Positive Semidefinite Matrix Completions on Chordal Graphs and Constraint Nondegeneracy in Semidefinite Programming

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Abstract

Let $G = (V, E)$ be a graph. In matrix completion theory, it is known that the following two conditions are equivalent: (i) G is a chordal graph; (ii) Every G-partial positive semidefinite matrix has a positive semidefinite matrix completion. In this paper, we relate these two conditions to constraint nondegeneracy condition in semidefinite programming and prove that they are each equivalent to (iii) For any G-partial positive definite matrix that has a positive semidefinite completion, constraint nondegeneracy is satisfied at each of its positive semidefinite matrix completions.

Key words: Chordal graph, constraint nondegeneracy, matrix completion, semidefinite programming.

AMS subject classifications. 90C25, 90C27, 90C33

1 Introduction

The positive semidefinite completion problem, a prominent example of the general matrix completion problem, has long been extensively studied, see the survey papers by Johnson [14], Alfakih and Wolkowicz [1], Laurent [18], Harrison [12] (and references therein), and [9, 11, 5, 3, 16, 17, 4]. In particular, its intersection with *semidefinite programming* (SDP) has proved to be productive in approximating solutions of some hard combinatorial optimization problems [15, 28]. In this paper, we enhance this intersection by contributing a new characterization of the positive semidefinite completion via constraint nondegeneracy in SDP.

In order to describe the new characterization, we first introduce some basic notations. Most of them are consistent with that used in [11]. Let $G = (V, E)$ be a finite undirected graph, where $V = \{1, 2, \ldots, n\}$ and E is the set of edges. The graph is assumed to be *simple*, i.e., it has no loops or parallel edges. The graph is a convenient tool to show the pattern of a partially known symmetric matrix. Let $\mathcal{S}^n$ and $\mathcal{S}^n_+$ be, respectively, the space of $n \times n$ symmetric real matrices and the cone of *positive semidefinite* matrices in $\mathcal{S}^n$. We use $X \succ (\succeq)\ 0$ to denote that $X \in \mathcal{S}^n$ is positive definite (positive semidefinite).

For a given graph $G = (V, E)$, define a G-partial symmetric matrix as a set of real numbers, denoted by $[A_{ij}]_G$ or $A(G)$, where all the diagonal elements A_{ii}, $i = 1, \ldots, n$ and $A_{ij} = A_{ji}$,

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$(i,j) \in E$ are known. A *completion* of $A(G)$ is a matrix $X \in \mathcal{S}^n$ which satisfies $X_{ij} = A_{ij}$ for all $(i,j) \in E$ and $X_{ii} = A_{ii}$, $i = 1, \ldots, n$. We say that $X \in \mathcal{S}^n$ is a *positive definite* completion (a *positive semidefinite* completion) of $A(G)$ if and only if X is a completion of $A(G)$ and $X \succ 0$ ($X \succeq 0$).

The *existence* issue in the positive (semi)definite completion problem asks the following question: Given a G-partial symmetric matrix $A(G)$, does it have a positive (semi)definite completion? The answer to this question leads to the *completability* of graph G, whose definition involves all the *cliques* of G. A clique is a subset $C \subset V$ having the property that $(x,y) \in E$ for all $x, y \in C$. A *cycle* in G is a sequence of pairwise distinct vertices $\gamma = (v_1, \ldots, v_s)$ having the property that (v_1, v_2), (v_2, v_3), $\ldots$, (v_{s-1}, v_s), $(v_s, v_1) \in E$, and s is referred to as the *length* of the cycle. A *chord* of the cycle γ is an edge $(v_i, v_j) \in E$ where $i \leq i < j \leq s$, $(i,j) \neq (1,s)$, and $|i - j| \geq 2$. The cycle γ is *minimal* if any other cycle in G has a vertex not in γ, or equivalently, γ has no chord. A graph G is *chordal* if there are no minimal cycles of length ≥ 4.

We say that a symmetric $A(G)$ is G-partial positive definite (G-partial positive semidefinite) matrix if for any clique C of G, the principal submatrix $[A_{ij} : \ i, j \in C]$ of $A(G)$ is positive definite (positive semidefinite). We say that the graph G is *completable* (positive semidefinite-*completable*[1]) if and only if any G-partial positive definite (G-partial positive semidefinite) matrix has a positive definite (semidefinite) completion. Although the completability and the positive semidefinite completability are defined independently, there is no need to distinguish them from each other as they are equivalent [11, Prop. 2].

The first non-trivial sufficient condition for the completability is that G is a band graph [9]. This result is greatly generalized to a sufficient and necessary condition in [11, Thm. 7]:

$$G \text{ is completable if and only if } G \text{ is a chordal graph.}$$

For other types of sufficient and necessary conditions, see [16].

Constraint nondegeneracy has long been known to be a generic property in *semidefinite programming* [2], where the name of *primal nondegeneracy* was used. It plays an important role in stability analysis in linear/nonlinear SDP [6, 24, 7, 19] and has algorithmic implications in interior-point methods for quadratic SDP [27, 26]. An early indication that completability and constraint nondegeneracy may be closely related comes from our study on the *nearest correlation matrix* problem [20], see also [13]. The constraint of the problem, in terms of the terminology in this paper, is the collection of all positive semidefinite completions of $A(G)$ with $A_{ii} = 1$, $i = 1, \ldots, n$ and $E = \emptyset$ (see Example 4.1). Constraint nondegeneracy is satisfied at the nearest correlation matrix [20, Lem. 3.3] and [27, Prop. 4.2]. The proofs can be extended to band graphs. The indication is greatly enhanced by observing that the positive semidefinite matrix completion problem can be cast as the correlation matrix completion problem, an approach adopted in [16]. We will show that constraint nondegeneracy condition is satisfied at any positive semidefinite matrix completion of $A(G)$, where G is chordal and $A(G)$ is any G-partial positive definite matrix.

The above result shows that constraint nondegeneracy forms a necessary condition for the completability of a graph. We may ask whether it is also sufficient. The answer is affirmative and our proof is heavily motivated by a widely known fact that is used among others to establish the equivalence between the completability and the chordal graph in [11]. The fact is that the

[1] In [11], the name of nonnegative-completable was used instead of the positive semidefinite-completable.

following G-partial positive semidefinite matrix

$$A(G) = \begin{bmatrix} 1 & 1 & ? & 0 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ 0 & ? & 1 & 1 \end{bmatrix}$$

has no positive semidefinite completion, where $?'s$ denote the unfilled elements in $A(G)$. We may ask the opposite question: For those $A(G)$ that have a positive semidefinite completion, what is the condition that all the positive semdefinite completions have to obey in order to remove the existence of cycles of length ≥ 4? We will show that the condition is exactly constraint nondegeneracy. By this, we establish its sufficiency.

The paper is organized as follows. In the next section, we collect some facts about constraint nondegeneracy in SDP. Section 3 establishes the equivalence between completability and constraint nondegeneracy. In Section 4, we present some by-products of the new characterization by studying what we called the nearest positive semidefinite matrix completion problem. We conclude the paper in Section 5.

Notation: We use := to mean "define". For a matrix X, X_{ij} denotes its $(i,j)^{\text{th}}$ elements. For $X, Y \in \mathcal{S}^n$, $\langle X,\ Y\rangle := \text{Trace}(XY)$ and $\|X\|^2 := \langle X,\ X\rangle$, the Frobenius norm induced by the standard inner product. For $i, j = 1, \ldots, n$, define $E^{ij} \in \mathcal{S}^n$ by

$$\left(E^{ij}\right)_{\ell t} := \begin{cases} 1 & (\ell, t) \in \{(i,j),\ (j,i)\} \\ 0 & \text{otherwise} \end{cases} \qquad \ell, t = 1, \ldots, n.$$

2 Constraint Nondegeneracy in SDP

Consider the semidefinite programming problem

$$\begin{array}{ll} \min & \langle C,\ X\rangle \\ \text{s.t.} & X \in \mathcal{C} := \{X \succeq 0,\ \langle A_i,\ X\rangle = b_i,\ i = 1, \ldots, m\}, \end{array} \tag{1}$$

where $C,\ A_i \in \mathcal{S}^n$ and $b_i \in \mathbb{R}$, for $i = 1, \ldots, m$ are known. The matrices $A_i, i = 1, \ldots, m$ are assumed to be linearly independent, i.e., they span an m-dimensional linear space in $\mathcal{S}^n$. Recall that $X \succeq 0$ means that $X \in \mathcal{S}^n_+$, the cone of positive semidefinite matrices in $\mathcal{S}^n$.

Suppose $X \in \mathcal{S}^n_+$. We let $T_{\mathcal{S}^n_+}(X)$ be the tangent cone of $\mathcal{S}^n_+$ at X. Because $\mathcal{S}^n_+$ is a closed convex cone, $T_{\mathcal{S}^n_+}(X)$ is the closure of the cone generated by $\mathcal{S}^n_+ - X$. We further let $\text{lin}(T_{\mathcal{S}^n_+}(X))$ be the largest space contained in $T_{\mathcal{S}^n_+}(X)$. Obviously, $\text{lin}(T_{\mathcal{S}^n_+}(X)) = \mathcal{S}^n$ for $X \succ 0$ (i.e., $T_{\mathcal{S}^n_+}(X) = \mathcal{S}^n$ when X is positive definite). Define the linear transformation $\mathcal{A} : \mathcal{S}^n \mapsto \mathbb{R}^m$ by

$$\mathcal{A}(X) = (\langle A_1,\ X\rangle, \ldots, \langle A_m,\ X\rangle)^T.$$

Now we are ready to introduce constraint nondegeneracy in SDP.

Definition 2.1 *([2, Def. 5], [7, Def. 9]) We say that $X \in \mathcal{C}$ is constraint nondegenerate with respect to the constraints defining $\mathcal{C}$ if*

$$\mathcal{A}\Big(\text{lin}(T_{\mathcal{S}^n_+}(X))\Big) = \mathbb{R}^m. \tag{2}$$

We note that condition (2) holds if and only if

$$\left\{\mathcal{A}\Big(\mathrm{lin}(T_{\mathcal{S}^n_+}(X))\Big)\right\}^{\perp} = \{0\}\,, \tag{3}$$

where $\{\mathcal{A}(\mathrm{lin}(T_{\mathcal{S}^n_+}(X)))\}^{\perp}$ denotes the orthogonal space of $\{\mathcal{A}(\mathrm{lin}(T_{\mathcal{S}^n_+}(X)))\}$. If X has a full rank (i.e., $X \succ 0$), then $\mathrm{lin}(T_{\mathcal{S}^n_+}(X)) = \mathcal{S}^n$. Condition (3) reduces to

$$\sum_{i=1}^{m} A_i y_i = 0 \quad \Longrightarrow \quad y_i = 0,\ i = 1, \ldots, m.$$

Hence, in this case, constraint nondegeneracy of X follows the linear independence of $A_i, i = 1, \ldots, m$.

Generally speaking, constraint nondegeneracy is a property of a feasible point of a constrained system. So the underlying constraints are very important to the verification of constraint nondegeneracy. In this paper, we only encounter linearly constrained systems of the type in (1). In [2], *primal nondegeneracy* instead of constraint nondegeneracy was used for linear constraints. For the general definition of constraint nondegeneracy for nonlinear semidefinite constraints, see [6, Eq 4.172].

Throughout the paper, when we say $X \in \mathcal{C}$ is nondegenerate we mean that X is constraint nondegenerate with respect to those constraints defining $\mathcal{C}$. For example, in the next section, when we say $X \in \mathcal{C}_{A(G)}$ (see (12) for its definition) is nondegenerate we mean that X is constraint nondegenerate with respect to those constraints defining $\mathcal{C}_{A(G)}$. It is always clear what constraints are underlying constraint nondegeneracy.

Constraint nondegeneracy has a nice characterization in terms of the eigenvectors of X. Suppose $X \in \mathcal{S}^n_+$ has the following spectral decomposition:

$$X = Q\mathrm{Diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)Q^T, \tag{4}$$

where $\lambda_1, \ldots, \lambda_r$ are positive eigenvalues of X and $QQ^T = I$. Let $Q = [Q_1, Q_2]$, where $Q_1 \in \mathbb{R}^{n\times r}$, $Q_2 \in \mathbb{R}^{n\times(n-r)}$ denote the first r columns and the last $(n-r)$ columns of Q, respectively. Then we have the following characterization.

Lemma 2.2 *[2, Thm. 6] Let X be a feasible point (i.e., $X \in \mathcal{C}$) of SDP (1) with the spectral decomposition (4). Then X is constraint nondegenerate if and only if the matrices*

$$B_k = \left[\begin{array}{cc} Q_1^T A_k Q_1 & Q_1^T A_k Q_2 \\ Q_2^T A_k Q_1 & 0 \end{array}\right], \quad k = 1, \ldots, m$$

are linearly independent in $\mathcal{S}^n$.

Another important characterization of constraint nondegeneracy at the optimal solution of SDP (1) is that the *strong second-order sufficient condition* (SSOSC) is satisfied at the unique optimal solution of the dual problem of (1), see [7, Prop. 15] and the companying comments. Since we are not going to make use of this characterization in our proofs, we omit its description here.

3 Constraint Nondegenercy and Completability

Let us consider the constraints in SDP (1):

$$\left\{X \in \mathcal{S}^n_+,\ \langle A_i,\ X\rangle = b_i,\ i = 1, \dots, m\right\}. \tag{5}$$

Our first result is to claim that constraint nondegeneracy is orthogonally invariant.

Lemma 3.1 *Let X be a feasible point of (5) and P be an orthogonal matrix in $\mathbb{R}^{n\times n}$ ($P^TP = I$). Then X is constraint nondegenerate for (5) if and only if $\widetilde{X} := P^TXP$ is constraint nondegenerate for the following constraint:*

$$\left\{X \in \mathcal{S}^n_+,\ \langle P^TA_iP,\ X\rangle = b_i,\ i = 1, \dots, m\right\}. \tag{6}$$

Proof. First we note that $\widetilde{X} = P^TXP$ is a feasible point of (6). Suppose X has the spectral decomposition (4). Then it is known [2] that

$$\mathrm{lin}(T_{\mathcal{S}^n_+}(X)) = \left\{ Q \begin{bmatrix} U & V \\ V^T & 0 \end{bmatrix} Q^T \;\middle|\; \begin{array}{l} U \in \mathbb{R}^{r\times r} \\ V \in \mathbb{R}^{r\times(n-r)} \end{array} \right\}.$$

Similarly, we have

$$\mathrm{lin}(T_{\mathcal{S}^n_+}(\widetilde{X})) = \left\{ P^TQ \begin{bmatrix} U & V \\ V^T & 0 \end{bmatrix} (P^TQ)^T \;\middle|\; \begin{array}{l} U \in \mathbb{R}^{r\times r} \\ V \in \mathbb{R}^{r\times(n-r)} \end{array} \right\} = P^T\Big(\mathrm{lin}(T_{\mathcal{S}^n_+}(X))\Big)P.$$

Let $\mathcal{A}$ be defined as in (2). Define $\widetilde{\mathcal{A}} : \mathcal{S}^n \mapsto \mathbb{R}^m$ by

$$\widetilde{\mathcal{A}}(X) := \begin{bmatrix} \langle P^TA_1P,\ X\rangle \\ \vdots \\ \langle P^TA_mP,\ X\rangle \end{bmatrix}.$$

We have the following chain of equivalences.

$$\begin{aligned} & \mathcal{A}\Big(\mathrm{lin}(T_{\mathcal{S}^n_+}(X))\Big) = \mathbb{R}^m \\ \Longleftrightarrow\quad & \widetilde{\mathcal{A}}\Big(P^T\Big(\mathrm{lin}(T_{\mathcal{S}^n_+}(X))\Big)P\Big) = \mathbb{R}^m \\ \Longleftrightarrow\quad & \widetilde{\mathcal{A}}\Big(\mathrm{lin}(T_{\mathcal{S}^n_+}(\widetilde{X}))\Big) = \mathbb{R}^m. \end{aligned}$$

The claim in the lemma follows Definition 2.1. □

Let $G = (V, E)$ be a given graph with $V = \{1, 2, \dots, n\}$. An *ordering* of G is a bijection $\rho : V \mapsto \{1, \dots, n\}$. A bijection ρ is called a *perfect elimination ordering* if for every $v \in V$, the set

$$\left\{ x \in V \;\middle|\; (v, x) \in E \ \text{ and } \ \rho(v) < \rho(x) \right\}$$

is a clique of G. We will need the following important result.

Lemma 3.2 *[23] G has a perfect elimination ordering if and only if G is chordal.*

Now assume that G is chordal and $A(G)$ is a G-partial positive definite matrix. By Lemma 3.2, G has a perfect elimination ordering ρ. Re-order the vertices of G in the following way:

$$v_k := \rho^{-1}(k), \quad k = 1, \ldots, n. \tag{7}$$

That is, V has a new ordering by $v_1, \ldots, v_n$. For each v_k, define the index sets

$$\begin{aligned} \mathcal{J}_k &:= \left\{ v \in \{v_{k+1}, \ldots, v_n\} \,\middle|\, (v_k, v) \in E \right\} && (8) \\ \mathcal{I}_k &:= \{v_k\} \cup \mathcal{J}_k. && (9) \end{aligned}$$

Then both $\mathcal{J}_k$ and $\mathcal{I}_k$ are cliques of G due to ρ being a perfect elimination ordering. Consequently, the principal submatrix $(A(G))_{\mathcal{I}_k\mathcal{I}_k}$ is positive definite.

Associated with ρ, there exists an $n \times n$ permutation matrix P such that

$$P^T \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \tag{10}$$

Define

$$\widetilde{A}(G) := P^T A(G) P. \tag{11}$$

Then $\widetilde{A}(G)$ is a symmetric permutation of $A(G)$ (i.e., an orthogonal congruence). Corresponding to the new ordering of vertices $\{v_1, \ldots, v_n\}$, every (existing) principal submatrix of $\widetilde{A}(G)$ is also positive definite.

Define

$$\begin{aligned} \mathcal{C}_{A(G)} &:= \left\{ X \in \mathcal{S}^n \,\middle|\, X \text{ is a positive semidefinite completion of } A(G) \right\} \\ &= \left\{ X \in \mathcal{S}^n_+ \,\middle|\, \begin{array}{l} X_{ij} = A_{ij}, \ (i,j) \in E \\ X_{ii} = A_{ii}, \ i = 1, \ldots, n \end{array} \right\} \\ &= \left\{ X \succeq 0 \,\middle|\, \begin{array}{l} \langle E^{ij}, \ X \rangle = 2A_{ij}, \ (i,j) \in E \\ \langle E^{ii}, \ X \rangle = A_{ii}, \ i = 1, \ldots, n \end{array} \right\}. \end{aligned} \tag{12}$$

The constant factor 2 before A_{ij} in (12) is due to the definition of E^{ij} and the property of the inner product in $\mathcal{S}^n$. $\mathcal{C}_{A(G)}$ is not empty because $A(G)$ is G-partial positive definite and G is chordal. In fact, $\mathcal{C}_{A(G)}$ contains at least one positive definite matrix. But, in general, it contains more than just positive definite matrices.

Suppose $X \in \mathcal{C}_{A(G)}$, then

$$\widetilde{X} := P^T X P$$

belongs to the set

$$\widetilde{\mathcal{C}}_{A(G)} := \left\{ X \succeq 0 \,\middle|\, \begin{array}{l} \langle P^T E^{ij} P, \ X \rangle = 2A_{ij}, \ (i,j) \in E \\ \langle P^T E^{ii} P, \ X \rangle = A_{ii}, \ i = 1, \ldots, n \end{array} \right\}. \tag{13}$$

That is, $\widetilde{X}$ is a positive semidefinite completion of the partial matrix $\widetilde{A}(G)$ defined in (11).

Now we use the index sets $\mathcal{I}_k$ and $\mathcal{J}_k$ defined in (9) and (8) to get rid of P used in $\widetilde{\mathcal{C}}_{A(G)}$. This will simplify our technical proof below. Noticing that P is the permutation matrix satisfying (10), $\widetilde{\mathcal{C}}_{A(G)}$ has the following equivalent description.

$$\begin{aligned}\widetilde{\mathcal{C}}_{A(G)} &:= \left\{ X \in \mathcal{S}^n_+ \;\middle|\; \begin{array}{l} X_{ij} = A_{v_iv_j},\ i = 1,\ldots,n-1,\ v_j \in \mathcal{J}_i \\ X_{ii} = A_{v_iv_i},\ i = 1,\ldots,n \end{array} \right\} \\ &:= \left\{ X \succeq 0 \;\middle|\; \begin{array}{l} \langle E^{ij},\ X\rangle = 2A_{v_iv_j}, i = 1,\ldots,n-1, \quad v_j \in \mathcal{J}_i \\ \langle E^{ii},\ X\rangle = A_{v_iv_i},\ i = 1,\ldots,n \end{array} \right\}. \end{aligned} \tag{14}$$

In other words, $\widetilde{A}(G)$ is a re-ordering of $A(G)$ according to the new ordering of vertices $\{v_1,\ldots,v_n\}$. Any $X \in \widetilde{\mathcal{C}}_{A(G)}$ is a positive semidefinite completion of $\widetilde{A}(G)$.

We have the following result, whose proof is a bit technical.

Lemma 3.3 *Suppose that the graph $G = (V,E)$ is chordal with $V = \{1,\ldots,n\}$ and $A(G)$ is G-partial positive definite. Suppose ρ is a (fixed) perfect elimination ordering. Let P be the permutation matrix satisfying (10). Let $\widetilde{\mathcal{C}}_{A(G)}$ be defined by (13). Then constraint nondegeneracy is satisfied at any $X \in \widetilde{\mathcal{C}}_{A(G)}$.*

Proof. We emphasize once again that by saying constraint nondegeneracy is satisfied at $X \in \widetilde{\mathcal{C}}_{A(G)}$ we mean that X is constraint nondegenerate with respect to the constraints defining $\widetilde{\mathcal{C}}_{A(G)}$.

As mentioned early on, $\widetilde{\mathcal{C}}_{A(G)} \neq \emptyset$. Let $X \in \widetilde{\mathcal{C}}_{A(G)}$ and let it have the spectral decomposition (4). Then elements in X can be represented in terms of the eigenvalues $\lambda_1,\ldots,\lambda_r$ and the eigenvectors in Q. In fact, we have

$$X = \Big(X_{ij} = \sum_{\ell=1}^{r} (\lambda_\ell Q_{i\ell} Q_{j\ell}) \Big).$$

In particular, we have

$$X_{ij} = \sum_{\ell=1}^{r} (\lambda_\ell Q_{i\ell} Q_{j\ell}) = A_{v_iv_j},\ i = 1,\ldots,n-1,\ j \in \mathcal{J}_i, \tag{15}$$

and

$$X_{ii} = \sum_{\ell=1}^{r} (\lambda_\ell Q^2_{i\ell}) = A_{v_iv_i},\ i = 1,\ldots,n. \tag{16}$$

To prove that X is nondegenerate, it suffices to prove by applying Lemma 2.2 to the constraints in (14) that the matrices

$$B^i := \left[\begin{array}{cc} Q_1^T E^{ii} Q_1 & Q_1^T E^{ii} Q_2 \\ Q_2^T E^{ii} Q_1 & 0 \end{array} \right],\ i = 1,\ldots,n;\ B^{ij} := \left[\begin{array}{cc} Q_1^T E^{ij} Q_1 & Q_1^T E^{ij} Q_2 \\ Q_2^T E^{ij} Q_1 & 0 \end{array} \right],\ \begin{array}{l} i = 1,\ldots,n-1 \\ v_j \in \mathcal{J}_i \end{array}$$

are linearly independent in $\mathcal{S}^n$. Suppose there exist $\{z_{ii}\}$ with $i = 1,\ldots,n$ and $\{z_{ij}\}$ with $i = 1,\ldots,n-1$ and $v_j \in \mathcal{J}_i$ such that

$$\sum_{i=1}^{n} z_{ii} B^i + \sum_{i=1}^{n} \sum_{v_j \in \mathcal{J}_i} z_{ij} B^{ij} = 0. \tag{17}$$

Our purpose is to show that $z_{ij} = 0$ for $i = 1, \ldots, n$ and $z_{ij} = 0$ for $v_j \in \mathcal{J}_i$ and $i = 1, \ldots, n-1$.

It follows from (17) that we have

$$\begin{aligned} 0 &= Q_1^T \left(\sum_{i=1}^n z_{ii} E^{ii} + \sum_{i=1}^n \sum_{v_j \in \mathcal{J}_i} z_{ij} E^{ij} \right) [Q_1,\ Q_2] \\ &= Q_1^T \left(\sum_{i=1}^n z_{ii} E^{ii} + \sum_{i=1}^n \sum_{v_j \in \mathcal{J}_i} z_{ij} E^{ij} \right) Q, \end{aligned}$$

which, by the nonsingularity of Q, implies that

$$Q_1^T \left(\sum_{i=1}^n z_{ii} E^{ii} + \sum_{i=1}^n \sum_{v_j \in \mathcal{J}_i} z_{ij} E^{ij} \right) = 0. \tag{18}$$

Now define a matrix $Z \in \mathcal{S}^n$ in the following way. We first define its upper triangular part; the lower triangular part is symmetric to the upper part. For $j \geq i$, $i = 1, \ldots, n$, define

$$Z_{ij} := \begin{cases} z_{ij} & \text{if } v_j \in \mathcal{I}_i, \\ 0 & \text{otherwise.} \end{cases}$$

For $j < i$, $i = 2, \ldots, n$, define

$$Z_{ji} := Z_{ij}.$$

Then the equation (18) is equivalent to

$$Q_1^T Z = 0. \tag{19}$$

It is sufficient to prove that $Z = 0$. Our first step toward this is to prove that the first column of Z is zero, followed by the second column being proved to be zero. The process goes on until all the columns are proved to be zero.

Step 1. Let us calculate the first column of $(Q_1^T Z)$ (recall $Q_1 \in \mathbb{R}^{n \times r}$). For $i = 1, \ldots, r$,

$$\begin{aligned} 0 &= (Q_1^T Z)_{i1} = \sum_{\ell=1}^n Q_{\ell i} Z_{\ell 1} \\ &= \sum_{v_\ell \in \mathcal{I}_1} Q_{\ell i} Z_{\ell 1} \quad (\text{because } Z_{\ell 1} = 0 \text{ for } v_\ell \notin \mathcal{I}_1). \end{aligned}$$

Therefore,

$$\lambda_i \Big((Q_1^T Z)_{i1}^2 \Big) = \sum_{v_\ell \in \mathcal{I}_1} \Big[\lambda_i Q_{\ell i}^2 Z_{\ell 1}^2 \Big] + 2 \sum_{v_\ell \in \mathcal{I}_1} \sum_{\substack{v_t \in \mathcal{I}_1 \\ t > \ell}} \Big[\lambda_i Q_{ti} Q_{\ell i} Z_{t1} Z_{\ell 1} \Big].$$

Summarizing over the index $i = 1, \ldots, r$ gives

$$
\begin{aligned}
0 &= \sum_{i=1}^{r} \Big[\lambda_i \Big((Q_1^T Z)_{i1}^2\Big)\Big] \\
&= \sum_{v_\ell \in \mathcal{I}_1} \left[\Big(\sum_{i=1}^{r} \lambda_i Q_{\ell i}^2 \Big) Z_{\ell 1}^2 \right] + 2 \sum_{v_\ell \in \mathcal{I}_1} \sum_{\substack{v_t \in \mathcal{I}_1 \\ t > \ell}} \left[\Big(\sum_{i=1}^{r} \lambda_i Q_{ti} Q_{\ell i} \Big) Z_{t1} Z_{\ell 1} \right] \\
&= \sum_{v_\ell \in \mathcal{I}_1} \left[A_{v_\ell v_\ell} Z_{\ell 1}^2 \right] + 2 \sum_{v_\ell \in \mathcal{I}_1} \sum_{\substack{v_t \in \mathcal{I}_1 \\ t > \ell}} \left[A_{v_\ell v_t} Z_{t1} Z_{\ell 1} \right] \qquad \text{(by (15) and (16))} \\
&= Z_{\mathcal{I}_1 1}^T A_{\mathcal{I}_1 \mathcal{I}_1} Z_{\mathcal{I}_1 1}, \qquad (20)
\end{aligned}
$$

where $Z_{\mathcal{I}_1 1}$ is the column vector defined by $[Z_{\ell 1} : v_\ell \in \mathcal{I}_1]$. Now recall that $A(G)$ is G-partial positive definite and $\mathcal{I}_1$ is a clique of G. The principal submatrix $A_{\mathcal{I}_1 \mathcal{I}_1}$ is positive definite. Then (20) forces $Z_{\mathcal{I}_1 1} = 0$, which means $Z_{\ell 1} = 0$ for all $v_\ell \in \mathcal{I}_1$. By the definition of Z, we proved that the first column (and hence the first row) of Z is zero.

Step 2. Now we prove the second column of Z to be zero. At this moment, it is very useful to bear in mind that the first column of Z has been proved to be zero. Calculate the second column of $(Q_1 Z)$: for $i = 1, \ldots, r$,

$$
\begin{aligned}
0 &= (Q_1^T Z)_{i2} = \sum_{\ell=1}^{n} Q_{\ell i} Z_{\ell 2} \\
&= \sum_{v_\ell \in \mathcal{I}_2} Q_{\ell i} Z_{\ell 2} \quad \text{(because } Z_{\ell 2} = 0 \text{ for } v_\ell \notin \mathcal{I}_2).
\end{aligned}
$$

We note that the fact that $Z_{\ell 2} = 0$ for $v_\ell \notin \mathcal{I}_2$ comes from the definition of Z as well as the fact that the first column of Z has already been proved to be zero.

Repeating the proof in Step 1 yields that

$$
0 = Z_{\mathcal{I}_2 2}^T A_{\mathcal{I}_2 \mathcal{I}_2} Z_{\mathcal{I}_2 2} \qquad Z_{\mathcal{I}_2 2} := [Z_{\ell 2} : v_\ell \in \mathcal{I}_2].
$$

Once again, the fact that $\mathcal{I}_2$ is a clique of G means that the principal submatrix $A_{\mathcal{I}_2 \mathcal{I}_2}$ is positive definite. This further implies $Z_{\mathcal{I}_2 2} = 0$, i.e., $Z_{\ell 2} = 0$ for $v_\ell \in \mathcal{I}_2$. The combination of this result, the definition of Z and the fact that the first column of Z has already been proved to be zero implies that the second column of Z is zero.

By repeating the above proof process, we can prove that $Z = 0$. This finishes the whole proof. □

We note that if X used in the proof has a full rank, i.e., $r = n$. Then $Q_1 = Q$ and the linear equation (19) automatically implies $Z = 0$. Hence, the above proof is mainly for those X with rank-deficiency (i.e., $r < n$). Now we are ready to present our first main result, which says that constraint nondegeneracy is a necessary condition for the completability.

Theorem 3.4 *Suppose $G = (V, E)$ is a chordal graph and $A(G)$ is G-partial positive definite. Then constraint nondegeneracy holds at any $X \in \mathcal{C}_{A(G)}$ defined in (12).*

Proof. Let ρ be a fixed perfect elimination ordering and the vertices of V are re-ordered by (7). Let P be the permutation matrix satisfying (10). Then $X \in \mathcal{C}_{A(G)}$ if and only if $P^T XP \in \widetilde{\mathcal{C}}_{A(G)}$. It is proved in Lemma 3.3 that any matrix in $\widetilde{\mathcal{C}}_{A(G)}$ is constraint nondegenerate. The result follows from Lemma 3.1. □

Now we address the possibility that constraint nondegeneracy may constitute a sufficient condition for G being chordal. It is known that if G is not chordal, $A(G)$ may not have a positive semidefinite completion even it is G-partial positive definite. Hence, it is sensible from now on to only consider those $A(G)$ which do have a positive semidefinite completion.

Consider the following simple example

$$G = (V, E) \text{ with } V = \{1, 2, 3, 4\},\ E = \{(1, 2), (2, 3), (3, 4), (4, 1)\}, \tag{21}$$

and

$$A(G) = \begin{bmatrix} 1 & 0 & ? & 0 \\ 0 & 1 & 0 & ? \\ ? & 0 & 1 & 0 \\ 0 & ? & 0 & 1 \end{bmatrix}. \tag{22}$$

Apparently, $A(G)$ is G-partial positive definite. $\mathcal{C}_{A(G)}$ defined in (12) becomes

$$\mathcal{C}_{A(G)} = \left\{ X \in \mathcal{S}^4_+,\ \begin{matrix} X_{ii} = 1, & i = 1, 2, 3, 4 \\ X_{12} = 0, & X_{23} = 0 \\ X_{34} = 0, & X_{41} = 0 \end{matrix} \right\}. \tag{23}$$

We have the following result.

Lemma 3.5 *Consider the graph and $A(G)$ given by (21) and (22) respectively. Let*

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then $X \in \mathcal{C}_{A(G)}$. Furthermore, X is degenerate with respect to the constraints in (23).

Proof. It is easy to verify that $X \in \mathcal{C}_{A(G)}$. Moreover, X has the spectral decomposition (4) with

$$Q = \begin{bmatrix} 0 & -\sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \end{bmatrix}, \quad \text{and } \lambda_1 = \lambda_2 = 2, \quad \lambda_3 = \lambda_4 = 0,$$

where $\lambda_i, i = 1, \ldots, 4$ are eigenvalues of X. We further have

$$Q_1 = \begin{bmatrix} 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 0 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{bmatrix}.$$

To prove X is degenerate, by Lemma 2.2 we need to prove that the matrices

$$B^i = \begin{bmatrix} Q_1^T E^{ii} Q_1 & Q_1^T E^{ii} Q_2 \\ Q_2^T E^{ii} Q_1 & 0 \end{bmatrix}, \ i = 1, \ldots, 4;$$

and

$$B^{ij} = \begin{bmatrix} Q_1^T E^{ij} Q_1 & Q_1^T E^{ij} Q_2 \\ Q_2^T E^{ij} Q_1 & 0 \end{bmatrix}, \quad (i,j) \in \left\{ \begin{array}{c} (1,2),(2,3) \\ (3,4),(4,1) \end{array} \right\}$$

are linearly dependent in $\mathcal{S}^4$, where $E^{ij} \in \mathcal{S}^4$ are defined as before. It is equivalent to prove for the following linear system (see (19))

$$Q_1^T Z = 0, \quad Z = \begin{bmatrix} z_{11} & z_{12} & 0 & z_{13} \\ z_{12} & z_{22} & z_{33} & 0 \\ 0 & z_{23} & z_{33} & z_{34} \\ z_{13} & 0 & z_{34} & z_{44} \end{bmatrix}$$

to have a solution $Z \neq 0$. It is easy to verify that

$$Z = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

is such a solution. Therefore, X is degenerate with respect to the constraints in (23). □

The following result formalizes constraint nondegeneracy as a sufficient condition for a graph to be chordal.

Theorem 3.6 *Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$. Suppose for any G-partial positive definite matrix $A(G)$ that has a positive semidefinite completion (i.e., consider those G-partial positive definite $A(G)$ with $\mathcal{C}_{A(G)} \neq \emptyset$), we have that constraint nondegeneracy holds at any $X \in \mathcal{C}_{A(G)}$. Then G is chordal.*

Proof. We assume that G has a minimal cycle γ of length ≥ 4. We will get a contradiction. We may assume without loss of generality that $\gamma = (1, 2, \ldots, k)$. Define a G-partial positive definite matrix $A(G)$ by

$$A_{ii} = 1, \ i = 1, \ldots, n; \quad A_{ij} = 0, \quad \forall\, (i,j) \in E.$$

In particular, we have $A_{1k} = A_{k1} = 0$. Therefore,

$$\mathcal{C}_{A(G)} = \left\{ X \succeq 0, \quad \begin{array}{ll} X_{ii} = 1, & \text{for } i = 1, \ldots, n \\ X_{ij} = 0, & \text{for } (i,j) \in E \end{array} \right\}.$$

Define the matrix $X \in \mathcal{S}^n$ by

$$\left\{ \begin{array}{ll} X_{ii} = 1, & \text{for } i = 1, \ldots, n \\ X_{ij} = 0, & \text{for } (i,j) \in E \\ X_{1(k-1)} = 1, \ X_{2k} = 1 & \\ X_{ij} = 0, & \text{otherwise.} \end{array} \right.$$

That is,

$$X = \begin{bmatrix} X_{KK} & 0 \\ 0 & I \end{bmatrix} \quad \text{with} \quad X_{KK} = \begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 1 \end{bmatrix} \in \mathcal{S}^{k\times k}$$

and I is the identity matrix in $\mathcal{S}^{n-k}$. Recall $k \geq 4$. Swap the $(k-1)^{\text{th}}$-column and k^{th}-column of X with the 3^{rd}-column and 4^{th}-column of X respectively, the resulting matrix can be written as

$$\widetilde{X} = P^T XP = \begin{bmatrix} \widetilde{X}_0 & 0 \\ 0 & I \end{bmatrix} \quad \text{with} \quad \widetilde{X}_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad I \in \mathcal{S}^{n-4},$$

where P is the permutation matrix satisfying

$$P^T \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ k-1 \\ k \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ k-1 \\ k \\ \vdots \\ 3 \\ 4 \\ \vdots \\ n \end{bmatrix} =: \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{k-1} \\ v_k \\ \vdots \\ v_n \end{bmatrix}.$$

Apparently, $\widetilde{X}$ is positive semidefinite, so is X, implying $X \in \mathcal{C}_{A(G)}$. As a fact that we used again and again, $\widetilde{X} \in \widetilde{\mathcal{C}}_{A(G)}$, which is given by

$$\begin{aligned} \widetilde{X} \in \widetilde{\mathcal{C}}_{A(G)} = P^T \mathcal{C}_{A(G)} P &= \left\{ X \succeq 0, \quad \begin{array}{l} \langle P^T E^{ii} P,\ X\rangle = 1, \ \text{for } i = 1, \ldots, n \\ \langle P^T E^{ij} P,\ X\rangle = 0, \ \text{for } (i,j) \in E \end{array} \right\} \\ &= \left\{ X \succeq 0, \quad \begin{array}{l} \langle E^{ii},\ X\rangle = 1, \ \text{for } i = 1, \ldots, n \\ \langle E^{ij},\ X\rangle = 0, \ \text{for } (v_i, v_j) \in E \end{array} \right\}. \end{aligned}$$

We now proceed to prove that $\widetilde{X} \in \widetilde{\mathcal{C}}_{A(G)}$ is degenerate. Due to the structure of $\widetilde{X}$ and the constraints in $\widetilde{\mathcal{C}}_{A(G)}$, it reduces to prove that $\widetilde{X}_0$ is degenerate with respect to the constraints

$$\left\{ X \in \mathcal{S}^4, \quad \begin{array}{l} X_{ii} = 1, \ i = 1, 2, 3, 4 \\ X_{12} = X_{23} = X_{34} = X_{41} = 0 \end{array} \right\}.$$

This has been proved in Lemma 3.5. Therefore, $\widetilde{X}$ is degenerate with respect to the constraint in $\widetilde{\mathcal{C}}_{A(G)}$. Apply Lemma 3.1 once again, we see that $X \in \mathcal{C}_{A(G)}$ is degenerate. This contradicts the assumption we made in the theorem and hence establishes that G is chordal. □

Putting the results in Theorems 3.4 and 3.6 and the main results [11, Prop. 2, Thm. 7] together we have the following characterization.

Theorem 3.7 *Suppose $G = (V, E)$ is a simple graph. Then the following are equivalent.*

(i) G is chordal.

(ii) [11, Thm. 7] Every G-partial positive definite matrix has a positive definite completion (completability).

(iii) [11, Prop. 2] Every G-partial positive semidefinite matrix has a positive semidefinite completion (positive semidefinite completability).

(iv) For any G-partial positive definite matrix that has a positive semidefinite completion, constraint nondegeneracy holds at each of its positive semidefinite completions.

One may ask whether the condition (iv) in Theorem 3.7 can be extended to G-partial positive semidefinite matrices. The following is a counterexample.

Example 3.8 *Consider the chordal graph $G = (V, E)$ given by*

$$V = \{1, 2, 3, 4\} \quad and \quad V = \{(1, 2), (2, 3), (3, 4)\}.$$

Define

$$A(G) = \begin{bmatrix} 1 & 1 & ? & ? \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ ? & ? & 1 & 1 \end{bmatrix} \qquad and \qquad X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Apparently, $A(G)$ is G-partial positive semidefinite and X is a positive semidefinite completion of $A(G)$. That is, $X \in \mathcal{C}_{A(G)}$. Furthermore, X has the spectral decomposition (4) with

$$Q_1^T = \frac{1}{2}[1, 1, 1, 1].$$

It is easy to verify by using Lemma 2.2 that X is degenerate even though G is chordal.

The above example shows that the G-partial positive definiteness is crucial in the condition (iv) of Theorem 3.7. In other words, that whether or not a graph G is completable depends only on such G-partial positive definite matrices that have positive semidefinite completions, rather than on the bigger set of G-partial positive semidefinite matrices.

4 Stability Implications

The relationship between the completability (i.e., the chordal graph) and constraint nondegeneracy established in the proceeding section has a number of important implications. It not only allows us to have a fresh look at matrix completion problems from the viewpoint of SDP, but also enables us to have a better understanding of some quadratic SDP whose constraints can be reformulated as positive semidefinite completions of some graphs.

This short section tries to point out certain stability implications of some optimization problems. We also point out that the often assumed *Slater* condition in those problems is actually the completability condition of some graphs.

Let us consider a general problem called *the nearest positive semidefinite completion* problem. For a given chordal graph $G = (V, E)$ and a G-partial positive definite matrix $A(G)$, let $\mathcal{C}_{A(G)}$ be the set of all positive semidefinite completions of $A(G)$. The nearest positive semidefinite completion problem can be phrased as follows:

$$\begin{array}{ll} \min & \frac{1}{2}\|X - C\|^2 \\ \text{s.t.} & X \in \mathcal{C}_{A(G)}, \end{array} \tag{24}$$

where $C \in \mathcal{S}^n$ is known and is supposedly not in $\mathcal{C}_{A(G)}$. Problem (24) serves as a general model for some existing problems.

We note that the objective function in (24) is quadratic and is strongly convex. As a consequence, (24) has a unique solution, denoted X^*. We may ask whether or not this solution is stable under data perturbation. By Theorem 3.7, X^* is constraint nondegenerate. The *strong second-order sufficient condition* (SSOSC) studied by Sun [24] is automatically satisfied at X^* because the problem is strongly convex. In the theory of nonlinear SDP, it is known [24, Thm. 4.1] that constraint nondegeneracy and the SSOSC together are equivalent to the strong regularity of the problem in the sense of Robinson [21]. A string of equivalent statements can be made about the stability of (24) and we refer to [24, Thm. 4.1] and [19, Thm. 3.5] for those statements.

It is interesting to see that problem (24) serves as a general model for a number of important problems. We present two of them here.

Example 4.1 *(The nearest correlation matrix problem [13]) The problem is to find the nearest correlation matrix to a given matrix $C \in \mathcal{S}^n$:*

$$\begin{array}{ll} \min & \frac{1}{2}\|X - C\|^2 \\ s.t. & X_{ii} = 1, \ i = 1, \ldots, n \\ & X \succeq 0. \end{array} \tag{25}$$

This corresponds to (24) with $G = (V, E)$ given by

$$V = \{1, 2, \ldots, n\} \quad \text{and} \quad E = \emptyset$$

and the G-partial positive definite matrix $A(G)$ given by

$$A_{ii} = 1, \quad i = 1, 2, \ldots, n.$$

Apparently, G is chordal (no cycle exists in G) and $A(G)$ is G-partial positive definite. Therefore, both constraint nondegeneracy and the SSOSC are satisfied at the optimal solution; so is the dual problem of (25), recovering [27, Prop. 4.2] and [19, Cor. 3.2].

Example 4.2 *(The local correlation stress testing problem [25]) For a given correlation matrix $C \in \mathcal{S}^n$, which has the following structure*

$$C = \begin{bmatrix} C_1 & C_2 \\ C_2^T & C_3 \end{bmatrix}, \quad C_1 \in \mathcal{S}^{m \times n}, \ C_2 \in \mathbb{R}^{m \times (n-m)} \quad \text{and} \quad C_3 \in \mathcal{S}^{(n-m) \times (n-m)},$$

the first step in the local correlation stress testing is to keep the first m rows of C unchanged, but stress testing correlations in C_3 to $\widehat{C}_3$. This gives a new matrix containing the stressed correlations, denoted by $\widehat{C}$:

$$\widehat{C} := \begin{bmatrix} C_1 & C_2 \\ C_2^T & \widehat{C}_3 \end{bmatrix}, \ C_3 \in \mathcal{S}^{(n-m) \times (n-m)}.$$

This new matrix $\widehat{C}$ is not necessarily a correlation matrix any more.

The second step in the local correlation matrix is to find the nearest correlation matrix to $\widehat{C}$. This is to solve the following problem:

$$\begin{array}{ll} \min & \frac{1}{2}\|X - \widehat{C}\|^2 \\ s.t. & X_{ij} = C_{ij},\ i = 1, \ldots, m,\ j = 1, \ldots, n \\ & X_{ii} = 1,\ i = m+1, \ldots, n \\ & X \succeq 0. \end{array} \tag{26}$$

This problem corresponds to (24) with $G = (V, E)$ given by

$$V = \{1, \ldots, n\}, \quad E = \Big\{(i, j) \ \Big| \ i = 1, \ldots, m,\ j = i+1, \ldots, n\Big\},$$

and the G-partial matrix $A(G)$ is given by

$$A_{ij} = \begin{cases} C_{ij}, & \text{for } i = 1, \ldots, m,\ j = i+1, \ldots, n \\ 1, & \text{for } i = j = m+1, \ldots, n. \end{cases}$$

Apparently, G is chordal. Note that C is a correlation matrix, which may be rank-deficient. Suppose that $A(G)$ is G-partial positive definite, then constraint nondegeneracy is satisfied at the optimal solution by Theorem 3.7 because G is chordal. This is a result not known before. We note that the Slater condition for this example is equivalent to assuming $A(G)$ being G-partial positive definite as G is chordal. Assuming C being of full rank is stronger than the Slater condition.

5 Conclusion

In this paper, we established the equivalence between the completability of a graph and constraint nondegeneracy widely used in SDP. This new result has some interesting implications. We use the nearest positive semidefinite completion problem to demonstrate its stability implications. A lot can also be said of its numerical implications. For example, Newton's method [20] and the interior-point method [27, 26] can be studied for the problem, especially on the convergence rate and the well-conditionedness of the matrices encountered in the methods. It is also interesting to study whether constraint nondegeneracy plays any roles in some of the matrix nearness problems recently studied in [8].

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