

New Sufficient Conditions for Global Robust Stability of Delayed Neural Networks

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Abstract—In this paper, we continue to explore application of nonsmooth analysis to the study of global asymptotic robust stability (GARS) of delayed neural networks. In combination with Lyapunov theory, our approach gives several new types of sufficient conditions ensuring GARS. A significant common aspect of our results is their low computational complexity. It is demonstrated that the reported results can be verified either by conducting spectral decompositions of symmetric matrices associated with the uncertainty sets of network parameters, or by solving a Semidefinite Programming problem (SDP). Nontrivial examples are constructed to compare with some closely related existing results.

Index Terms—Delayed neural networks, equilibrium point, global asymptotic robust stability, nonsingularity, Lyapunov function.

I. INTRODUCTION

THE level of reliability of delayed neural networks depends on the global uniqueness of an equilibrium point as well as its global asymptotic stability, known as the GAS property. There is a large body of publications of addressing stability properties of different classes of neural networks with delay, to just name a few, see [1], [2], [3], [9], [10], [11], [13], [14], [16], [18], [21], [23], [24], [25], [26], which also aspire our investigation in this paper.

The delayed neural networks considered in this paper can be modelled by the differential equation:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau)) + u, \quad (1)$$

where $x(t) = (x_1(t), \dots, x_n(t))^T \in \mathbb{R}^n$, $C = \text{diag}(c_i > 0)_{n \times n}$ is a positive diagonal matrix, $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$, $u = (u_1, \dots, u_n)^T \in \mathbb{R}^n$, $f(x(t)) = (f_1(x_1(t)), \dots, f_n(x_n(t)))^T \in \mathbb{R}^n$ and $f(x(t - \tau)) = (f_1(x_1(t - \tau_1)), \dots, f_n(x_n(t - \tau_n)))^T \in \mathbb{R}^n$. The matrices C, A, B are often referred to as the network parameters, τ_i is the delay associated with the i th neuron, f_i is an activation function and u_i is a constant input to the neuron i .

There are statistically justified reasons [4], [29] (due to, for example, physical implementation of the networks and/or experimental errors) that the network parameters often appear in random nature. In other words, (C, A, B) may be subject to random errors. For example, A may take the following form:

$$A = E(A) \pm \Delta A,$$

where $E(A)$ denotes the expectation of random variable A and ΔA is the possible error. To put into a different form, the network parameters (C, A, B) are often contained in the following sets:

$$\mathcal{C} = \{C = \text{diag}(c_i) \mid 0 < \underline{C} \leq C \leq \bar{C}, \text{ i.e., } 0 < \underline{c}_i \leq c_i \leq \bar{c}_i\}$$

$$\mathcal{A} = \{A = (a_{ij}) \mid \underline{A} \leq A \leq \bar{A}, \text{ i.e., } \underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij}\}$$

$$\mathcal{B} = \{B = (b_{ij}) \mid \underline{B} \leq B \leq \bar{B}, \text{ i.e., } \underline{b}_{ij} \leq b_{ij} \leq \bar{b}_{ij}\}$$

for all $i, j = 1, \dots, n$, where the matrices $\underline{C}, \bar{C}, \underline{A}, \bar{A}$ and \underline{B}, \bar{B} are known. Those are the sets that contain uncertainty (hence, often referred to as uncertainty sets). This results in neural networks with uncertainty sets \mathcal{C}, \mathcal{A} and \mathcal{B} :

$$\begin{cases} \dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau)) + u \\ (C, A, B) \in \mathcal{C} \times \mathcal{A} \times \mathcal{B} \text{ and } u \in \mathbb{R}^n \end{cases} \quad (2)$$

The robust stability (in the sense of GAS) of this system has recently been studied in [7], [8], [19], [20], [22], [27]. In this paper, we will report new sufficient conditions for the global asymptotic robust stability (GARS) of model (2). We also develop a semidefinite programming problem (SDP) to verify one of our main results, while the others can be verified by spectral decompositions of a small number of symmetric matrices. SDP verification is not shared by many of existing similar results. Another purpose of this paper is to promote application of nonsmooth analysis in the stability study of neural networks. Two basic lemmas that we used are derived via nonsmooth analysis. We refer the reader to [23] for much detailed discussion on them. We organize our paper as follows.

In Section II, we give a formal definition of GARS and include some technical lemmas, which are frequently used in Section III to derive several sufficient conditions of new type for GARS. In Section IV, we conduct comparison with several closely related known results. A significant common aspect of our new conditions is their low computational complexity. We demonstrate this low cost complexity in Section V that our new conditions can be verified either by conducting spectral decompositions of certain symmetric matrices associated with the uncertainty sets \mathcal{A} and \mathcal{B} , or by solving a semidefinite programming problem. We conclude the paper in Section VI.

Notation: We assume that all the activation functions belong to \mathcal{K} , which denotes the class of nondecreasing Lipschitz functions, i.e., $f \in \mathcal{K}$ if there exist some positive constants k_i such that

$$0 \leq \frac{f_i(x) - f_i(y)}{x - y} \leq k_i, \quad \forall x \neq y \in \mathbb{R}, \quad i = 1, \dots, n.$$

$:=$ means ‘define’. For a symmetric matrix M , $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the largest eigenvalue and the smallest eigenvalue of M , respectively. For $M \in \mathbb{R}^{n \times n}$ (not

necessarily symmetric), $\mu_2(M) := 0.5\lambda_{\max}(M + M^T)$, i.e., $\mu_2(M)$ is the largest eigenvalue of the symmetric part of M . $M \succ (\prec) 0$ means that M is symmetric and positive (negative) definite. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the Euclidean norm of v and for a matrix M , $\|M\|_2$ denotes the matrix norm induced by the vector norm $\|\cdot\|$. Equivalently, $\|M\|_2 = \sqrt{\lambda_{\max}(MM^T)}$. Furthermore, $|v| := (|v_1|, \dots, |v_n|)^T \in \mathbb{R}^n$. I is the identity matrix of appropriate dimension. Define $K := \text{diag}(k_1, \dots, k_n)$, $\bar{r} := \min(\underline{c}_i/k_i)$ and $\underline{c} := \min(\underline{c}_i)$. Finally, we let $\Lambda := \text{diag}(\underline{c}_i/k_i)$.

II. BASIC DEFINITIONS AND TECHNICAL LEMMAS

We first recall that for any given data C, A, B and u , a state $x^* \in \mathbb{R}^n$ is called an equilibrium point of (1) if it satisfies

$$-Cx^* + (A + B)f(x^*) + u = 0.$$

Central to the study of model (1) is its *global asymptotic stability* (GAS), which is widely studied in literature. It is known that GAS requires both existence and uniqueness of an equilibrium point for any $u \in \mathbb{R}^n$. The following results, adapted to model (1) and proved by Qi and Qi in [23] via nonsmooth analysis, are for this purpose.

Lemma 2.1: [23, Theorem 1] Suppose $f \in \mathcal{K}$ and the network parameters C, A and B are given. Let

$$\mathcal{W} = \left\{ W \in \mathbb{R}^{n \times n} \mid \begin{array}{l} W = C - (A + B)D, \\ D = \text{diag}(d_1, \dots, d_n), \\ 0 \leq d_i \leq k_i, \\ \text{for all } i = 1, \dots, n \end{array} \right\}.$$

If every matrix $W \in \mathcal{W}$ is nonsingular, then model (1) has a unique equilibrium point for any $u \in \mathbb{R}^n$.

Lemma 2.2: [23, Theorem 2 (iii)] Suppose $f \in \mathcal{K}$ and the network parameters C, A and B are given. If

$$\mu_2(A + B) < \min_{1 \leq i \leq n} (c_i/k_i),$$

then model (1) has a unique equilibrium point for any $u \in \mathbb{R}^n$.

When the network parameters contain errors, we need a stronger property of the robust stability to ensure the reliability of the networks.

Definition 2.1: [8] (GARS) Model (2) is *globally asymptotically robust stable* if for any instance $(C, A, B) \in \mathcal{C} \times \mathcal{A} \times \mathcal{B}$, model (1) is globally asymptotically stable.

We also need the following inequalities of norms of matrices concerning the uncertainty sets \mathcal{A} and \mathcal{B} :

Lemma 2.3: [8, Lemma 3] For any $A \in [\underline{A}, \bar{A}]$ and $B \in [\underline{B}, \bar{B}]$, the following inequalities hold:

$$\begin{aligned} \|A\|_2 &\leq \|A^*\|_2 + \|A_*\|_2, \\ \|B\|_2 &\leq \|B^*\|_2 + \|B_*\|_2, \\ \|\Delta A\|_2 &\leq \|A_*\|_2, \quad \|\Delta B\|_2 \leq \|B_*\|_2, \end{aligned}$$

where $\Delta A = A - A_*$, $\Delta B = B - B_*$, $A^* = \frac{1}{2}(\underline{A} + \bar{A})$, $A_* = \frac{1}{2}(\bar{A} - \underline{A})$, $B^* = \frac{1}{2}(\bar{B} + \underline{B})$ and $B_* = \frac{1}{2}(\bar{B} - \underline{B})$.

The boundedness of the sets \mathcal{A} and \mathcal{B} also implies the following

Lemma 2.4: There exist $\bar{\lambda}_A > 0$ and $\bar{\lambda}_B > 0$ such that

$$\lambda_{\max}(A^T A) \leq \bar{\lambda}_A \quad \text{and} \quad \lambda_{\max}(B^T B) \leq \bar{\lambda}_B \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

III. NEW CONDITIONS FOR GARS

As hinted in Section 2, the analysis of GARS is carried out in two steps. For any $(C, A, B) \in \mathcal{C} \times \mathcal{A} \times \mathcal{B}$, we need to address (i) the existence and uniqueness of an equilibrium point, say x^* , of (1) for any $u \in \mathbb{R}^n$; and (ii) that the equilibrium point x^* attracts all the solutions $x(t)$ of (1). We will use Lemmas 2.1 and 2.2 to address (i). To facilitate the proof of (ii), we shift x^* to the origin through the transformation:

$$z(t) = x(t) - x^* \quad \text{and} \quad z(t - \tau) = x(t - \tau) - x^*.$$

Model (1) then can be equivalently written as the following system:

$$\dot{z}(t) = -Cz(t) + A\Phi(z(t)) + B\Phi(z(t - \tau)) \quad (3)$$

where

$$z(\cdot) = (z_1(\cdot), \dots, z_n(\cdot))^T,$$

$$\Phi(z(\cdot)) = (\phi_1(z_1(\cdot)), \dots, \phi_n(z_n(\cdot)))^T,$$

and

$$\phi_i(z_i(\cdot)) = f_i(z_i(\cdot) + x_i^*) - f_i(x_i^*).$$

For $f \in \mathcal{K}$, it is easy to see that

$$\|\Phi(z(\cdot))\|^2 \leq \sum_{i=1}^n k_i z_i(\cdot) \phi_i(z_i(\cdot)) \quad (4)$$

and

$$\phi_i(0) = 0, \quad \forall i = 1, \dots, n. \quad (5)$$

Hence, to show that x^* is GAS for model (1) is equivalently to show that the origin is GAS for model (3).

The proof of our first result uses, apart from Lemma 2.1, a Lyapunov function proposed by Ozcan and Arik [22].

Theorem 3.1: Let $f \in \mathcal{K}$. The neural network model (2) is GARS if there exists a positive diagonal matrix $P = \text{diag}(p_i > 0)$ such that

$$\begin{aligned} \bar{\Omega} &:= 2P\Lambda - (PA^* + (A^*)^T P) \\ &\quad - 2\|P\|_2 (\|B^*\|_2 + \|B_*\|_2 + \|A_*\|_2) I \succ 0, \end{aligned}$$

where $\Lambda := \text{diag}(\underline{c}_i/k_i)$.

Proof. For any $(C, A, B) \in \mathcal{C} \times \mathcal{A} \times \mathcal{B}$, we shall prove that model (1) is globally asymptotically stable. The proof is in two steps.

Step 1 (Existence and uniqueness of an equilibrium point). In this part we prove any matrix $W \in \mathcal{W}$ in Lemma 2.1 is nonsingular so that Lemma 2.1 implies that model (1) has a unique equilibrium point for any $u \in \mathbb{R}^n$. Since $W \in \mathcal{W}$, there must be a diagonal matrix $D = \text{diag}(d_i)$ with $0 \leq d_i \leq k_i$, $i = 1, \dots, n$, such that

$$W = C - (A + B)D.$$

Suppose W is singular, we shall derive a contradiction. Obviously, the matrix

$$PWK^{-1} = PCK^{-1} - P(A + B)DK^{-1}$$

is also singular. Let

$$Q := DK^{-1} = \text{diag}(q_1, \dots, q_n) \quad \text{with } q_i := d_i/k_i.$$

We immediately know that $0 \leq q_i \leq 1$ for all i . Since PWK^{-1} is singular, there must be a vector $0 \neq x \in C^n$ (the field of complex numbers) such that

$$PCK^{-1}x - P(A+B)Qx = 0. \quad (6)$$

First we note that $Qx \neq 0$. Otherwise equation (6) would imply $x = 0$, which contradicts the fact $x \neq 0$.

Multiplying equation (6) from left side by $\bar{x}^T Q$ (here \bar{x} denotes the conjugate vector of x) and substituting A with $A^* + \Delta A$, we have

$$\begin{aligned} 0 &= \bar{x}^T Q P C K^{-1} x - \bar{x}^T Q P (A+B) Q x \\ &\geq \bar{x}^T Q P \Lambda x - \bar{x}^T Q (P A^*) Q x - \bar{x}^T Q P (\Delta A + B) Q x \\ &\geq \bar{x}^T Q P \Lambda Q x - \frac{1}{2} \bar{x}^T Q (P A^* + (A^*)^T P) Q x \\ &\quad - \|P\|_2 (\|\Delta A\|_2 + \|B\|_2) \|Qx\|^2 \\ &\geq (Q\bar{x})^T P \Lambda (Qx) - \frac{1}{2} (Q\bar{x})^T (P A^* + (A^*)^T P) (Qx) \\ &\quad - \|P\|_2 (\|A^*\|_2 + \|B^*\|_2 + \|B_*\|_2) (Q\bar{x})^T I (Qx) \\ &= \frac{1}{2} (Q\bar{x})^T \bar{\Omega} (Qx). \end{aligned}$$

The second inequality above uses the property $0 \leq Q^2 \leq Q$ and P and Λ are positive diagonal matrices, and the third inequality uses Lemma 2.3. Because $\bar{\Omega}$ is positive definite, we must have $Qx = 0$, which contradicts the fact $Qx \neq 0$. This contradiction establishes the nonsingularity of W . Therefore, by Lemma 2.1, model (1) has a unique equilibrium for any given $(C, A, B) \in \mathcal{C} \times \mathcal{A} \times \mathcal{B}$.

Step 2 (Global asymptotic convergence.) In this part, we use the Lyapunov theory to prove the global convergence of any solution of (1) to its unique equilibrium point x^* . This is equivalent to show that the origin is GAS for system (3). The positive definite Lyapunov function we are about to employ is the one proposed by Ozcan and Arik [22] (hence in our proof we will use several technical inequalities, without proofs of them, of concerning the function derived in [22]):

$$\begin{aligned} V(z(t)) &= \|z(t)\|^2 + 2\alpha \sum_{i=1}^n \int_0^{z_i(t)} p_i \phi_i(s) ds \\ &\quad + (\alpha\gamma + \beta) \sum_{i=1}^n \int_{t-\tau_i}^t \phi_i^2(z_i(\xi)) d\xi, \end{aligned}$$

where α, β and γ are some positive constants to be determined later on. The time derivative of $V(z(t))$ along the trajectories of equation (3) is calculated as follows:

$$\begin{aligned} \dot{V}(z(t)) &= -2z^T(t)Cz(t) + 2z^T(t)A\Phi(z(t)) \\ &\quad + 2z^T(t)B\Phi(z(t-\tau)) - 2\alpha\Phi^T(z(t))PCz(t) \\ &\quad + 2\alpha\Phi^T(z(t))PA\Phi(z(t)) \\ &\quad + 2\alpha\Phi^T(z(t))PB\Phi(z(t-\tau)) \\ &\quad + \alpha\gamma (\|\Phi(z(t))\|^2 - \|\Phi(z(t-\tau))\|^2) \\ &\quad + \beta (\|\Phi(z(t))\|^2 - \|\Phi(z(t-\tau))\|^2) \end{aligned} \quad (7)$$

We recall $\underline{c} := \min(c_i)$ and note the following inequalities:

$$-z^T(t)Cz(t) \leq -\underline{c}\|z(t)\|^2. \quad (8)$$

$$\begin{aligned} & -\underline{c}\|z(t)\|^2 + 2z^T(t)A\Phi(z(t)) \\ & \leq (1/\underline{c})\Phi^T(z(t))A^T A\Phi(z(t)) \\ & \leq (1/\underline{c})\lambda_{\max}(A^T A)\|\Phi(z(t))\|^2. \end{aligned} \quad (9)$$

$$\begin{aligned} & -\underline{c}\|z(t)\|^2 + 2z^T(t)B\Phi(z(t-\tau)) \\ & \leq (1/\underline{c})\Phi^T(z(t-\tau))B^T B\Phi(z(t-\tau)) \\ & \leq (1/\underline{c})\lambda_{\max}(B^T B)\|\Phi(z(t-\tau))\|^2. \end{aligned} \quad (10)$$

The following inequality can be proved either independently or by following its counterparts Eq. (17) on [22, Page 169]:

$$\begin{aligned} & 2\alpha\Phi^T(z(t))PB\Phi(z(t-\tau)) \\ & \leq \alpha\|P\|_2\|B\|_2 (\|\Phi(z(t))\|^2 + \|\Phi(z(t-\tau))\|^2). \end{aligned} \quad (11)$$

By using (4) and the fact $f \in \mathcal{K}$, we can easily have

$$-2\alpha\Phi^T(z(t))PCz(t) \leq -2\alpha\Phi^T(z)PCK^{-1}\Phi(z). \quad (12)$$

We also note the following inequality

$$\begin{aligned} & 2\alpha\Phi^T(z(t))PA\Phi(z(t)) \\ & = 2\alpha\Phi^T(z(t))(P(A^* + \Delta A))\Phi(z(t)) \\ & \leq \alpha\Phi^T(z(t))(PA^* + (A^*)^T P)\Phi(z(t)) \\ & \quad + 2\alpha\|P\|_2\|\Delta A\|_2\|\Phi(z(t))\|^2, \end{aligned} \quad (13)$$

where $A = A^* + \Delta A$. Using (8)-(13) in (7) yields

$$\begin{aligned} \dot{V}(z(t)) &\leq (1/\underline{c})\lambda_{\max}(A^T A)\|\Phi(z(t))\|^2 \\ &\quad + (1/\underline{c})\lambda_{\max}(B^T B)\|\Phi(z(t-\tau))\|^2 \\ &\quad - 2\alpha\Phi^T(z)PCK^{-1}\Phi(z) \\ &\quad + \alpha\|P\|_2\|B\|_2 (\|\Phi(z(t))\|^2 + \|\Phi(z(t-\tau))\|^2) \\ &\quad + \alpha\Phi^T(z(t))(PA^* + (A^*)^T P)\Phi(z(t)) \\ &\quad + 2\alpha\|P\|_2\|\Delta A\|_2\|\Phi(z(t))\|^2 \\ &\quad + \alpha\gamma (\|\Phi(z(t))\|^2 - \|\Phi(z(t-\tau))\|^2) \\ &\quad + \beta (\|\Phi(z(t))\|^2 - \|\Phi(z(t-\tau))\|^2). \end{aligned}$$

Let $\kappa_1 := 1/\underline{c}$. Using inequalities in Lemmas 2.3 and 2.4, we further estimate $\dot{V}(z(t))$ as follows:

$$\begin{aligned} \dot{V}(z(t)) &\leq (\kappa_1\bar{\lambda}_A + \beta) \|\Phi(z(t))\|^2 \\ &\quad + (\kappa_1\bar{\lambda}_B - \beta) \|\Phi(z(t-\tau))\|^2 \\ &\quad - 2\alpha\Phi^T(z)PCK^{-1}\Phi(z) \\ &\quad + \alpha(\|P\|_2(\|B^*\|_2 + \|B_*\|_2) \\ &\quad + 2\|P\|_2\|A^*\|_2 - \gamma) \|\Phi(z(t))\|^2 \\ &\quad + \alpha(\|P\|_2(\|B^*\|_2 + \|B_*\|_2) - \gamma) \|\Phi(z(t-\tau))\|^2 \\ &\quad + \alpha\Phi^T(z(t))(PA^* + (A^*)^T P)\Phi(z(t)). \end{aligned}$$

Now let $\beta := \kappa_1\bar{\lambda}_B$ and $\gamma := \|P\|_2(\|B^*\|_2 + \|B_*\|_2)$. We further use the relation $\Lambda \preceq CK^{-1}$ to get the following simplified estimation of $\dot{V}(z(t))$:

$$\begin{aligned} \dot{V}(z(t)) &= \kappa_1 (\bar{\lambda}_A + \bar{\lambda}_B) \|\Phi(z(t))\|^2 - \alpha\Phi^T(z(t))\bar{\Omega}\Phi(z(t)) \\ &\leq \kappa_1 (\bar{\lambda}_A + \bar{\lambda}_B) \|\Phi(z(t))\|^2 - \alpha\lambda_{\min}(\bar{\Omega})\|\Phi(z(t))\|^2. \end{aligned}$$

Note that $\bar{\Omega}$ is positive definite, the choice

$$\alpha > \frac{\kappa_1 (\bar{\lambda}_A + \bar{\lambda}_B)}{\lambda_{\min}(\bar{\Omega})}$$

ensures that $\dot{V}(z(t))$ is negative definite for all $\Phi(z(t)) \neq 0$. This implies that the origin of (3) is GAS. Detailed proof for this claim based on $\dot{V}(z(t)) < 0$ when $\Phi(z(t)) \neq 0$ can be found on [22, Page 170]. We omit the details. This completes the proof. \square

We have the following corollary.

Corollary 3.1: Suppose $f \in \mathcal{K}$. The neural network model (2) is GARS if there exists a positive diagonal matrix $P = \text{diag}(p_i)$ such that

$$\tilde{\Omega} := 2rI - (PA^* + (A^*)^T P) - 2\|P\|_2 (\|B^*\|_2 + \|B_*\|_2 + \|A_*\|_2) I \succ 0,$$

where $r := \min(p_i c_i / k_i)$. Moreover, $\tilde{\Omega} \succ 0$ if the following two bounds hold:

- 1) $\mu_2(A^*) + \|A_*\|_2 \leq \bar{r}$, and
- 2) $\|B^*\|_2 + \|B_*\|_2 < \bar{r} - (\mu_2(A^*) + \|A_*\|_2)$,

where $\bar{r} := \min(c_i / k_i)$.

Proof. By the definition of r , it is straightforward to see that $2rI \preceq 2P\Lambda$. Hence, $\tilde{\Omega} \succ 0$ must imply $\bar{\Omega} \succ 0$, which in turn by Theorem 3.1 implies the GARS of (2).

Now let $P = I$ in $\tilde{\Omega}$. We then know $\bar{r} = r$. $\tilde{\Omega} \succ 0$ means

$$2\bar{r}I - (A^* + (A^*)^T) - 2(\|B^*\|_2 + \|B_*\|_2 + \|A_*\|_2)I \succ 0.$$

Equivalently

$$\frac{1}{2} (A^* + (A^*)^T) \prec (\bar{r} - (\|B^*\|_2 + \|B_*\|_2 + \|A_*\|_2))I,$$

i.e., the largest eigenvalue of $(A^* + (A^*)^T)/2$ must be less than the quantity $\bar{r} - (\|B^*\|_2 + \|B_*\|_2 + \|A_*\|_2)$, which gives

$$\mu_2(A^*) < \bar{r} - (\|B^*\|_2 + \|B_*\|_2 + \|A_*\|_2).$$

Equivalently

$$\|B^*\|_2 + \|B_*\|_2 < \bar{r} - (\mu_2(A^*) + \|A_*\|_2).$$

The last condition also implies that

$$\mu_2(A^*) + \|A_*\|_2 \leq \bar{r}$$

as $\|B^*\|_2 + \|B_*\|_2 \geq 0$. \square

In Section IV, we will compare this result with a closely related result [22, Theorem 1] of Ozcan and Arik. It is interesting to point out that Corollary 3.1 is actually the best among a class of sufficient conditions depending on a parameter β . We list this class of conditions below with only a sketch of its proof.

Theorem 3.2: Suppose $f \in \mathcal{K}$. The neural network model (2) is GARS if the following conditions hold:

- 1) $\mu_2(A^*) + \|A_*\|_2 \leq \bar{r}$, and
- 2) $\|B^*\|_2 + \|B_*\|_2 < \beta \sqrt{(2/\beta)(\bar{r} - \mu_2(A^*) - \|A_*\|_2) - 1}$ for some $\beta \in (0, 2(\bar{r} - \mu_2(A^*) - \|A_*\|_2))$.

Proof. For any $(C, A, B) \in \mathcal{C} \times \mathcal{A} \times \mathcal{B}$, we shall prove that model (1) is globally asymptotically stable under the sets of conditions. As before, the proof is in two steps.

Step 1 (Existence and uniqueness of an equilibrium point). In this part we use Lemma 2.2. It follows from

$$(\beta - (\bar{r} - \mu_2(A^*) - \|A_*\|_2))^2 \geq 0$$

that

$$\mu_2(A^*) + \|A_*\|_2 + \beta \sqrt{(2/\beta)(\bar{r} - \mu_2(A^*) - \|A_*\|_2) - 1} \leq \bar{r}.$$

This inequality and the condition in 2) give us:

$$\begin{aligned} \mu_2(A + B) &= \mu_2(A^* + \Delta A + B^* + \Delta B) \\ &\leq \mu_2(A^*) + \|A_*\|_2 + \|B_*\|_2 + \|B^*\|_2 \\ &< \mu_2(A^*) + \|A_*\|_2 \\ &\quad + \beta \sqrt{(2/\beta)(\bar{r} - \mu_2(A^*) - \|A_*\|_2) - 1} \\ &\leq \bar{r} \leq \min(c_i / k_i). \end{aligned}$$

Then Lemma 2.2 implies that there exists a unique equilibrium point for any $(C, A, B) \in \mathcal{C} \times \mathcal{A} \times \mathcal{B}$ and $u \in \mathbb{R}^n$.

Step 2 (Global asymptotic convergence.) Once again, we use the Lyapunov theory to prove the global convergence of any solution of (1) to its unique equilibrium point x^* . Let us consider the Lyapunov function of the type proposed in [2]:

$$\begin{aligned} V(z(t)) &= \alpha \|z(t)\|^2 + \frac{2}{\beta} \sum_{i=1}^n \int_0^{z_i(t)} \phi_i(s) ds \\ &\quad + \sum_{i=1}^n \int_{t-\tau_i}^t \left(\phi_i^2(\xi) + \frac{\alpha}{\underline{c}} \Phi^T(z(\xi)) B^T B \Phi(z(\xi)) \right) d\xi, \end{aligned}$$

where $\alpha > 0$ is to be chosen appropriately and β is the constant appeared in Theorem 3.2. The remaining task is to estimate the time derivative of $V(z)$ as we did for Theorem 3.1 or Theorem 3.3 below, or on [24, Page 1703]. We omit this part. What interests us is that Corollary 3.1 is the best version of this theorem, as we show below. \square

The bound in the right hand side of 2) in Theorem 3.2 depends on parameter β . We now seek the biggest possible bound offered by this condition. Let

$$h(\beta) := \beta \sqrt{(2/\beta)(\bar{r} - \mu_2(A^*) - \|A_*\|_2) - 1},$$

with $\beta \in (0, 2(\bar{r} - \mu_2(A^*) - \|A_*\|_2))$. We want to find the largest value of $h(\beta)$ on the interval $(0, 2(\bar{r} - \mu_2(A^*) - \|A_*\|_2))$. Since the square of $h(\beta)$ does not change the location of the maximum of $h(\beta)$, we consider

$$\ell(\beta) := h^2(\beta) = 2\beta(\bar{r} - \mu_2(A^*) - \|A_*\|_2) - \beta^2.$$

$\ell(\cdot)$ is concave on the interval and hence the optimality condition $\ell'(\beta) = 0$ gives the maximum

$$\beta^* = \bar{r} - \mu_2(A^*) - \|A_*\|_2.$$

Substituting this value in $h(\beta)$, we get the largest bound

$$h(\beta^*) = \bar{r} - \mu_2(A^*) - \|A_*\|_2.$$

This shows that Corollary 3.1 is the best version of Theorem 3.2. It is also interesting to point out that Corollary 3.1 is reached via two different routes, one as a consequence of Theorem 3.1 (with $P = I$); the other is through Theorem 3.2. Both theorems are proved by different techniques. \square

Our next results give sufficient conditions involving the largest eigenvalues of the symmetric part of the matrices $A^* + B^*$ and $A^* - B^*$. We recall that $\bar{r} := \min(c_i / k_i)$.

Theorem 3.3: Let $f \in \mathcal{K}$. The neural networks model (2) is GARS if one of the following two conditions holds:

(i)

$$2\mu_2(A^* + B^*) + (\|B^* - I\|_2 + \|B_*\|_2)^2 + 2(\|A_*\|_2 + \|B_*\|_2) < 2\bar{r}. \quad (14)$$

(ii)

$$\mu_2(A^* + B^*) + (\|A_*\|_2 + \|B_*\|_2) < \bar{r} \quad (15)$$

and

$$2\mu_2(A^* - B^*) + (\|B^* + I\|_2 + \|B_*\|_2)^2 + 2(\|A_*\|_2 + \|B_*\|_2) < 2\bar{r}. \quad (16)$$

Proof. First we note that condition (14) in (i) also implies condition (15) in (ii). This fact suggests that we are able to give the proofs for the two cases altogether.

For any $(C, A, B) \in \mathcal{C} \times \mathcal{A} \times \mathcal{B}$, we shall prove that model (1) is globally asymptotically stable under either set of the conditions. The proof is in two steps.

Step 1 (Existence and uniqueness of an equilibrium point). This part is based on Lemma 2.2:

$$\begin{aligned} \mu_2(A + B) &= \mu_2(A^* + \Delta A + B^* + \Delta B) \\ &\leq \mu_2(A^* + B^*) + \mu_2(\Delta A + \Delta B) \\ &\leq \mu_2(A^* + B^*) + \|\Delta A\|_2 + \|\Delta B\|_2 \\ &\leq \mu_2(A^* + B^*) + \|A_*\|_2 + \|B_*\|_2 \\ &< \bar{r} \leq \min(c_i/k_i), \end{aligned}$$

where we used $A = A^* + \Delta A$ and $B = B^* + \Delta B$ with $\Delta A \in [-A_*, A_*]$ and $\Delta B \in [-B_*, B_*]$ and inequalities in Lemma 2.3 were applied. This string of inequalities means that Lemma 2.2 is satisfied under the condition of (15) and hence step 1 is finished for case (ii). The proof for case (i) is straightforward because condition (14) implies condition (15). We note that condition (15) alone in Case (ii) already guarantees the existence and uniqueness of an equilibrium point. Condition (16) will ensure the global convergence as we will see shortly.

Step 2 (Global asymptotic convergence.) Once again, we use the Lyapunov theory to prove the global convergence of any solution of (1) to its unique equilibrium point x^* . This is equivalent to show that the origin is GAS for system (3). Now assume (14) in (i) holds. Let us consider the Lyapunov function of the type originally used in [3]:

$$\begin{aligned} V(z(t)) &= \|z(t)\|^2 + 2\alpha \sum_{i=1}^n \int_0^{z_i(t)} \phi_i(s) ds \\ &\quad + (\alpha + \beta) \sum_{i=1}^n \int_{t-\tau_i}^t \phi_i^2(z_i(\xi)) d\xi, \end{aligned}$$

where α and β are some positive constants chosen appropriately later on. The time derivative of $V(z(t))$ along the trajectories of equation (3) is calculated as follows:

$$\begin{aligned} \dot{V}(z(t)) &= -2z^T(t)Cz(t) + 2z^T(t)A\Phi(z(t)) \\ &\quad + 2z^T(t)B\Phi(z(t-\tau)) - 2\alpha\Phi^T(z(t))Cz(t) \\ &\quad + 2\alpha\Phi^T(z(t))A\Phi(z(t)) \\ &\quad + 2\alpha\Phi^T(z(t))B\Phi(z(t-\tau)) \\ &\quad + (\alpha + \beta) (\|\Phi(z(t))\|^2 - \|\Phi(z(t-\tau))\|^2). \end{aligned} \quad (17)$$

We note the following inequalities:

$$\begin{aligned} &-\alpha\|\Phi(z(t-\tau))\|^2 + 2\alpha\Phi^T(z(t))B\Phi(z(t-\tau)) \\ &= -\alpha\|\Phi(z(t-\tau)) - B^T\Phi(z(t))\|^2 \\ &\quad + \alpha\Phi^T(z(t))BB^T\Phi(z(t)) \end{aligned} \quad (18)$$

$$-2\alpha\Phi^T(z(t))Cz(t) \leq -2\alpha\bar{r}\|\Phi(z(t))\|^2. \quad (19)$$

Rearranging the terms in (17) and using inequalities (8)-(10), (18), and (19), we obtain

$$\begin{aligned} \dot{V}(z(t)) &\leq \kappa_1\Phi^T(z(t))A^TA\Phi(z(t)) \\ &\quad + \kappa_1\Phi^T(z(t-\tau))B^TB\Phi(z(t-\tau)) \\ &\quad + 2\alpha\Phi^T(z(t))A\Phi(z(t)) - \alpha(2\bar{r} - 1)\|\Phi(z(t))\|^2 \\ &\quad + \alpha\Phi^T(z(t))BB^T\Phi(z(t)) \\ &\quad + \beta(\|\Phi(z(t))\|^2 - \|\Phi(z(t-\tau))\|^2) \\ &\leq \kappa_1\lambda_{\max}(A^TA)\|\Phi(z(t))\|^2 \\ &\quad + \kappa_1\lambda_{\max}(B^TB)\|\Phi(z(t-\tau))\|^2 \\ &\quad + 2\alpha\Phi^T(z(t))A\Phi(z(t)) - \alpha(2\bar{r} - 1)\|\Phi(z(t))\|^2 \\ &\quad + \alpha\Phi^T(z(t))BB^T\Phi(z(t)) \\ &\quad + \beta(\|\Phi(z(t))\|^2 - \|\Phi(z(t-\tau))\|^2). \end{aligned}$$

Using the facts in Lemma 2.4 and letting $\beta := \kappa_1\bar{\lambda}_B$, we have

$$\begin{aligned} \dot{V}(z(t)) &\leq \kappa_1(\bar{\lambda}_A + \bar{\lambda}_B)\|\Phi(z(t))\|^2 \\ &\quad + 2\alpha\Phi^T(z(t))A\Phi(z(t)) - \alpha(2\bar{r} - 1)\|\Phi(z(t))\|^2 \\ &\quad + 2\alpha\Phi^T(z(t))B\Phi(z(t)) \\ &\quad - 2\alpha\Phi^T(z(t))B\Phi(z(t)) \\ &\quad + \alpha\Phi^T(z(t))BB^T\Phi(z(t)). \end{aligned} \quad (20)$$

Using the fact that

$$\begin{aligned} &-2\alpha\Phi^T(z(t))B\Phi(z(t)) + \alpha\Phi^T(z(t))BB^T\Phi(z(t)) \\ &= \alpha\|B^T\Phi(z(t)) - \Phi(z(t))\|^2 - \alpha\|\Phi(z(t))\|^2 \\ &\leq \alpha\|B - I\|_2^2\|\Phi(z(t))\|^2 - \alpha\|\Phi(z(t))\|^2 \\ &= -\alpha(1 - \|B - I\|_2^2)\|\Phi(z(t))\|^2. \end{aligned}$$

We then have from (20) that

$$\begin{aligned} \dot{V}(z(t)) &\leq \kappa_1(\bar{\lambda}_A + \bar{\lambda}_B)\|\Phi(z(t))\|^2 \\ &\quad - \alpha(1 - \|B - I\|_2^2)\|\Phi(z(t))\|^2 \\ &\quad - \alpha(2\bar{r} - 1)\|\Phi(z(t))\|^2 \\ &\quad + \alpha\Phi^T(z(t))(A + A^T + B + B^T)\Phi(z(t)) \\ &= \kappa_1(\bar{\lambda}_A + \bar{\lambda}_B)\|\Phi(z(t))\|^2 \\ &\quad - \alpha(2\bar{r} - \|B - I\|_2^2)\|\Phi(z(t))\|^2 \\ &\quad + \alpha\Phi^T(z(t))(A^* + (A^*)^T + B^* + (B^*)^T)\Phi(z(t)) \\ &\quad + 2\alpha\Phi^T(z(t))(\Delta A + \Delta B)\Phi(z(t)) \\ &\leq \kappa_1(\bar{\lambda}_A + \bar{\lambda}_B)\|\Phi(z(t))\|^2 \\ &\quad - \alpha(2\bar{r} - \|B^* - I + \Delta B\|_2^2)\|\Phi(z(t))\|^2 \\ &\quad + 2\alpha\mu_2(A^* + B^*)\|\Phi(z(t))\|^2 \\ &\quad + 2\alpha\|\Delta A + \Delta B\|_2\|\Phi(z(t))\|^2 \\ &\leq \kappa_1(\bar{\lambda}_A + \bar{\lambda}_B)\|\Phi(z(t))\|^2 \\ &\quad - \alpha(2\bar{r} - \kappa_2)\|\Phi(z(t))\|^2, \end{aligned} \quad (21)$$

where

$$\kappa_2 := 2\mu_2(A^* + B^*) + (\|B^* - I\|_2 + \|B_*\|)^2 + 2(\|A_*\|_2 + \|B_*\|_2).$$

Now we consider the following three cases:

Case 1: $\Phi(z(t)) \neq 0$ and $z(t) \neq 0$. It then follows from (21) and condition (14) that the choice

$$\alpha > \kappa_1(\bar{\lambda}_A + \bar{\lambda}_B)/(2\bar{r} - \kappa_2)$$

ensures that $\dot{V}(z(t))$ is negative.

Case 2: $\Phi(z(t)) = 0$, but $z(t) \neq 0$. Then it follows from (17) and (8) that

$$\begin{aligned} \dot{V}(z(t)) &\leq -2\underline{c}\|z(t)\|^2 + 2z^T(t)B\Phi(z(t-\tau)) \\ &\quad - (\alpha + \beta)\|\Phi(z(t-\tau))\|^2 \\ &= -\underline{c}\|z(t)\|^2 + 2z^T(t)B\Phi(z(t-\tau)) - \underline{c}\|z(t)\|^2 \\ &\quad - (\alpha + \beta)\|\Phi(z(t-\tau))\|^2 \\ &\leq \frac{1}{\underline{c}}\Phi(z(t-\tau))B^TB\Phi(z(t-\tau)) - \underline{c}\|z(t)\|^2 \\ &\quad - (\alpha + \beta)\|\Phi(z(t-\tau))\|^2 \\ &\leq -(\alpha + \beta - \bar{\lambda}_B/\underline{c})\|\Phi(z(t-\tau))\|^2 - \underline{c}\|z(t)\|^2. \end{aligned}$$

We recall that $\beta = \bar{\lambda}_B/\underline{c}$, which obviously implies the first term in the last inequality is nonpositive and the second term is strictly negative. Hence, $\dot{V}(z(t)) < 0$ for this case.

Case 3: $z(t) = 0$. Clearly, $\Phi(z(t)) = 0$ due to the fact (5). In this case, $\dot{V}(z(t))$ is given by

$$\dot{V}(z(t)) = -(\alpha + \beta)\|\Phi(z(t-\tau))\|^2.$$

Hence, $\dot{V}(z(t))$ is negative if $\Phi(z(t-\tau)) \neq 0$ and $\dot{V}(z(t)) = 0$ if and only if it happens in the last case where

$$z(t) = \Phi(z(t)) = \Phi(z(t-\tau)) = 0.$$

We recall that $V(z(t))$ is radially unbounded. According to [17, Corollary 3.2, Ch.3] that the origin of (3) or equivalently the equilibrium point x^* of (1) is GAS. This proves that model (2) is GARS under condition (14).

Now assume conditions in (ii) hold. Starting from (20) and using the fact that

$$\begin{aligned} &2\alpha\Phi^T(z(t))B\Phi(z(t)) + \alpha\Phi^T(z(t))BB^T\Phi(z(t)) \\ &= \alpha\|B^T\Phi(z(t)) + \Phi(z(t))\|^2 - \alpha\|\Phi(z(t))\|^2 \\ &\leq \alpha\|B + I\|_2^2\|\Phi(z(t))\|^2 - \alpha\|\Phi(z(t))\|^2 \\ &= -\alpha(1 - \|B + I\|_2^2)\|\Phi(z(t))\|^2. \end{aligned}$$

We then have from (20) that

$$\begin{aligned} \dot{V}(z(t)) &\leq \kappa_1(\bar{\lambda}_A + \bar{\lambda}_B)\|\Phi(z(t))\|^2 \\ &\quad - \alpha(1 - \|B + I\|_2^2)\|\Phi(z(t))\|^2 \\ &\quad - \alpha(2\bar{r} - 1)\|\Phi(z(t))\|^2 \\ &\quad + \alpha\Phi^T(z(t))(A + A^T - B - B^T)\Phi(z(t)) \\ &= \vdots \\ &\leq \kappa_1(\bar{\lambda}_A + \bar{\lambda}_B)\|\Phi(z(t))\|^2 \\ &\quad - \alpha(2\bar{r} - \kappa_3)\|\Phi(z(t))\|^2, \end{aligned}$$

where

$$\kappa_3 := 2\mu_2(A^* - B^*) + (\|B^* + I\|_2 + \|B_*\|)^2 + 2(\|A_*\|_2 + \|B_*\|_2).$$

Just repeating the arguments for the three subcases above, we are able to show that model (2) is GARS under conditions in (ii). This completes our proof. \square

An interesting consequence of Theorem 3.3 is when the uncertainty sets each contains only one element, i.e., $A_* = 0$ and $B_* = 0$:

Corollary 3.2: Let $f \in \mathcal{K}$. The neural network model (1) is GAS if one of the following two conditions holds:

(i)

$$2\mu_2(A + B) + \|B - I\|_2^2 < 2\bar{r}. \quad (22)$$

(ii)

$$\mu_2(A + B) < \bar{r} \quad (23)$$

and

$$2\mu_2(A - B) + \|B + I\|_2^2 < 2\bar{r}. \quad (24)$$

We note that condition (22) recovers the main result [23, Theorem 3] and conditions (23) and (24) appear new from many known results for GAS property of model (1).

IV. COMPARISON

In this section, we conduct comparison with several existing and closely related results reported in [8], [22]. Our comparison clearly shows the significance of the results reported in the last section.

As we promised early on, we now compare Theorem 3.1 with following result by Ozcan and Arik.

Theorem 4.1: [22, Theorem 1] Let $f \in \mathcal{K}$. then, the neural network model (2) is GARS if there exists a positive diagonal matrix $P = \text{diag}(p_i > 0)$ such that

$$\Omega := 2rI + S - 2\|P\|_2(\|B^*\|_2 + \|B_*\|_2)I \succ 0,$$

where $S = (s_{ij})_{n \times n}$ with $s_{ii} = -2p_i\bar{a}_{ii}$ and $s_{ij} = -\max(|p_i\bar{a}_{ij} + p_j\bar{a}_{ji}|, |p_i\bar{a}_{ij} + p_j\bar{a}_{ji}|)$ for $i \neq j$.

An interesting corollary of the above theorem is the following

Corollary 4.1: [22, Corollary 2] Let $f \in \mathcal{K}$. Then, the neural network model (2) is GARS if there exists a positive diagonal matrix $P = \text{diag}(p_i > 0)$ such that

- 1) the symmetric matrix S is positive definite, i.e., $S \succ 0$, and
- 2) $\|B^*\|_2 + \|B_*\|_2 \leq r/\|P\|_2$.

A surprising aspect of this corollary is that it covers a seemingly different result by Cao and Wang [8], see the discussion before Theorem 8 in [22].

Theorem 4.2: [8, Theorem 2] Let $f \in \mathcal{K}$. Suppose also that f is bounded. Then, the neural network model (2) is GARS if there exists a positive diagonal matrix $P = \text{diag}(p_i > 0)$ and a positive definite matrix $D \succ 0$ such that

- 1) the symmetric matrix S is positive definite, i.e., $S \succ 0$, and
- 2) $\|B^*\|_2 + \|B_*\|_2 \leq (2r - \|D\|_2)/(\|D^{-1}\|_2\|P\|_2^2)$.

Remarks. In fact, Ozcan and Arik present a group of similar conditions to Corollary 4.1, all involve the positive definiteness of S and various bounds on $\|B^*\|_2 + \|B_*\|_2$. The positive definiteness of S and the bounds are aimed to ensure $\Omega \succ 0$. The proof of showing the existence and uniqueness of an equilibrium point under this condition is based on a

homeomorphism theorem of Forti and Tesi [10] and is quite involved in analysis. It is worth to point out that Lemma 2.1 provides comparatively short and compact a proof.

Suppose $\Omega \succ 0$. We want to prove every matrix W in \mathcal{W} is nonsingular. Since $W \in \mathcal{W}$, there must be a diagonal matrix $D = \text{diag}(d_i)$ with $0 \leq d_i \leq k_i$ for all $i = 1, \dots, n$ such that $W = C - (A + B)D$. Suppose W is singular, we shall derive a contradiction. There must be a vector $0 \neq x \in C^n$ (the field of complex numbers) such that

$$PCK^{-1}x - P(A + B)Qx = 0, \quad (25)$$

where $Q = DK^{-1}$. First we note that $Qx \neq 0$. Otherwise equation (25) would imply $x = 0$, which contradicts the fact $x \neq 0$.

Multiplying equation (25) from left by $\bar{x}^T Q$, we have

$$\begin{aligned} 0 &= \bar{x}^T Q P C K^{-1} x - \bar{x}^T Q P (A + B) Q x \\ &\geq \bar{x}^T Q x - \frac{1}{2} \bar{x}^T Q (P A + A^T P) Q x - \bar{x}^T Q P B Q x \\ &\geq r \bar{x}^T Q^2 x - \frac{1}{2} \sum_{i=1}^n 2 p_i a_{ii} (Q \bar{x})_i (Q x)_i \\ &\quad - \frac{1}{2} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n (p_i a_{ij} + p_j a_{ji}) (Q \bar{x})_i (Q x)_j \\ &\quad - \|P\|_2 \|B\|_2 \|Q x\|^2 \\ &\geq r \|Q x\|^2 + \frac{1}{2} |Q \bar{x}|^T S |Q x| \\ &\quad - \|P\|_2 (\|B^*\| + \|B_*\|_2) \|Q x\|^2 \\ &= \frac{1}{2} |Q x|^T (2rI + S - 2\|P\|_2 (\|B^*\| + \|B_*\|_2) I) |Q x| \\ &= \frac{1}{2} |Q x|^T \Omega |Q x|. \end{aligned}$$

The second inequality above uses the property $0 \leq Q^2 \leq Q$. Because Ω is positive definite, we must have $Qx = 0$, which contradicts the fact $Qx \neq 0$. Hence, every W in \mathcal{W} is nonsingular. Lemma 2.1 implies the existence and uniqueness of an equilibrium point under the condition $\Omega \succ 0$. \square

Although the appearance of the conditions in Theorem 3.1 and Theorem 4.1 looks similar, they are quite different as demonstrated by the example below.

Example 4.1: Let the reference matrix A^* and the perturbation matrix E be:

$$A^* = -\frac{1}{2} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let

$$\underline{A} = A^* - \epsilon E, \quad \bar{A} = A^* + \epsilon E$$

and

$$\underline{B} = -aE, \quad \bar{B} = aE,$$

where $\epsilon, a \geq 0$. We also let

$$\underline{C} = \bar{C} = K = P = I,$$

so that $r = 1$ and $\|P\|_2 = 1$. It is also easy to see that

$$\|A_*\|_2 = 3\epsilon, \quad \|B_*\|_2 = 3a \quad \text{and} \quad \|B^*\|_2 = 0.$$

$$S = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} - 2\epsilon E$$

and $-(A^* + (A^*)^T) = -2A^*$. We now calculate $\bar{\Omega}$ and Ω respectively and obtain:

$$\bar{\Omega} = \begin{pmatrix} 4 - 6a - 6\epsilon & 1 & 1 \\ 1 & 4 - 6a - 6\epsilon & 1 \\ 1 & 1 & 3 - 6a - 6\epsilon \end{pmatrix}$$

and

$$\Omega = \begin{pmatrix} 4 - 6a - 2\epsilon & -1 - 2\epsilon & -1 - 2\epsilon \\ -1 - 2\epsilon & 4 - 6a - 2\epsilon & -1 - 2\epsilon \\ -1 - 2\epsilon & -1 - 2\epsilon & 3 - 6a - 2\epsilon \end{pmatrix}.$$

We note that for this example $\bar{\Omega} = \tilde{\Omega}$ defined in Corollary 3.1. To simplify our calculation, we let $\epsilon = 0$, which gives

$$\bar{\Omega} = \begin{pmatrix} 4 - 6a & 1 & 1 \\ 1 & 4 - 6a & 1 \\ 1 & 1 & 3 - 6a \end{pmatrix}$$

and

$$\Omega = \begin{pmatrix} 4 - 6a & -1 & -1 \\ -1 & 4 - 6a & -1 \\ -1 & -1 & 3 - 6a \end{pmatrix}.$$

Using the famous root formulae of cubic equations, we find that when $a \geq 0$,

$$\bar{\Omega} \succ 0 \quad \text{if and only if} \quad a \leq 0.37799$$

and

$$\Omega \succ 0 \quad \text{if and only if} \quad a \leq 0.26429.$$

Therefore, Theorem 3.1 provides a better bound for this example than Theorem 4.1. We also note that since the roots are continuous functions of ϵ , we can easily extend the case $\epsilon = 0$ to cases where $\epsilon > 0$ is small enough.

The next example shows the difference of our result Corollary 3.1 from Corollary 4.1. First we recall that $\bar{r} = \min(\underline{c}_i/k_i)$ and $r = \min(p_i \underline{c}_i/k_i)$.

Example 4.2: Suppose $\underline{C} = \bar{C} = K = P = I$ so that $\bar{r} = r = 1$. Now, conditions in Corollary 3.1 become

$$\mu_2(A^*) + \|A_*\|_2 \leq 1 \quad \text{and} \quad \|B^*\|_2 + \|B_*\|_2 < 1 - \mu_2(A^*) - \|A_*\|_2,$$

and conditions in Corollary 4.1 become

$$S \succ 0 \quad \text{and} \quad \mu_2(A^*) + \|A_*\|_2 \leq 1.$$

Let

$$E_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We define the network parameters as follows:

$$\underline{A} = -I - \epsilon E_1, \quad \bar{A} = I - \epsilon E_1, \quad \underline{B} = -a E_2 \quad \text{and} \quad \bar{B} = a E_2,$$

where $\epsilon, a \geq 0$. Then we have

$$A^* = -\epsilon E_1, \quad A_* = I, \quad B^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_* = a E_2.$$

On the one hand, we have

$$S = \begin{pmatrix} -2(1 - \epsilon) & -\epsilon \\ -\epsilon & -2(1 - \epsilon) \end{pmatrix},$$

which is never positive definite for $\epsilon \leq 2$. Hence, Corollary 4.1 does not apply to this case. On the other hand, we have

$$\mu_2(A^*) = -\frac{1}{2}\epsilon, \quad \|A_*\|_2 = 1, \quad \|B^*\|_2 = 0, \quad \|B_*\|_2 = 2a.$$

It is easy to calculate that

$$\mu_2(A^*) + \|A_*\|_2 = 1 - \frac{1}{2}\epsilon \leq 1$$

and

$$\|B^*\|_2 + \|B_*\|_2 = 2a < 1 - (\mu_2(A^*) + \|A_*\|_2) = \frac{1}{2}\epsilon$$

for $a < \frac{1}{4}\epsilon$. Hence, Corollary 3.1 implies that model (2) is GARS for this example with $0 \leq \epsilon \leq 2$ and $0 \leq a < \frac{1}{4}\epsilon$.

Remarks. The purpose of Example 4.2 is to show that Corollary 3.1 does not need S to be positive definite. We also note that a significant property of this corollary is that the quantity $\mu_2(A^*) + \|A_*\|_2$ may be negative, leaving much freedom for B^* and B_* as the bound $(1 - (\mu_2(A^*) + \|A_*\|_2))$ for $\|B^*\|_2 + \|B_*\|_2$ may exceed 1. This possibility is ruled out if we use Corollary 4.1. We verify this observation by continuously examining Example 4.1. \square

Example 4.3: (Example 4.1 continued) For this example, we calculate that

$$\mu_2(A^*) = -0.5 \times 0.2679 = -0.13395.$$

Hence,

$$\mu_2(A^*) + \|A_*\|_2 = -0.13395 + 3\epsilon \leq 1 \quad \text{for } \epsilon \leq 1.13395/3.$$

In particular, $\mu_2(A^*) + \|A_*\|_2 \leq 0$ for $\epsilon \leq 0.13395/3$. It is easy to see that

$$\|B^*\|_2 + \|B_*\|_2 = 3a < 1 - (\mu_2(A^*) + \|A_*\|_2) = 1.13395 - 3\epsilon$$

for $a \leq (1.13395 - 3\epsilon)/3$. In particular, when $\epsilon < 0.13395/3$, $1 - (\mu_2(A^*) + \|A_*\|_2) > 1$. To summer up, Corollary 3.1 means this example has GARS property when

$$0 \leq \epsilon \leq 1.13395/3 \quad \text{and} \quad 3a < 1.13395 - 3\epsilon.$$

Our last example shows that the two conditions in Theorem 3.3 are also different. We only give an example that no errors of network parameters are in presence.

Example 4.4: Let $\underline{C} = \bar{C} = K = P = I$ and

$$M = \begin{pmatrix} 0.2 & 0 \\ 0.4 & 0.2 \end{pmatrix}.$$

The network parameters are

$$\underline{A} = \bar{A} = M, \quad \underline{B} = \bar{B} = M^T.$$

The we have

$$A + B = 0.4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B - I = \begin{pmatrix} -0.8 & 0.4 \\ 0 & -0.8 \end{pmatrix}$$

and

$$A - B = \begin{pmatrix} 0 & -0.4 \\ 0.4 & 0 \end{pmatrix}, \quad B + I = \begin{pmatrix} 1.2 & 0.4 \\ 0 & 1.2 \end{pmatrix}.$$

It is easy to see that

$$\mu_2(A + B) = 0.8, \quad \|B - I\|_2 \approx 1.0246$$

and

$$\mu_2(A - B) = 0, \quad \|B + I\|_2 \approx 1.4166.$$

Hence, condition (22) is not satisfied, while conditions (23) and (24) are satisfied for this example.

If we let $\underline{B} = \bar{B} = -M^T$ and keep other parameters unchanged, then by symmetry we see that condition (22) is satisfied, while conditions (23) and (24) are not satisfied.

We also note that Theorem 3.3 represents a different class of conditions from those reported and cited in this paper. It is not too hard to construct examples showing the differences. We simply omit those examples. It is also worth to note that there are several results reported in [7], [19], [20], [27] of concerning GARS of model (2). It has been shown in [22] that those results are different from Theorem 4.1 and Corollary 4.1. It is also possible to follow the ideas of [22] to construct examples to show that our results are different as well.

V. VERIFICATION

The significance of the reported results is their low computational cost that is needed to verify them. To be specific, results in Corollary 3.1, Theorems 3.2 and 3.3 all involve calculation of the largest symmetric matrices, which can be done via a small number of spectral decompositions of symmetric matrices involved. Verification of Theorem 3.1 can be done by formulating its condition as a semidefinite programming problem (SDP), to which efficient softwares are available [6]. It is not known, however, if other similar results cited in this paper, say those by Ozcan and Arik [22], can be verified by SDP, because most of their results involve a matrix S (see Theorem 4.1 in last section) which is a nonlinear function of P . In this section, we will first give a brief discussion how spectral decomposition is sufficient for some cases, followed by a detailed derivation how SDP can be used to verify Theorem 3.1.

Verification via Spectral Decomposition We use Condition (i) in Theorem 3.3 as an example to demonstrate how the spectral decomposition is sufficient to verify the condition. For other results except Theorem 3.1, similar arguments hold. For a square matrix A , define its symmetric part $\text{Sym}(A) = (A + A^T)/2$. Then for Condition (i), we need the spectral decomposition

$$\text{Sym}(A^* + B^*) = Q \text{diag}(\lambda_1, \dots, \lambda_n) Q^T,$$

where $\lambda_1 \geq \dots \geq \lambda_n$ be the eigenvalues of $\text{Sym}(A^* + B^*)$ and Q are the matrix of corresponding eigenvectors. Then $\mu_2(A^* + B^*) = \lambda_1$. Hence, to verify Condition (i), we need 4 spectral decompositions, namely for $\text{Sym}(A^* + B^*)$, $(B^* - I)((B^*)^T - I)$, $B_* B_*^T$, and $A_* A_*^T$. An alternative for doing spectral decomposition for products like $A_* A_*^T$ is to do a singular value decomposition of A_* , see [12]. One can actually avoid spectral/singular-value decomposition of those matrices because what we really need in Condition (i) is the largest eigenvalues of those matrices. There are good cheaper (compared to spectral/singular-value decomposition) methods that only calculate the largest eigenvalue of a symmetric matrix. For our case, we are contented since we only need

to do spectral decompositions of at most four symmetric matrices.

Verification via SDP The condition in Theorem 3.1 involves a feasibility issue that asks whether there exists a positive diagonal matrix P satisfying $\bar{\Omega} \succ 0$. It cannot be verified through easy spectral decomposition like we did for other cases. Fortunately, we are able to show how powerful SDP can be applied to verify the positive definiteness of $\bar{\Omega}$.

Recall

$$\bar{\Omega} := 2P\Lambda - (PA^* + (A^*)^T P) - 2\|P\|_2 \tau^* I \succ 0 \quad (26)$$

where $\tau^* := \|B^*\|_2 + \|B_*\|_2 + \|A_*\|_2$. Note that $\|P\|_2 = \max(p_i)$. Dividing $\|P\|_2$ in both sides of (26) yields

$$2\bar{P}\Lambda - (\bar{P}A^* + (A^*)^T \bar{P}) - 2\tau^* I \succ 0$$

where $\bar{P} := P/\|P\|_2$. Because P is positive and diagonal, \bar{P} has to satisfy

$$\bar{P} \succ 0, \quad \bar{P} \preceq I \quad \text{and} \quad \Pi_{i=1}^n (1 - \bar{p}_i) = 0. \quad (27)$$

The last identity in (27) means that at least one of the diagonal elements of \bar{P} has to be 1. Hence condition (26) is satisfied for some $P = \text{diag}(p_i)$ if and only if following conditions are satisfied for some $P = \text{diag}(p_i)$.

$$2P\Lambda - (PA^* + (A^*)^T P) - 2\tau^* I \succ 0, \quad (28)$$

$$P \succ 0, \quad P \preceq I,$$

and

$$\Pi_{i=1}^n (1 - p_i) = 0. \quad (29)$$

Note that we replaced \bar{P} by P for notation simplicity. It is easy to see (29) holds if and only if

$$p_i = 1 \quad \text{for some } i.$$

Moreover, (28) is equivalent to

$$P(A^* - \Lambda) + (A^* - \Lambda)^T P + 2\tau^* I \prec 0.$$

To summarize, to check condition (26) is to check if there exist a positive diagonal matrix $P = \text{diag}(p_i)$ such that

$$\begin{cases} P(A^* - \Lambda) + (A^* - \Lambda)^T P + 2\tau^* I < 0 \\ -P < 0 \\ P \preceq I \\ \text{for some } i, \quad p_i = 1 \end{cases} \quad (30)$$

We note that to assess there is no positive diagonal matrix P satisfying (30) one has to check n linear system of (30) each corresponding to one $p_i = 1$. Fortunately the size of the variable in this system is just n because P is diagonal. The difficulty in checking the feasibility of (30) is greatly relieved by the following fact.

It is widely known in semidefinite programming [6] that the feasibility of the linear system of the type (30) can be checked by reformulating it as the following SDP:

$$\begin{aligned} \min \quad & t \\ \text{s. t.} \quad & P(A^* - \Lambda) + (A^* - \Lambda)^T P + 2\tau^* I \preceq tI \\ & -P \preceq tI \\ & P \preceq I \\ & \text{for some } i, \quad p_i = 1 \end{aligned} \quad (31)$$

A minor warning for this SDP reformulation is that the constraint $P \preceq I$ cannot be replaced by a relaxed $P \preceq I + tI$ as would be suggested in [6]. This is because it would be contradictory for $t < 0$ between the relaxed $P \preceq I + tI$ and the last equality constraint $p_i = 1$ for some i . We note that (31) has an optimal solution. First, it is feasible. In fact, one can easily verify that

$$(P = I, t = 1 + 2 \max\{\mu_2(A^* - \Lambda + \tau^* I), 1\})$$

is a feasible point. Second, since $p_i = 1$ for some i , we must have $t \geq -1$ for any feasible pair (P, t) . Hence an optimal solution (P^*, t^*) exists for this problem.

Standard SDP softwares are then used to produce a certificate stating that the linear system (30) is infeasible if $t^* \geq 0$ and it is feasible if $t^* < 0$. In summary, the feasibility of $\bar{\Omega} \succ 0$ can be low-costly verified by SDP.

VI. CONCLUSION

The paper may be viewed as continuation of our pursuit in applying nonsmooth analysis to study of stabilities of neural networks initiated in [23]. This time, we work with models containing network parameter errors. The reported results demonstrate again the potential of nonsmooth analysis approach when combined with different Lyapunov functions (three different Lyapunov functions are used in this paper). One important common property of all of the reported results is that they are easy to verify (in the sense of computational complexity) either by spectral decomposition of symmetric matrices or by the semidefinite programming, which can be polynomially solved by interior-point methods [6]. We feel that the development of SDP verification provides new incentive to extending the nonsmooth analysis approach to other problems. Since LMI is often closely related to SDP, we conclude this paper by pointing out that it is worth to explore the application of the nonsmooth analysis with LMI approach [5], [8], [18], [25] to derive new conditions for GARS of model (2).

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