Deriving Sufficient Conditions for Global Asymptotic Stability of Delayed Cellular Neural Networks via Nonsmooth Analysis II

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Abstract: Following our recent approach of nonsmooth analysis, we report a new set of sufficient conditions and its implications for the global asymptotic stability of delayed cellular neural networks (DCNN). The new conditions not only unify a string of previous stability results, but also yield strict improvement over them by allowing the symmetric part of the feedback matrix positive definite, hence enlarging the application domain of DCNNs. Advantages of the new results over existing ones are illustrated with examples. We also compare our results with those related results obtained via LMI approach.

Index Terms. Neural networks, equilibrium point, global asymptotic stability, Lipschitzian functions, Nonsmooth analysis.

1 Introduction

Since its introduction by Chua and Yang [8, 9], the Cellular Neural Network (CNN, for short) has found wide range of applications in many areas such as signal processing, pattern recognition and moving image processing, to name a few. In the implementation of CNNs in those applications, it is sometimes necessary to introduce delays in the signals transmitted among cells [7]. This is how the Delayed Cellular Neural Network (DCNN, for short) comes [24, 25]. The level of reliability of DCNNs depends on the global uniqueness of the equilibrium point as well as its asymptotic stability. Due to this reason, many papers are dedicated to the study of stability issue of DCNNs, see [1, 3, 4, 5, 6, 10, 13, 14, 18, 19, 20, 21, 23, 26, 27] and references therein. It turns out that the stability analysis for DCNNs is much more difficult than for conventional CNNs.

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The general class of delayed neural networks considered in this paper can be described by the following state equation:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + A^{\tau} f(x(t-\tau)) + u \tag{1}$$

where, $x(t) = (x_1(t), \ldots, x_n(t))^T \in \Re^n$ is the state vector; n is the number of units in a neural network; $f(x(t)) = (f_1(x_1(t)), \ldots, f_n(x_n(t)))^T \in \Re^n$ is the output with each activation function $f_i : \Re \to \Re$; $C = \mathrm{diag}(c_1, \ldots, c_n)$ is a diagonal matrix with each $c_i > 0$ controlling the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $A = (a_{ij})$ and $A^{\tau} = (a_{ij}^{\tau})$ are the feedback and the delayed feedback matrix respectively; $u = (u_1, \ldots, u_n)^T \in \Re^n$ is a constant input vector and τ is the time delay. Since our results are all independent of time delays, for the sake of simplicity we use the same scale of delay τ rather than various scales $\tau_1, \ldots, \tau_n$. Throughout, we assume that each of the activation functions possess the following property:

(H) $f_i : \Re \to \Re$, $i = 1, \ldots, n$, is nondecreasing and globally Lipschitzian, i.e., there exists a number $k_i > 0$ such that $|f_i(\nu_1) - f_i(\nu_2)| \le k_i |\nu_1 - \nu_2|$ for all $\nu_1, \nu_2 \in \Re$.

Two well known activation functions which satisfy the property (H) are:

$$f_i(\theta) = \frac{1}{2}(|\theta + 1| - |\theta - 1|) \tag{2}$$

and

$$f_i(\theta) = \max\{0, \theta\} \qquad \theta \in \Re. \tag{3}$$

Both functions are nondecreasing and have a Lipschitz modulus of 1, i.e, $k_i = 1$, but they are different in two aspects: Function (2) is bounded and has a unique zero point; while the function in (3) is unbounded and has the left half real line $(-\infty, 0]$ as the set of zero points. Those features of uniqueness vs non-uniqueness of zero points and boundedness vs unboundedness show the great variety of instances covered by the assumption (H).

To address the global asymptotic stability of (1), a nonsmooth analysis approach was developed in our previous paper [23]. We only mention the precis of this approach. Let $F : \Re^n \mapsto \Re^n$ be locally Lipschitzian. F is differentiable almost everywhere by Rademacher's theorem. Then the generalized Jacobian of F at any given point x, denoted by $\partial F(x)$, in the sense of Clarke [11] is well defined. One important tool in nonsmooth analysis approach is the Lipschitzian Hadamard theorem [22], which says that if there exists a positive constant $\kappa > 0$ such that $\partial F(x)$ is invertible and $\|W^{-1}\| \le \kappa$ for all $x \in \Re^n$ and all $W \in \partial F(x)$, then F is a homeomorphism from $\Re^n$ to $\Re^n$. We remark that the Hadamard theorem plays a similar role as the homeomorphism theorem developed by Forti and Tesi [12]. With $c := \min\{c_1, \ldots, c_n\}$ and $k := \max\{k_1, \ldots, k_n\}$, and applying the Hadamard theorem to (1), [23] obtained the following result: If

$$\mu_2(A + A^{\tau}) < c/k, \tag{4}$$

then (1) has a unique equilibrium point for any input vector $u \in \Re^n$, where for a matrix $M \in \Re^{n \times n}$, $\mu_2(M) = 0.5\lambda_{\max}(M + M^T)$, i.e., $\mu_2(M)$ is the largest eigenvalue of the symmetric part of M. We also let $\|M\|_2$ denote the norm given by $\|M\|_2 = \sqrt{\lambda_{\max}(M^T M)}$. It is well-known that $\mu_2(M) \leq \|M\|_2$.

In this paper, we continue to pursue if condition (4) is also sufficient for (1) to be globally asymptotically stable (GAS). Since there are instances A and A^τ such that

$$\mu_2(A) + \|A^\tau\|_2 = \mu_2(A) + \mu_2(A^\tau) = \mu_2(A + A^\tau),$$

a reasonable version of (4) seems to be

$$\mu_2(A) < c/k \ \text{ and } \ \|A^\tau\|_2 < c/k - \mu_2(A). \tag{5}$$

Surprisingly, (5) turns out to be a set of sufficient conditions for GAS of (1) (see, Thm. 1, with $\beta = c/k - \mu_2(A)$.) In fact, Thm. 1 gives a set of sufficient conditions which depends on a parameter β. With different choice of β (say $\beta = 1, 2$ or $1 - \mu_2(A)$), we recover a series of known results that are widely used by many researchers. We organize our paper as follows.

In section 2, we present our main result (Thm. 1) and states its three corollaries, corresponding to $\beta = 1, 2$ and $1 - \mu_2(A)$ respectively. In order to establish the significance of our general result, in section 3 we conduct extensive comparison with existing criteria that are widely used. In particular, we establish (i) Corollary 1 unifies and improves a string of results, initiated by Arik and Tavsanoglu [4] and extended by several others such as Arik [2], Liao and Wang [17, 18] and Cao [5] (see, Prop. 1); (ii) Corollary 2 is equivalent to a recent result of Arik [2] (see, Prop. 2); and most importantly, (iii) Corollary 3 is "best" from the point of nonsmooth analysis (see, Prop. 3). We will also conduct comparison with several results obtained via LMI approach reported in [19, 20, 21, 26, 27] and results in [23]. In particular, comparison with a very general result by Singh [26] and Lu, Rong and Chen [21] shows the significance of Cor. 3 (see, Props. 4 and 5.)

2 The Main Result and Its Corollaries

We state our main result as follows. It depends on a positive parameter β . With different choices of β, our main result gives a series of results that we are going to discuss in some depth. The second part of the proof makes use of the Lyapunov function of Arik [2].

Theorem 1 *Suppose that each activation function f_i satisfies the property (H) and the following conditions hold:*
(i) $\mu_2(A) < c/k$; and
(ii) $\|A^\tau\|_2 < \beta\sqrt{\frac{2}{\beta}(c/k - \mu_2(A)) - 1}$ for some $\beta \in (0, 2(c/k - \mu_2(A)))$.
Then, for each input vector $u \in \Re^n$, (1) has a unique equilibrium point which is GAS.

Proof. We note that conditions in Thm. 1 are well defined. Under condition (i), the interval $(0, 2(c/k - \mu_2(A))$ is nonempty and for any β from this interval, the constant $\frac{2}{\beta}(c/k - \mu_2(A)) - 1$ is positive, implying the set of A^τ satisfying (ii) nonempty.

Step 1. (Existence and uniqueness of equilibrium points) In this part we prove condition (4) holds so that according to [23, Thm. 2 (iii)] that (1) has a unique and equilibrium point for any input $u \in \Re^n$. First we note that the largest eigenvalue function $\lambda_{\max}(\cdot)$ on the space of symmetric matrices is convex. Hence we have

$$\begin{aligned}
\mu_2(A + A^\tau) &= \frac{1}{2}\lambda_{\max}(A + A^T + A^\tau + (A^\tau)^T) \\
&\leq \frac{1}{2}\lambda_{\max}(A + A^T) + \frac{1}{2}\lambda_{\max}(A^\tau + (A^\tau)^T) \\
&= \mu_2(A) + \mu_2(A^\tau) \leq \mu_2(A) + \|A^\tau\|_2 \\
&< \mu_2(A) + \beta\sqrt{\frac{2}{\beta}(c/k - \mu_2(A)) - 1}.
\end{aligned} \tag{6}$$

The strict inequality above used the condition (ii) in Thm. 1. From the inequality

$$(\beta - (c/k - \mu_2(A)))^2 \geq 0,$$

we obtain that

$$(c/k - \mu_2(A))^2 \geq 2\beta(c/k - \mu_2(A)) - \beta^2.$$

Since both sides of the above inequality are nonnegative, we have

$$c/k - \mu_2(A) \geq \sqrt{2\beta(c/k - \mu_2(A)) - \beta^2},$$

which gives

$$\mu_2(A) + \beta\sqrt{\frac{2}{\beta}(c/k - \mu_2(A)) - 1} \leq c/k.$$

Hence, under conditions of Thm. 1, inequalities in (6) implies condition (4).

Step 2. (Global convergence) In this part, we use the Lyapunov theory to prove the global convergence of any solution of Eq. (1) to its unique equilibrium point, denoted by x^*. For simplicity, we shift x^* to the origin through the transformation:

$$z(t) = x(t) - x^* \qquad \text{and} \qquad z(t-\tau) = x(t-\tau) - x^*.$$

Eq. (1) then can be equivalently written as the following system

$$\dot{z}(t) = -C(z(t)) + A\Phi(z(t)) + A^\tau\Phi(z(t-\tau)) \tag{7}$$

where $z(\cdot) = (z_1(\cdot), \ldots, z_n(\cdot))^T$, $\Phi(z(\cdot)) = (\phi_1(z_1(\cdot)), \ldots, \phi_n(z_n(\cdot)))^T$, and

$$\phi_i(z_i(\cdot)) = f_i(z_i(\cdot) + x_i^*) - f_i(x_i^*).$$

We now show that the origin is GAS of (7).

It is easy to see

$$\phi_i(0) = 0, \ \forall i = 1, 2, \ldots, n \quad \text{and} \quad \|\Phi(z(\cdot))\|^2 \leq kz^T(\cdot)\Phi(z(\cdot)). \tag{8}$$

The second relation in (8) follows from the assumption (H). We now employ the Lyapunov function used by Arik [2]

$$\begin{aligned}
V(z(t)) = \alpha\|z(t)\|^2 + \frac{2}{\beta}\sum_{i=1}^{n}\int_0^{z_i}\phi_i(s)ds \quad &+ \quad \sum_{i=1}^{n}\int_{t-\tau}^{t}\phi_i^2(z_i(\xi))d\xi \\
&+ \quad \frac{\alpha}{c}\int_{t-\tau}^{t}\Phi^T(z(\xi))((A^\tau)^T A^\tau)\Phi(z(\xi))d\xi,
\end{aligned}$$

with $\alpha > 0$ being chosen appropriately later on and β is the constant in Thm. 1. We first point out that $V(z(\cdot))$ is positive except at the origin, and it is radially unbounded in the sense that $V(z(t)) \to \infty$ as $\|z(t)\| \to \infty$. Next evaluating the time derivative of $V(z)$ along the trajectories of the system (7), we obtain

$$\begin{aligned}\dot{V}(z(t)) \;=\;& -2\alpha z^T(t)Cz(t) + 2\alpha z^T(t)A\Phi(z(t)) + 2\alpha z^T(t)A^\tau\Phi(z(t-\tau)) \\ & -\frac{2}{\beta}\Phi^T(z(t))Cz(t) + \frac{2}{\beta}\Phi^T(z(t))A\Phi(z(t)) + \frac{2}{\beta}\Phi^T(z(t))A^\tau\Phi(z(t-\tau)) \\ & +\Phi^T(z(t))\Phi(z(t)) - \Phi^T(z(t-\tau))\Phi(z(t-\tau)) \\ & +\frac{\alpha}{c}\Phi^T(z(t))(A^\tau)^TA^\tau\Phi(z(t)) - \frac{\alpha}{c}\Phi^T(z(t-\tau))(A^\tau)^TA^\tau\Phi(z(t-\tau)). \qquad (9)\end{aligned}$$

It follows from the inequalities

$$z^T(t)Cz(t) \geq c\|z(t)\|^2 \quad \text{and} \quad \Phi^T(z(t))Cz(t) \geq \frac{c}{k}\|\Phi(z(t))\|^2$$

that

$$\begin{aligned}\dot{V}(z(t) \;\leq\;& -2c\alpha\|z(t)\|^2 + 2\alpha z^T(t)A\Phi(z(t)) + 2\alpha z^T(t)A^\tau\Phi(z(t-\tau)) \\ & -\frac{2}{\beta}\frac{c}{k}\|\Phi(z(t))\|^2 + \frac{2}{\beta}\Phi^T(z(t))A\Phi(z(t)) + \frac{2}{\beta}\Phi^T(z(t))A^\tau\Phi(z(t-\tau)) \\ & +\Phi^T(z(t))\Phi(z(t)) - \Phi^T(z(t-\tau))\Phi(z(t-\tau)) \\ & +\frac{\alpha}{c}\Phi^T(z(t))(A^\tau)^TA^\tau\Phi(z(t)) - \frac{\alpha}{c}\Phi^T(z(t-\tau))(A^\tau)^TA^\tau\Phi(z(t-\tau)). \qquad (10)\end{aligned}$$

Noticing that

$$\begin{aligned}& -c\alpha\|z(t)\|^2 + 2\alpha z^T(t)A\Phi(z(t)) \\ & = -\alpha\|\sqrt{c}z(t) - \frac{1}{\sqrt{c}}A\Phi(z(t))\|^2 + \frac{\alpha}{c}\Phi^T(z(t))A^TA\Phi(z(t)),\end{aligned}$$

$$\begin{aligned}& -c\alpha\|z(t)\|^2 + 2\alpha z^T(t)A^\tau\Phi(z(t-\tau)) \\ & = -\alpha\|\sqrt{c}z(t) - \frac{1}{\sqrt{c}}A^\tau\Phi(z(t-\tau))\|^2 + \frac{\alpha}{c}\Phi^T(z(t-\tau))(A^\tau)^TA^\tau\Phi(z(t-\tau)),\end{aligned}$$

and

$$\begin{aligned}& -\|\Phi(z(t-\tau))\|^2 + \frac{2}{\beta}\Phi^T(z(t))A^\tau\Phi(z(t-\tau)) \\ & = -\|\Phi(z(t-\tau)) - \frac{1}{\beta}(A^\tau)^T\Phi(z(t))\|^2 + \frac{1}{\beta^2}\Phi^T(z(t))A^\tau(A^\tau)^T\Phi(z(t)).\end{aligned}$$

Rearranging the terms in (10) and using above inequalities, we obtain

$$\dot{V}(z(t)) \;\leq\; \frac{\alpha}{c}\Phi^T(z(t))A^TA\Phi(z(t)) - \frac{2}{\beta}\frac{c}{k}\|\Phi(z(t))\|^2 + \frac{2}{\beta}\Phi^T(z(t))A\Phi(z(t))$$

$$
\begin{aligned}
& +\|\Phi(z(t))\|^2 + \frac{1}{\beta^2}\Phi^T(z(t))A^\tau(A^\tau)^T\Phi(z(t)) + \frac{\alpha}{c}\Phi^T(z(t))(A^\tau)^T A^\tau\Phi(z(t)) \\
\leq\; & \frac{\alpha}{c}\left(\|A\|_2^2 + \|A^\tau\|_2^2\right)\|\Phi(z(t))\|^2 \\
& -\frac{2}{\beta}\frac{c}{k}\|\Phi(z(t))\|^2 + \frac{1}{\beta}\Phi^T(z(t))(A+A^T)\Phi(z(t)) \\
& +\|\Phi(z(t))\|^2 + \frac{1}{\beta^2}\|A^\tau\|_2^2\|\Phi(z(t))\|^2 \\
\leq\; & \frac{\alpha}{c}\left(\|A\|_2^2 + \|A^\tau\|_2^2\right)\|\Phi(z(t))\|^2 + \left(\frac{1}{\beta^2}\|A^\tau\|_2^2 + 1 - \frac{2}{\beta}\frac{c}{k} + \frac{2}{\beta}\mu_2(A)\right)\|\Phi(z(t))\|^2.
\end{aligned}
$$

Let

$$
\begin{aligned}
\kappa(\beta) \;&:=\; \frac{1}{\beta^2}\|A^\tau\|_2^2 + 1 - \frac{2}{\beta}\frac{c}{k} + \frac{2}{\beta}\mu_2(A) \\
&=\; \frac{1}{\beta^2}\|A^\tau\|_2^2 + \left(1 - \frac{2}{\beta}\left(\frac{c}{k} - \mu_2(A)\right)\right).
\end{aligned}
$$

Obviously, $\kappa(\beta) < 0$ under the conditions in Thm. 1. Now we consider the following three cases.

(i) $\Phi(z(t)) \neq 0$. In this case, $z(t) \neq 0$. The choice of α satisfying

$$
0 < \alpha < -\frac{c\kappa(\beta)}{\|A\|_2^2 + \|A^\tau\|_2^2}
$$

ensures that $\dot V(z(t))$ is negative.

(ii) $\Phi(z(t)) = 0$, but $z(t) \neq 0$. Then it follows from (10) that

$$
\begin{aligned}
\dot V(z(t)) \;&\leq\; -2c\alpha\|z(t)\|^2 + 2\alpha z^T(t)A^\tau\Phi(z(t-\tau)) - \|\Phi(z(t-\tau))\|^2 \\
&\quad -\frac{\alpha}{c}\Phi^T(z(t-\tau))(A^\tau)^T A^\tau\Phi(z(t-\tau)) \\
&=\; -c\alpha\|z(t)\|^2 - \alpha\|\sqrt{c}z(t) - \frac{1}{\sqrt{c}}A^\tau\Phi(z(t-\tau))\|^2 - \|\Phi(z(t-\tau))\|^2 \\
&<\; 0.
\end{aligned}
$$

(iii) $z(t) = 0$. Clearly, $\Phi(z(t)) = 0$ due the fact (8). In this case, $\dot V(z(t))$ is given by

$$
\dot V(z(t)) = -\|\Phi(z(t-\tau))\|^2 - \frac{\alpha}{c}\Phi^T(z(t-\tau))(A^\tau)^T A^\tau\Phi(z(t-\tau)).
$$

Hence, $\dot V(z(t))$ is negative if $\Phi(z(t-\tau)) \neq 0$, and $\dot V(z(t)) = 0$ if and only if it happens in the last case where

$$
z(t) = \Phi(z(t)) = \Phi(z(t-\tau)) = 0.
$$

We recall that $V(z(t))$ is radially unbounded. According to Cor. 3.2 in [15, Ch.3] or [16] that the origin of (7) or equivalently the equilibrium point x^* of (1) is GAS. □

The significance of Thm. 1 can be clearly seen when f_i is taken of the form (2) and/or (3) where $c = k = 1$, and when β takes some specific values.

Corollary 1 *(with $\beta = 1$) Suppose that each activation function f_i is given by (2) and/or (3) with $C = I$, the identity matrix in $\Re^{n\times n}$, and the following conditions hold:*
(i) $\mu_2(A) < 1/2$; and
(ii) $\|A^\tau\|_2 < \sqrt{1 - 2\mu_2(A)}$.
Then, for each input vector $u \in \Re^n$, (1) has a unique equilibrium point which is GAS.

Corollary 2 *(with $\beta = 2$) Suppose that each activation function f_i is given by (2) and/or (3) with $C = I$, the identity matrix in $\Re^{n\times n}$, and the following conditions hold:*
(i) $\mu_2(A) < 0$; and
(ii) $\|A^\tau\|_2 < 2\sqrt{-\mu_2(A)}$.
Then, for each input vector $u \in \Re^n$, (1) has a unique equilibrium point which is GAS.

Corollary 3 *(with $\beta = 1 - \mu_2(A)$) Suppose that each activation function f_i is given by (2) and/or (3) with $C = I$, the identity matrix in $\Re^{n\times n}$, and the following conditions hold:*
(i) $\mu_2(A) < 1$; and
(ii) $\|A^\tau\|_2 < 1 - \mu_2(A)$.
Then, for each input vector $u \in \Re^n$, (1) has a unique equilibrium point which is GAS.

We remark that those corollaries follows from Thm. 1 by just verifying that the value of $\beta = 1, 2, 1 - \mu_2(A)$ belongs to the interval $(0, 2(1 - \mu_2(A)))$ under respective assumptions in those corollaries.

3 Comparison and Examples

To make our comparison as simple as possible, we take the simplest version of Thm. 1 when f is taken of the form of (2) and $C = I$. Hence, comparison is given on Corollaries 1-3 with some other existing results reported in [2, 4, 5, 17, 18, 19, 21, 23, 26, 27].

3.1 Comparison on Corollary 1

In this subsection, we compare Cor. 1 with a string of results, initiated by Arik and Tavsanoglu [4] and extended by Liao and Wang [17, 18] and Cao [5]. We list this string of results for easy reference.

Theorem 2 *[4] Let f_i take the form (2) and $C = I$. If the following conditions hold*
(i) $-(A + A^T)$ is positive definite; and
(ii) $\|A^\tau\|_2 \leq 1$;
then (1) has a unique equilibrium point which is GAS.

Conditions in Thm. 2 are weakened to the following by Liao and Wang [18].

Theorem 3 *[18] Let f_i take the form (2) and $C = I$. If there exists a number $r \geq 0$ such that the following conditions hold*
(i) $-(A + A^T + rI)$ is positive definite; and
(ii) $\|A^\tau\|_2 \leq \sqrt{1 + r}$;
then (1) has a unique equilibrium point which is GAS.

An intermediate version of Thm. 3 appeared in [17], where the condition $\|A^\tau\|_2 \leq \sqrt{1+r}$ is replaced by $\|A^\tau\|_2 \leq \sqrt{1+r/2}$. The latest version of Thm. 2 is reported by Cao [5].

Theorem 4 *[5] Let f_i take the form (2) and $C = I$. If there exists a number $r \geq 0$ such that the following conditions hold*
(i) $-(A + A^T + rI)$ is positive definite; and
(ii) $\|A^\tau\|_2 < \sqrt{1+r+s}$, where $s = \lambda_{\min}[-(A + A^T + rI)] > 0$;
then (1) has a unique equilibrium point which is GAS.

Note the difference that condition (ii) in Thm. 4 is in strict 'less than' inequality (i.e., $<$), while Thm. 2 and Thm. 3 assume 'not greater than' inequality (i.e., $\leq$), and we stress that the strict inequality in Thm. 4 (ii) cannot be relaxed to 'not greater than' inequality. It is also easy to see that conditions in Thm. 4 are the weakest among the three results, but they share a common feature that the matrix $(A + A^T)$ is negative definite. The following lemma shows that conditions in Cor. 1 are weaker than that in Thm. 4.

Proposition 1 *Conditions in Thm. 4 imply conditions in Cor. 1.*

Proof. Suppose that conditions in Thm. 4 hold. Since the matrix $-(A + A^T + rI)$ is positive definite and $r \geq 0$, the matrix $(A + A^T)$ must be negative definite. Its largest eigenvalue must be negative, i.e., $\mu_2(A) < 0$. It obviously holds $\mu_2(A) \leq 1/2$. Condition (i) in Cor. 1 holds. We also observe that

$$\begin{aligned} s &= \lambda_{\min}[-(A + A^T + rI)] \\ &= -\lambda_{\max}(A + A^T + rI) \\ &= -\lambda_{\max}(A + A^T) - r \\ &= -2\mu_2(A) - r. \end{aligned}$$

Hence, we have from condition (ii) of Thm. 4 that

$$\|A^\tau\|_2 < \sqrt{1+r+s} = \sqrt{1 + r - 2\mu_2(A) - r} = \sqrt{1 - 2\mu_2(A)}.$$

That is, condition (ii) in Cor. 1 holds. This completes the proof. □

According to the proof, under the condition that the matrix $(A+A^T)$ is negative definite, we have $\sqrt{1 - 2\mu_2(A)} = \sqrt{1+r+s}$. That is, Thm. 4 and Cor. 1 are equivalent. However, Cor. 1 allows the matrix $(A + A^T)$ positive definite as well as non-negative definite. This fact is reflected in examples 1 and 3 below. Therefore, when f_i takes the form (2) and $C = I$, the following implication holds:

$$\text{Cs in Thm. 2} \implies \text{Cs in Thm. 3} \implies \text{Cs in Thm. 4} \implies \text{Cs in Cor. 1},$$

where Cs stands for Conditions. In other words, Conditions in Cor. 1 are the weakest among all those results. We now employ examples to show that Cor. 1 yields strict improvement over this string of results.

Example 1: Consider the following matrices:

$$A = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix} \quad \text{and} \quad A^{\tau} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

Since the matrix $(A + A^T)$ is positive definite, the matrix $-(A + A^T + rI)$ is never positive definite for all $r \geq 0$. Hence results in Theorems 2 to 4 cannot be applied to this example. However, the conditions of Cor. 1 are satisfied, since

$$\mu_2(A) = 0.2 < 0.5 \quad \text{and} \quad \|A^{\tau}\|_2 = 0.5 < \sqrt{0.6} = \sqrt{1 - 2\mu_2(A)}.$$

Hence, Eq. (1) has a unique and globally asymptotically stable equilibrium point. □

Example 2: This example was taken from [18] and has been used in a couple of papers to illustrate the theory involved. Consider the following matrices:

$$A = \begin{pmatrix} -a & -1 \\ 1 & -b \end{pmatrix} \quad \text{and} \quad A^{\tau} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$$

where a, b are two positive constants. It is easy to see $\|A^{\tau}\|_2 = \sqrt{2}$. Since the matrix $(A + A^T)$ is negative definite, Cor. 1 and Thm. 4 are equivalent. We note that

$$\mu_2(A) = \frac{1}{2}\lambda_{\max}(A + A^T) = \max(-a, -b) = -\min(a, b) < 0 < 0.5,$$

and the condition $\|A^{\tau}\|_2 < \sqrt{1 - 2\mu_2(A)}$ gives $a > 1/2$ and $b > 1/2$. Hence the stability region, according to Cor. 1 (also Thm. 4) is

$$S_{ab}^{>} = \{(a, b) \mid a > 1/2, \ b > 1/2\}.$$

This region contains the stability region S_{ab}:

$$S_{ab} = \{(a, b) \mid a > 1, \ 1 < b < 1 + \sqrt{2}, \ 2ab + (2 - \sqrt{2})(a + b) < 4\sqrt{2}\}$$

which was reported in [18] by using their result of Thm. 3. □

Example 3: We slightly modify *Example 2* so that the parameters (a, b) can take any values. Let

$$A = \begin{pmatrix} -a & -1 \\ 1 & -b \end{pmatrix} \quad \text{and} \quad A^{\tau} = \frac{1}{4}\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}.$$

It is easy to see $\|A^{\tau}\|_2 = \sqrt{2}/2$. We first consider the case that there is at least one of a and b taking non-positive values, i.e., $\max(-a, -b) \geq 0$. Since the matrix $(A + A^T)$ is not negative definite for this case, all results in Thm. 2 to Thm. 4 can not be applied to this case. Notice that

$$\mu_2(A) = \max(-a, -b).$$

According to Cor. 1, we let

$$\mu_2(A) \le \frac{1}{2} \quad \text{and} \quad \|A^\tau\|_2 < \sqrt{1 - 2\mu_2(A)}$$

which amounts to $\mu_2(A) < 1/4$, i.e.,

$$a > -1/4 \quad \text{and} \quad b > -1/4.$$

Now we consider the case where $a > 0, b > 0$. In this case, the matrix $(A + A^T)$ is negative definite and $\mu_2(A) < 0$. Furthermore, the condition $\|A^\tau\|_2 < \sqrt{1 - 2\mu_2(A)}$ holds for this case. We summarize that the stability region for this example is

$$S_{ab} = \{(a, b) \mid a > -1/4,\ b > -1/4\},$$

which allows a and b take negative values. $\square$

3.2 Comparison on Corollary 2

In a different format from $\|A^\tau\|_2 \le \sqrt{1 + r\mu_2(A)}$ (r is a constant), a result was recently reported by Arik [2].

Theorem 5 *[2] Let f_i take the form (2) and $C = I$. If there exists a number $r > 0$ such that the following conditions hold*
(i) $-(A + A^T + rI)$ is positive definite; and
(ii) $\|A^\tau\|_2 \le \sqrt{2r}$;
then (1) has a unique equilibrium point which is GAS.

In fact, this result as we show below is equivalent to Cor. 2

Proposition 2 *Cor. 2 and Thm. 5 are equivalent.*

Proof. First we assume that (A, A^τ) satisfy conditions in Cor. 2. Let

$$r := \frac{1}{2}\|A^\tau\|_2^2.$$

Following Condition (ii) in Cor. 2, we have

$$\sqrt{2r} = \|A^\tau\| < 2\sqrt{-\mu_2(A)}.$$

Hence, $r < -2\mu_2(A)$, implying that

$$\lambda_{\min}(-(A + A^T + rI)) = -\lambda_{\max}(A + A^T) - r = -2\mu_2(A) - r > 0.$$

That is, we have found a positive r such that conditions in Thm. 5 hold.

Conversely, let (A, A^τ) satisfy conditions in Thm. 5. Certainly, $\mu_2(A) < -r/2 < 0$ and

$$\|A^\tau\| \leq \sqrt{2r} < \sqrt{-4\mu_2(A)} = 2\sqrt{-\mu_2(A)}.$$

That is, conditions in Cor. 2 hold. This finishes our proof. □

Apart from the fact that Cor. 1 allows $(A + A^T)$ to be indefinite while Cor. 2 does not, they are even different for the same case of negative definite $(A + A^T)$ because we have

$$\sqrt{1 - 2\mu_2(A)} \leq 2\sqrt{-\mu_2(A)} \quad \text{for } \mu_2(A) \leq -\frac{1}{2}$$

and

$$\sqrt{1 - 2\mu_2(A)} \geq 2\sqrt{-\mu_2(A)} \quad \text{for } -\frac{1}{2} \leq \mu_2(A) \leq 0.$$

In other words, for $\mu_2(A) \in (-\infty, -1/2]$ Cor. 2 gives a better Criterion than Cor. 1 and Cor. 1 gives a better criterion than Cor. 2 for $\mu_2(A) \in [-1/2, 0]$.

3.3 Comparison on Corollary 3

The most interesting aspect of Cor. 3 is its simplicity. We first establish a fact that $1-\mu_2(A)$ is the best possible bound we can have for $\|A^\tau\|_2$ from the point of nonsmooth analysis.

Proposition 3 $\|A^\tau\|_2 < 1 - \mu_2(A)$ *is the best bound we can have from Thm. 1.*

Proof. Let $h(\beta)$ be the bound on $\|A^\tau\|_2$ in Thm. 1 for the case $c = k = 1$, i.e.,

$$h(\beta) := \beta\sqrt{\frac{2}{\beta}(1 - \mu_2(A)) - 1}, \quad \beta \in (0, 2(1 - \mu_2(A)).$$

We want to find the largest value of $h(\beta)$ on the interval $(0, 2(1 - \mu_2(A))$. Since the square of $h(\beta)$ does not change the location of the maximum of $h(\beta)$, we consider

$$\ell(\beta) := h^2(\beta) = 2\beta(1 - \mu_2(A)) - \beta^2,$$

and note that the condition

$$0 = \ell'(\beta) = 2(1 - \mu_2(A)) - 2\beta$$

gives $\beta = 1 - \mu_2(A)$. Since $\ell(\beta)$ is concave, $\beta = 1 - \mu_2(A)$ gives the largest value of $h(\beta)$ over the interval $(0, 2(1 - \mu_2(A))$. This choice of β gives the bound

$$\|A^\tau\|_2 < h(\beta) = 1 - \mu_2(A),$$

which is the best we can have from Thm. 1. □

We note that from the optimality condition in the proof we have

$$h(1 - \mu_2(A)) \geq h(1) \quad \text{and} \quad h(1 - \mu_2(A)) \geq h(2).$$

Hence, Cor. 3 covers Corollaries 1 and 2. We can look at the bound from another point of view. Nonsmooth analysis yields possibly the best sufficient condition $\mu_2(A + A^\tau) < 1$ (cf. (4)) that ensures the existence and uniqueness of an equilibrium point. Following the inequalities

$$\mu_2(A + A^\tau) \le \mu_2(A) + \mu_2(A^\tau) \le \mu_2(A) + \|A^\tau\|_2$$

and the fact that there are matrices A and A^τ making those inequalities become equalities, we know that the best bound on $\|A^\tau\|_2$ is $1 - \mu_2(A)$.

Remark 1. The main result Thm. 1 is also different from the following close-related result, which is also obtained from (4) in [23].

Theorem 6 *[23, Thm. 3] Suppose that each activation function f_i satisfies the property (H) and the following condition (H) holds*

$$2\mu_2(A + A^\tau) + \|A^\tau - I\|_2^2 < \frac{2c}{k}.$$

Then for each $u \in \Re^n$, (1) has a unique equilibrium point which is also GAS.

To see the difference between Thm. 1 and Thm. 6, we make use of *Example 2.*

Example 2: (again) We recall from Example 2 that

$$\|A^\tau\|_2 = \sqrt{2} \ \text{ and } \ \mu_2(A) = -\min(a, b).$$

The condition $\mu_2(A) < 1$ in Cor. 3 holds automatically, and the condition $\|A^\tau\|_2 < 1 - \mu_2(A)$ gives $\min(a, b) > \sqrt{2} - 1$. Hence the stability region, according to Cor. 3 is:

$$S_{ab}^{\ge} := \left\{(a, b) \middle| \ \min(a, b) > \sqrt{2} - 1\right\}.$$

We also recall that the stability region of Example 2 by Thm. 6 is S_{ab}^{+} [23, Remark 6]. It is easy to see that the region $S_{ab}^{\ge}$ is not contained in the region S_{ab}^{+}, and vice versa. In fact, we have

$$(0.6, 0.6) \in S_{ab}^{\ge} \setminus S_{ab}^{+} \quad \text{and} \quad (\sqrt{2} - 1, 3\sqrt{2} + 1) \in S_{ab}^{+} \setminus S_{ab}^{\ge}.$$

Hence the region $S_{ab}^{\ge}$ and S_{ab}^{+} are not contained each other. □

Remark 2. Under the condition that

$$0 < \frac{f_i(\nu_1) - f_i(\nu_2)}{\nu_1 - \nu_2} \le k_i, \quad \forall \nu_1, \nu_2 \in \Re \text{ and } i = 1, \dots, n \tag{11}$$

and the assumption that an equilibrium point x^* exists for (1), it is proved [19, Cor. 6] that x^* is GAS if the following condition holds

$$-I + A + A^T + (A^\tau)^T A^\tau \preceq 0, \tag{12}$$

where a symmetric matrix $X \preceq 0$ means $-X$ is positive semidefinite (i.e., $-X \succeq 0$). We employ Example 2 once again to illustrate the difference of this result from Cor. 3.

Example 2:(again) With this example,

$$-I + A + A^T + (A^\tau)^T A^\tau = \begin{pmatrix} -2a & 1 \\ 1 & -2b \end{pmatrix}.$$

Hence condition (12) holds if and only if $4ab \geq 1$, i.e., the stability region by (12) is:

$$S_{ab}^{\succeq} := \{(a,b) |\ 4ab \geq 1\}.$$

It is an easy task to see that the stability region $S_{ab}^{\geq}$ by Cor. 3 and $S_{ab}^{\succeq}$ do not contain each other. We emphasize other distinctions between Cor. 3 and [19, Cor. 6] in two aspects: First, assumption (11) applies to functions not including many well-known activation functions, e.g., functions (2) and (3), while Cor. 3 applies to these two functions; Second, it is yet to know if conditions (11) and (12) are sufficient to ensure existence as well as uniqueness of equilibrium points of (1), while conditions in Cor. 3 are. □

A slightly different sufficient condition by Singh [27] for the function (2) is

$$-D + (DA + A^T D) + (A^\tau)^T D A^\tau \prec 0, \tag{13}$$

where $X \prec 0$(respectively, $\succ 0$) means that X is negative (respectively, positive) definite and D is a positive definite diagonal matrix. Note that when $D = I$, condition (13) becomes (12) with $\preceq$ replaced by $\prec$. We can also construct examples that condition (13) fails while Cor. 3 is satisfied.

Another closely related result from LMI approach is the one given by Singh [26, Thm. 1]:

Theorem 7 *[26] Suppose there exist positive definite symmetric matrices $P, Q \in \Re^{n\times n}$ and a positive definite diagonal matrix $D \in \Re^{n\times n}$ such that*

$$M = \begin{pmatrix} 2P & -PA & -PA^\tau \\ -A^T P & 2D - Q - DA - A^T D & -DA^\tau \\ -(A^\tau)^T P & -(A^\tau)^T D & Q \end{pmatrix} \succ 0. \tag{14}$$

Then (1) with f given by (2) has a unique equilibrium point which is GAS.

The flexibility of this theorem is with the free choice of P, Q and D. In particular, if

$$P = \alpha I,\ D = \beta_1 I, \ \text{ and } \ Q = \left(\alpha (A^\tau)^T A^\tau + I\right)$$

where α and β_1 are positive constants, Condition (14) becomes

$$2\beta_1 I - \alpha (A^\tau)^T A^\tau - I - \beta_1 (A + A^T) - \alpha A^T A - \beta_1^2 A^\tau (A^\tau)^T \succ 0. \tag{15}$$

This condition covers the known conditions in Theorems 3 and 5 as its special cases [26, Part III]. It is this condition that we want to compare with. First we note that the condition $\mu_2(A) < 1$ is a necessary condition for (15). In fact, (15) certainly implies the matrix $2\beta_1 I - \beta_1 (A + A^T) \succ 0$, which must imply $\mu_2(A) < 1$. Nevertheless, We have the following

Proposition 4 *Condition (15) and Cor. 3 are different from each other.*

Proof. To facilitate our discussion, we define two sets of pairs (A, A^τ) defined respectively by Cor. 3 and Condition (15):

$$\begin{aligned} \mathcal{C} &:= \{(A, A^\tau) |\ \mu_2(A) < 1 \text{ and } \|A^\tau\|_2 < 1 - \mu_2(A)\} \\ \mathcal{C}_1 &:= \{(A, A^\tau) |\ (15) \text{ holds for some } \beta_1 > 0,\ \alpha > 0\}. \end{aligned}$$

We first prove $\mathcal{C} \not\subseteq \mathcal{C}_1$. This is equivalent to prove that for any given $\alpha > 0$ and $\beta_1 > 0$, we can find a pair (A, A^τ) which satisfies Cor. 3 but fails to satisfy (15).

Let $t \in (0, 1)$ be fixed and let $A = tI$ so that

$$0 < \mu_2(A) = t < 1 \quad \text{and} \quad \|A\|_2 = t.$$

We also choose A^τ to be diagonal so that the least eigenvalue $\lambda_{\min}$ of the left-hand side matrix in (15) is

$$\lambda_{\min} = 2\beta_1 - 1 - 2\beta_1\mu_2(A) - \beta_1^2\|A^\tau\|_2^2 - \alpha\left(\|A\|_2^2 + \|A^\tau\|_2^2\right).$$

Choose a constant $\kappa \in (0, 1)$ such that

$$\kappa^2 > 1 - \frac{\alpha\|A\|_2^2}{\beta_1^2(1-t)^2} = 1 - \frac{\alpha t^2}{\beta_1^2(1-t)^2}. \tag{16}$$

Since $\frac{\alpha t^2}{\beta_1^2(1-t)^2} > 0$, such κ always exists. Now we choose A^τ such that

$$\|A^\tau\|_2 = \kappa(1-t) < 1 - t = 1 - \mu_2(A) \qquad (\text{since } \kappa < 1).$$

Hence conditions in Cor. 3 are satisfied by such choice of (A, A^τ). Now we calculate $\lambda_{\min}$.

$$\begin{aligned} \lambda_{\min} &= 2\beta_1 - 1 - 2\beta_1 t - \beta_1^2\kappa^2(1-t)^2 - \alpha\left(\|A\|_2^2 + \|A^\tau\|_2^2\right) \\ &< 2\beta_1(1-t) - 1 - \beta_1^2(1-t)^2 + \beta_1^2(1-t)^2(1-\kappa^2) - \alpha\|A\|_2^2 \\ &= -(1-\beta_1(1-t))^2 - \beta_1^2(1-t)^2\left(\kappa^2 - \left(1 - \frac{\alpha t^2}{\beta_1^2(1-t)^2}\right)\right) \\ &< 0. \end{aligned}$$

The first strict inequality holds because we throw away the negative part $-\alpha\|A^\tau\|_2^2$ and the second strict inequality uses (16). Hence with such choice of (A, A^τ) the left-hand side matrix of (15) is not positive definite. We proved $\mathcal{C} \not\subseteq \mathcal{C}_1$.

We now prove the reverse relation: $\mathcal{C}_1 \not\subseteq \mathcal{C}$. This is equivalent to prove that there exist $\beta_1 > 0$ and $\alpha > 0$ such that there exists a pair $(A, A^\tau) \in \mathcal{C}_1$, but $(A, A^\tau) \notin \mathcal{C}$. We construct the following example:

$$\beta_1 = \frac{25}{8}, \ \alpha = \frac{25}{564}, \ A = \begin{pmatrix} -1/2 & -1 \\ 1 & 1/2 \end{pmatrix}, \ A^\tau = \frac{\sqrt{2}}{5}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is easy to see that $\mu_2(A) = 0.5$ and

$$\|A^\tau\|_2 = 2\sqrt{2}/5 > 0.5 = 1 - 0.5 = 1 - \mu_2(A).$$

Hence $(A, A^\tau) \notin \mathcal{C}$. Let Δ denote the left-hand side matrix in (15). After some calculation we have

$$\Delta = \frac{1}{16}\begin{pmatrix} 109 & -25 \\ -25 & 9 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} 1 & 116/141 \\ 116/141 & 1 \end{pmatrix} \succ 0.$$

Hence $(A, A^\tau) \in \mathcal{C}_1$. This proves $\mathcal{C}_1 \not\subseteq \mathcal{C}$. □

Probably the most closest result to our in the existing literature is the following by Lu, Rong and Chen [21].

Theorem 8 *[21, Prop. 2] Suppose there exists a positive r such that*

$$A + A^T + rI \prec 0 \quad \text{and} \quad \|A^\tau\|_2 \le 1 + r/2. \tag{17}$$

Then (1) with f given by (2) has a unique equilibrium point which is GAS.

It is easy to see the following claim.

Proposition 5 *Sufficient condition (17) is a special case of Cor. 3.*

Proof. It follows from the first condition of (17) that $r < -2\mu_2(A)$ and $\mu_2(A) < 0$, which, together with the second condition in (17) implies $\|A^\tau\| < 1 - \mu_2(A)$. Hence (17) satisfies Cor. 3. □

We end our comparison with a remark that each of [21] and [26] gave a very general conditions (in terms of LMIs) which ensures the GAS of (1). It remains unknown how those LMI conditions are related to each other. In their most useful (probably the simplest) cases, we see from Props. 4 and 5 that they are different from Cor. 3.

4 Discussion

The standard way in showing the global asymptotic stability of a delayed neural network is in two steps: First, one needs to show the existence and uniqueness of equilibrium points of the network; then by employing Lyapunov function, it is to show the global convergence of any solution to the unique equilibrium point. While it seems that there are many Lyapunov functions available for use, not many tools are available in dealing with the existence and uniqueness of equilibrium point. The paper follows the approach of nonsmooth analysis recently introduced in [23] and derive a new set of sufficient conditions for GAS of DCNNs. It appears to us that the nonsmooth analysis approach may also be applied to other neural networks apart from that considered in this paper.

It is interesting to summarize the existing well-known sufficient conditions and state their relations known to us.

$$\begin{aligned}
\mathcal{C}_2 &:= \left\{(A, A^\tau)|\ \mu_2(A) < 1/2 \text{ and } \|A^\tau\|_2 < \sqrt{1-2\mu_2(A)}\right\} \text{ (by Cor. 2)} \\
\mathcal{C}_3 &:= \left\{(A, A^\tau)|A + A^T + rI \prec 0 \text{ and } \|A^\tau\|_2 \leq 1 + r/2\} \text{ for some } r > 0\right\} \text{ ([21])} \\
\mathcal{C}_4 &:= \left\{(A, A^\tau)|A + A^T + rI \prec 0 \text{ and } \|A^\tau\|_2 < \sqrt{1+r} \text{ for some } r > 0\right\} \text{ ([18])} \\
\mathcal{C}_5 &:= \left\{(A, A^\tau)|A + A^T + rI \prec 0 \text{ and } \|A^\tau\|_2 \leq \sqrt{2r} \text{ for some } r > 0\right\} \text{ ([2])} \\
&= \left\{(A, A^\tau)|\ \mu_2(A) < 0 \text{ and } \|A^\tau\|_2 < 2\sqrt{\mu_2(A)}\right\} \text{ (by Prop. 2).}
\end{aligned}$$

Results in Section 3 give the following relations:

$$\mathcal{C} \supseteq \mathcal{C}_3 \supseteq \mathcal{C}_4 \cup C_5, \quad \text{and} \quad \mathcal{C} \supseteq \mathcal{C}_2 \supseteq \mathcal{C}_4.$$

We note that [26] gives $\mathcal{C}_1 \supseteq \mathcal{C}_4 \cup \mathcal{C}_5$. There are many papers reporting new sufficient conditions for GAS. Generally speaking, it is very hard to judge (without much prejudice) which condition is "good". We think that one may be not asking too much if we require such a condition to include $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$ individually or union of them as its subsets, because those sets contain really ideal pairs of (A, A^τ) in certain sense (e.g., negative definiteness).

From the viewpoint of nonsmooth analysis, we argue that Cor. 3 provides the best possible bounds on $\|A^\tau\|_2$. We also compare it with a number of closed related results derived by the LMI approach and known to be able to unify several existing results. It is interesting to see if nonsmooth analysis can be combined with LMI approach to develop more flexible combined conditions on A and A^τ. We leave this with our future research.

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