor. 3.3. We omit its details. \( \square \)
References


