Improved Iterative Hard Thresholding for Sparsity and Nonnegativity Constrained Optimization

Lili Pan1,2, Shenglong Zhou3, Naihua Xiu1, Houduo Qi3

Abstract

The iterative hard thresholding (IHT) algorithm is a popular greedy-type method in (linear and nonlinear) compressed sensing and sparse optimization problems. In this paper, we give an improved iterative hard thresholding algorithm for solving the sparsity and nonnegativity constrained optimization (SNCO) by employing the Armijo-type stepsize rule, which automatically adjusts the stepsize and support set and leads to a sufficient decrease of the objective function each iteration. Consequently, the improved IHT algorithm enjoys several convergence properties under standard assumptions. Those include the convergence to α-stationary point (also known as L-stationary point in literature if the objective function has Lipschitzian gradient) and the finite identification of the true support set. We also characterize when the full sequence converges to a local minimizer of SNCO and establish its linear convergence rate. Extensive numerical experiments are included to demonstrate the good performance of the proposed algorithm.

Keywords: sparsity constrained optimization, improved iterative hard thresholding, convergence, convergence rate, numerical experiment

Mathematical Subject Classification: 90C26, 90C30, 90C90

1 Introduction

In this paper, we are concerned with efficient numerical methods for the sparsity and nonnegativity constrained optimization problem (SNCO for short):

\[
\min f(x), \quad \text{s.t.} \quad x \in S \cap \mathbb{R}_+^n,
\]

where \( f(x) : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable and is bounded from below, \( s < n \) is a positive integer and defines the sparse set \( S := \{ x \in \mathbb{R}^n \mid \| x \|_0 \leq s \} \) with \( \| x \|_0 \) being the \( l_0 \)-norm of \( x \) (the number of nonzero elements in \( x \)) and \( \mathbb{R}_+^n \) is the nonnegative orthant in \( \mathbb{R}^n \). Important examples covered by (1.1) include the linear compressed sensing problem of \( f(x) = f_A(x) := \| Ax - b \|_2^2 \) with \( A \in \mathbb{R}^{m \times n} \) being a linear measurement matrix and \( b \in \mathbb{R}^m \) is the observation vector (see, e.g., [19] for an exclusive treatment of this problem) and the nonlinear compressed sensing problem of \( f(x) = f_\Phi(x) := \| \Phi(x) - b \|_2^2 \) with \( \Phi : \mathbb{R}^n \mapsto \mathbb{R}^m \) being a nonlinear measurement function [8]. Another important example is \( f(x) \) being a regularized logistic regression cost function [2, Sect. 4]. Those examples of (1.1) may or may not have the nonnegativity constraint.

*The work was supported in part by the National Natural Science Foundation of China (11431002, 71271021).
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We include it mainly because of the two reasons. One is that in many real-world problems the underlying parameters represent quantities that can only take on nonnegative values, e.g., amounts of materials, chemical concentrations, pixel intensities, to name a few [16]. Another reason is that it is one of the prototype examples of the symmetric set considered by Beck and Hallak [6] in their nonlinear sparse optimization. We hope that by including the nonnegativity constraint some of our results may have their counterparts when a more general symmetric set is used instead of $\mathbb{R}_+^n$.

It is usually expected that numerical methods for the linear (nonlinear) compressed sensing should be naturally extended to solve (1.1). A class of such methods are of the greedy methods. One advantage of these methods is that they are generally faster than the relaxation approaches. Another advantage is that many of them have stable recovery properties under some conditions [15]. A variety of greedy methods have been proposed in compressed sensing, such as matching pursuit (MP) [28], orthogonal MP (OMP) [18], compressive sampling matching pursuit (CoSaMP) [30], subspace pursuit (SP) [17], hard thresholding pursuit (HTP) [20], conjugate gradient iterative hard thresholding (CGIHT) [7], to name just a few. Understandably, some of those methods have been extended to the sparsity constrained nonlinear optimization. For example, Bahmani et al. [2] proposed a gradient hard-thresholding method which generalizes CoSaMP. Yuan et al. [38] generalized HTP to the sparsity constrained convex optimization. Yuan and Liu [39] proposed a Newton greedy pursuit (NTGP) method to approximately minimize a twice differentiable function over the sparsity constraint.

In particular, the iterative hard thresholding (IHT) algorithm, a popular greedy method which was proposed for the linear compressed sensing problem by Blumensath and Davies in [9, 10] (later was extended to the nonlinear case by Blumensath [8]), has attracted much attention due to its nice recovery properties. For example, when the matrix $A$ in defining $f_A$ is of full row-rank and its spectral norm satisfies $\|A\|_2 < 1$, IHT converges to a local minimum [9]. Furthermore, it was observed in [11] that the algorithm may fail to converge if the spectral norm condition is violated. They then proposed a normalised IHT (NIHT) with an adaptive stepsize by the line search strategy and proved its convergence to a local minimum if $A$ is of full row-rank and is $s$-regular (i.e., any $s$ columns of $A$ are linearly independent). A latest result of Cartis and Thompson [15] showed that NIHT converges to a local minimum if the matrix $A$ is 2s-regular.

There recently emerges a new line of research on those problems mainly attempted from the optimization community [1, 5, 6, 33, 34, 26], which tend to ask the following fundamental questions:

(Q1) Towards what stationary points that a generated sequence converges?

(Q2) Under what conditions that such a stationary point may become a local/global minimizer?

(Q3) What is the convergence rate to a local/global minimizer if the convergence is taking place?

There are two key elements that seem to be indispensable in the delicate analysis among the existing literature in answering those questions. One is on introducing a well characterized stationarity appropriate to the data in hand and the other is on a well defined stepsize rule that is to force certain sufficient decrease in the merit function used in the respective algorithms. For example, assuming that the function $f$ has Lipschitzian gradient, Beck and Eldar [5] introduced
the $L$-stationarity (among others) and by using a fixed or the accurate minimization stepsize rule, they established the convergence to the $L$-stationary point of various algorithms including IHT. See [6] for further results along this line on the sparse optimization problem with a symmetric constraint set. The results in [6] were further significantly enhanced by Lu [26] by employing a nonmonotone line search stepsize rule. When $f$ is nondifferentiable, Attouch et al. [1] introduced a concept of critical point and showed that a few classes of algorithms actually converge to such a critical point. In particular, a variant of IHT with a fixed (or varying) stepsize on the linear compressed sensing problem is proved to converge to a critical point [1, Example 5.4], which in this special case is also the $L$-stationary point of Beck and Eldar [5]. We note that (Q3) is hardly addressed in those researches.

In contrast to the researches reviewed above, the stationarities studied by Pan et al. [33, ?] followed the classical derivation of optimality conditions for nonconvex programming and are based on Bouligand or Clarke tangent cones for nonconvex sets (see [12, Section 6.3] for the definitions of those two cones). This leads to $B$-, $C$- and $\alpha$-stationarities. Their relationships to the $L$-stationarity (and others) have been briefly discussed in [6, Remark 5.3]. The blank assumption used in [33, 34] is that $f$ is continuously differentiable (its gradient is not necessarily Lipschitzian).

In this paper, we continue the research of [34, 33] by applying their stationarities to the algorithm of IHT with the Armijo stepsize rule to solve (1.1). In answering the questions (Q1)-(Q3), we asked whether our obtained results have been as general as they can be. This effort has led to the important relationships among the global/local minimizer and the three stationary points ($\alpha$-, $B$-, and $C$-stationarities) in Theorem 2.1 and Figure 1, which clearly show what extra conditions are required for one to become another. This theorem is fundamental to our algorithmic analysis later on. It turns out that the extra conditions needed are satisfied by the restricted strong convexity and restricted strong smoothness of $f$. Both of the concepts are introduced and popularized in [31]. The resulting IHT enjoys a number of very nice convergence properties. We single out a few that partially answered the questions (Q1)-(Q3):

(i) (for Q1) Any accumulation point of iterative sequence is the $\alpha$-stationary point of SNCO if the objective function $f$ is $2s$-restricted strongly smooth (Theorem 3.1).

(ii) (for Q2) The full iterative sequence converges to a local minimizer of SNCO if $f$ is $2s$-restricted strongly smooth and $2s$-restricted strongly convex (Theorem 3.2).

(iii) (for Q3) The sequence of functional values converges at a sublinear rate if $f$ is $2s$-restricted strongly smooth and $2s$-restricted strongly convex (Theorem 3.3). Furthermore, the sequence of iterates converges at a Q-linear rate under the condition that the sparsity constraint is tight at the solution (Theorem 3.4).

In addition, the numerical performance of our improved IHT is also very satisfactory for a large number of commonly tested problems. Finally, we would like to emphasize that some of our convergence results are similar to what have been reported in [6, 26], but under different assumptions and on different algorithms. Our basic assumption is on the continuity of the gradient of $f$. When the gradient is also Lipschitzian, our $\alpha$-startionarity becomes the $L$-stationarity. We will make more comments on the similarities when we report our results in the paper.
This paper is organized as follows. Section 2 presents some technical results on the optimality conditions of (1.1). Section 3 contains the IIHT algorithm for (1.1) and proves its convergence properties. Numerical results are given in Section 4. The last section makes some concluding remarks. For the sake of easy reading, we introduce some notations to end this section.

Table 1: Notations used in the paper.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>The sparse set ${x \in \mathbb{R}^n : |x|_0 \leq s}$;</td>
</tr>
<tr>
<td>$S_+$</td>
<td>The feasible region of (1.1), i.e., $S \cap \mathbb{R}^n_+$;</td>
</tr>
<tr>
<td>supp$(x)$</td>
<td>The support set of $x \in \mathbb{R}^n$, i.e., ${i \in {1, \cdots, n} : x_i \neq 0}$;</td>
</tr>
<tr>
<td>$\Gamma^*$ (or $\Gamma^k$)</td>
<td>The support set of $x^* \in \mathbb{R}^n$ (or $x^k \in \mathbb{R}^n$);</td>
</tr>
<tr>
<td>$\Gamma_{xy}$</td>
<td>The union of support sets between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, i.e., $\text{supp}(x) \cup \text{supp}(y)$;</td>
</tr>
<tr>
<td>$</td>
<td>\Gamma</td>
</tr>
<tr>
<td>$x_{\Gamma}$</td>
<td>The subvector consisting of entries of $x \in \mathbb{R}^n$ indexed by $\Gamma$;</td>
</tr>
<tr>
<td>$A_{\Gamma}$</td>
<td>The submatrix consisting of columns of $A \in \mathbb{R}^{m \times n}$ indexed by $\Gamma$;</td>
</tr>
<tr>
<td>$\nabla f$</td>
<td>The gradient of $f(x)$ on $\mathbb{R}^n$, i.e., $\nabla f(x)$;</td>
</tr>
<tr>
<td>$\nabla_{\Gamma} f(x)$</td>
<td>The subvector of $\nabla f(x)$ indexed on $\Gamma$, i.e., $(\nabla f(x))_{\Gamma}$;</td>
</tr>
<tr>
<td>$N(x, \delta)$</td>
<td>The neighbor region of $x \in \mathbb{R}^n$ with radius $\delta &gt; 0$, i.e., ${y \in \mathbb{R}^n : |y - x| &lt; \delta}$;</td>
</tr>
<tr>
<td>$e_i$</td>
<td>The vector in $\mathbb{R}^n$ whose $i$th component is one and others are zeros;</td>
</tr>
<tr>
<td>$x_{\Gamma}^\perp$</td>
<td>The $i$th largest (in absolute value) element of $x \in \mathbb{R}^n$.</td>
</tr>
<tr>
<td>$\mathbb{R}^n_{\Gamma}$</td>
<td>The subspace of $\mathbb{R}^n$ spanned by ${e_i : i \in \Gamma}$, i.e., span${e_i : i \in \Gamma}$</td>
</tr>
</tbody>
</table>

2 Characterizations of Various Stationarities

In this section, we will give detailed characterizations of the relationships among the three stationary points (namely, $\alpha$-, $B$-, and $C$-stationary point) and the local/global minimizers of (1.1). We will also report some consequences of those characterizations under some additional conditions such as the restricted strong convexity/smoothness of $f$. Those results will be used in the convergence analysis of the improved IIHT algorithm in later sections.

2.1 On the three stationarities

In this part, we assume that $f$ is continuously differentiable. We will use the orthogonal projection onto a closed set $\Omega \subseteq \mathbb{R}^n$ defined as follows:

$$P_{\Omega}(x) := \arg \min \left\{ \|y - x\|^2 \right\}, \quad \text{s.t. } y \in \Omega,$$

where $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^n$. Since $\Omega$ is not convex, there may be multiple optimal solutions. In this case, $P_{\Omega}(x)$ can be any one of them. In particular, when $\Omega = S$, $P_{\Omega}(x)$ can be obtained by setting all but $s$ largest absolute components of $x$ to zero ($P_{\Omega}(x)$ is also known as the support project of $x$ onto $S$). Furthermore, it was proved in [34, Prop. 3.1] that

$$P_{S_+}(x) = P_S\left(P_{\mathbb{R}^n_+}(x)\right).$$
Moreover,

\[ x_{\Gamma_x} = y_{\Gamma_x} \quad \text{for} \quad x = P_{S_+}(y). \]

The B- and C-stationary points are respectively defined through the orthogonal projection onto the Bouligand and Clarke tangent cones of \( S_+ \). In our analysis, we will only use an equivalent characterization of the each cone and will not need their original definitions, which are described in [12, Sect. 6.3] and [34, Sect. 2.1]. We note that the Bouligand tangent cone below can also be derived following [4].

**Proposition 2.1** [34, Thms. 2.1 and 2.2] (Characterizations of Bouligand and Clarke tangent cones). Recall from Table 1, \( \Gamma^* \) is the support set of \( x^* \in \mathbb{R}^n \). If \( x^* \in S \), the Bouligand and Clarke tangent cones of \( S \) at \( x^* \), respectively denoted by \( T^B_S(x^*) \) and \( T^C_S(x^*) \) are given by

\[
T^B_S(x^*) = \begin{cases} 
\mathbb{R}^n_{\Gamma^*}, & \text{if } |\Gamma^*| = s, \\
\bigcup_{\Upsilon \supseteq \Gamma^* : |\Upsilon| = s} \mathbb{R}^n_{\Upsilon}, & \text{if } |\Gamma^*| < s,
\end{cases}
\]

\[
T^C_S(x^*) = \mathbb{R}^n_{\Gamma^*}.
\]

Furthermore, if \( x^* \in S_+ \) we have

\[
T^B_{S_+}(x^*) = T^B_S(x^*) \cap T^R_{\mathbb{R}^n_+}(x^*), \quad T^C_{S_+}(x^*) = T^C_S(x^*),
\]

where \( T^R_{\mathbb{R}^n_+}(x^*) := \{d \in \mathbb{R}^n : d_i \geq 0, i \notin \Gamma^* \} \) is the usual tangent cone of \( \mathbb{R}^n_+ \) at \( x^* \).

The \( \alpha \)-stationary point defined below is actually the \( L \)-stationary point [5] when \( f \) has Lipschitzian gradient with the Lipschitz constant \( L_f \). The difference lies in that \( \alpha \) in our definition is allowed to take any positive value, while it is restricted within \( 0 < \alpha \leq 1/L_f \) in [5].

**Definition 2.1** Let \( x^* \in S_+ \) be a given feasible point of (1.1).

(i) We say that \( x^* \) is an \( \alpha \)-stationary point if there exists \( \alpha > 0 \) such that

\[
x^* \in P_{S_+}(x^* - \alpha \nabla f(x^*)).
\]

(ii) We say the \( x^* \) is a B-stationary point if

\[
0 \in P_{T^B_{S_+}(x^*)}(-\nabla f(x^*)).
\]

(iii) We say the \( x^* \) is a C-stationary point if

\[
0 = P_{T^C_{S_+}(x^*)}(-\nabla f(x^*)).
\]

The following table is extracted from [34, Table 3], which is very useful in helping us understand the subtle differences among the definitions. We will frequently use those characterizations in our analysis below.
It is well known that sparse optimization in general fundamentally differs from the classical optimization. One way to appreciate such difference, as well demonstrated in [5], is that the classical variational inequality
\[
\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in S
\]
is not a necessary optimality condition. Interestingly, as proved below, within sufficiently small a neighbourhood of \(x^*\), the variational inequality is equivalent to the \(B\)-stationary point.

**Proposition 2.2** Let \(x^* \in S_+\). Then the following results hold.

(i) \(x^*\) is a \(B\)-stationary point of (1.1) if and only if there exists \(\delta\) satisfying \(0 < \delta < \min\{x^* \i: i \in \Gamma^*\}\) such that
\[
\langle \nabla f(x^*), x - x^* \rangle \begin{cases} 
= 0, & \text{if } \|x^*\|_0 = s \\
\geq 0, & \text{if } \|x^*\|_0 < s 
\end{cases}
\]
holds for any \(x \in N(x^*, \delta) \cap S_+\).

(ii) In particular, if \(\|x^*\|_0 < s\) (the sparse constraint is not tight), then \(x^*\) is a \(B\)-stationary point of (1.1) if and only if
\[
\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in S_+.
\]

**Proof** (i) (Only if part) Suppose first that \(x^* \in S_+\) is a \(B\)-stationary point of (1.1). We prove (2.3) for any \(x \in N(x^*, \delta) \cap S_+\) with some \(\delta > 0\) by considering two cases.

**Case 1.** \(\|x^*\|_0 = s\) and \(x^* \geq 0\). Take \(\delta\) satisfying \(0 < \delta < \min\{x^*_i : i \in \Gamma^*\}\), for any \(x \in N(x^*, \delta) \cap S_+\) and \(i \in \Gamma^*\), we have
\[
x_i = x_i^* - (x_i^* - x_i) \geq x_i^* - \|x_i^* - x_i\| > x_i^* - \delta > 0,
\]
which yields that \(\Gamma^* \subseteq \text{supp}(x)\). By \(\|x\|_0 \leq s\) and \(|\Gamma^*| = \|x^*\|_0 = s\), we can obtain
\[
\text{supp}(x) \equiv \Gamma^*, \quad \forall x \in N(x^*, \delta) \cap S_+.
\]
Since $x^*$ is a $B$-stationary point of (1.1), by Table 2, we have
\[
\nabla_i f(x^*) \begin{cases} = 0, & \text{for } i \in \Gamma^*, \\ \in \mathbb{R}, & \text{for } i \notin \Gamma^*, \end{cases}
\]
which together with (2.4) yields that
\[
\langle \nabla f(x^*), x - x^* \rangle = \sum_{i \in \Gamma^*} \nabla_i f(x^*)(x_i - x_i^*) + \sum_{i \notin \Gamma^*} \nabla_i f(x^*)(x_i - x_i^*) = 0.
\]

**Case 2.** $\|x^*\|_0 < s$ and $x^* \geq 0$. Since $x^*$ is a $B$-stationary point of (1.1), we have
\[
\nabla_i f(x^*) \begin{cases} = 0, & \text{for } i \in \Gamma^*, \\ \in \mathbb{R}_+, & \text{for } i \notin \Gamma^*. \end{cases}
\]
It follows that for any $\delta > 0$ and $x \in N(x^*, \delta) \cap S_+$,
\[
\langle \nabla f(x^*), x - x^* \rangle = \sum_{i \in \Gamma^*} \nabla_i f(x^*)(x_i - x_i^*) + \sum_{i \notin \Gamma^*} \nabla_i f(x^*)(x_i - x_i^*) \geq 0,
\]
where the last inequality follows from the facts that (a) $\nabla_i f(x^*) = 0$, for $i \in \Gamma^*$, and (b) $x_i \geq 0$, $x_i^* = 0$, $\nabla_i f(x^*) \geq 0$ for $i \notin \Gamma^*$. We note that this part of the proof also applies to all $x \in S_+$ without having to be restricted in a neighbourhood of $x^*$.

(i) (The if part) Conversely, suppose that $x^* \in S_+$ satisfies (2.3) for any $x \in N(x^*, \delta) \cap S_+$ and $0 < \delta < \min\{x^*_i : i \in \Gamma^*\}$. We show $x^*$ is a $B$-stationary point of (1.1) also by two cases.

**Case 1.** $\|x^*\|_0 = s$ and $x^* \geq 0$. For any $i \in \Gamma^*$ and $\delta$ satisfying $0 < \delta < \min\{x^*_i : i \in \Gamma^*\}$, by letting $x = x^* + \delta e_i/2$, we have $x \in N(x^*, \delta) \cap S_+$. It follows from (2.3) that
\[
0 = \langle \nabla f(x^*), x - x^* \rangle = \langle \nabla f(x^*), \delta e_i/2 \rangle = \delta \nabla_i f(x^*)/2.
\]
Hence, $\nabla_i f(x^*) = 0$ for $i \in \Gamma^*$ and $\nabla_i f(x^*)$ for $i \notin \Gamma^*$ is not restricted.

**Case 2.** $\|x^*\|_0 < s$ and $x^* \geq 0$. If $i \in \Gamma^*$, using the same proof as Case 1 above, we obtain $\nabla_i f(x^*) = 0$. If $i \notin \Gamma^*$, let $x = x^* + \delta e_i/2$. Then $x \in N(x^*, \delta) \cap S_+$. It follows from (2.3) that
\[
0 \leq \langle \nabla f(x^*), x - x^* \rangle = \langle \nabla f(x^*), \delta e_i/2 \rangle = \delta \nabla_i f(x^*)/2.
\]
Hence, $\nabla_i f(x^*) \geq 0$ for $i \notin \Gamma^*$. It follows from Table 2 that $x^*$ is a $B$-stationary point of (1.1).

(ii) The only-if part follows from Case 2 of the only-if part of (i), where it was noted that the proof does not rely on the neighbourhood of $x^*$ used. The if-part proof follows from Case 2 of the if-part of (i), where the current condition in (ii) necessarily implies the condition within a neighborhood used in (i). \(\square\)

Our next major result is to establish the relationships among the three stationary points and the global/local minimizers of (1.1). Some of the relationships need a certain kind of convexity. We choose to use the one of the restricted strong convexity introduced in [31, 21]. Slightly different forms of this concept were also presented in [2, 8, 38]. Note that these properties are all analogous to the restricted isometry property (RIP) [14] in the standard (linear) compressed sensing. For easy reference, we include a definition.
**Definition 2.2** A function \( f \) is called \( s \)-restricted strongly smooth (s-RSS) with parameter \( L_s > 0 \), if for any \( x, y \in \mathbb{R}^n \) satisfying \( |\gamma_{xy}| \leq s \), it holds that

\[
(2.6) \quad f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq \frac{L_s}{2} \| y - x \|^2.
\]

We say that the function \( f \) is \( s \)-restricted strongly convex (s-RSC) with parameter \( l_s > 0 \), if for any \( x, y \in \mathbb{R}^n \) satisfying \( |\gamma_{xy}| \leq s \), it holds that

\[
(2.7) \quad f(y) - f(x) - \langle \nabla f(x), y - x \rangle \geq \frac{l_s}{2} \| y - x \|^2.
\]

In particular, if \( l_s = 0 \), the function \( f \) is said to be \( s \)-restricted convex (s-RC).

We note that \( f \) being s-RSS is a weaker condition than that \( f \) having a Lipschitzian gradient, and s-RSC may not imply the convexity of \( f \) on \( \mathbb{R}^n \). We are ready to report our main result below.

**Theorem 2.1** For \((1.1)\) and \( x^* \in S_+ \), consider three conditions: (a) \( \|x^*\|_0 = s \); (b) \( \|x^*\|_0 < s \); (c) \( f \) is 2s-RSC. Then we have the following (1) – (14) relationships shown in Figure 1 among the \( \alpha \)-, \( B \)-, \( C \)-stationary points and global/local minimizers. For example, for the relationship (3), an \( \alpha \)-stationary point will be a global minimizer of \((1.1)\) under the conditions (b) and (c).

- (1)
- (2): (b) (c)
- (3): (b) (c)
- (4): (b)
- (5): (c)
- (6): (b)
- (7)
- (8): (c)
- (9)
- (10): (a) (c)
- (11)
- (12): (c)
- (13)
- (14)

Figure 1: Relationships among \( \alpha \)-, \( B \)-, \( C \)-stationary points and global/local minimizers.

**Proof** Clearly, (1) holds. By using Table 2, (11)-(14) can be verified directly. We actually only need to prove (3), (7) and (8). In fact, if (3), (7) and (8) hold, then (12) and (8) \( \Rightarrow \) (5); (7) and (11) \( \Rightarrow \) (6); (6) and (3) \( \Rightarrow \) (2); (1) and (6) \( \Rightarrow \) (4); (7) and (14) \( \Rightarrow \) (9); (13) and (8) \( \Rightarrow \) (10).
For (3), if $f$ is $2s$-restricted convex, then for any $x \in S_+$ which implies $|\Gamma_{x^*}| = |\text{supp}(x) \cup \Gamma^*| \leq 2s$, we have

$$f(x) \geq f(x^*) + \langle \nabla f(x^*), x - x^* \rangle$$

$$= f(x^*) + \sum_{i \in \Gamma^*} \nabla_i f(x^*)(x_i - x_i^*) + \sum_{i \notin \Gamma^*} \nabla_i f(x^*)x_i$$

$$\geq f(x^*),$$

where the last inequality is from expression of $\alpha$-stationary point in Table 2 for the case $\|x^*\|_0 < s$. This proves that $x^*$ is a global minimizer of (1.1).

For (7), if $x^* \in S_+$ is a local minimizer of (1.1), then there is a constant $\delta > 0$ such that

$$f(x^*) \leq f(x), \forall x \in N(x^*, \delta) \cap S_+.$$

If $\|x^*\|_0 < s$, then for any $i \in \Gamma^*$, we have $x^* + te_i \in N(x^*, \delta) \cap S_+$ with sufficiently small $t > 0$ or $t < 0$ such that

$$f(x^*) \leq f(x^* + te_i) = f(x^*) + t\nabla_i f(x^*) + o(t),$$

thus $\nabla_i f(x^*) = 0$ for $i \in \Gamma^*$. For any $i \not\in \Gamma^*$, the above inequality holds for sufficiently small $t > 0$, which yields $\nabla_i f(x^*) \geq 0$. If $\|x^*\|_0 = s$, the same argument leads to $\nabla_i f(x^*) = 0$ for any $i \in \Gamma^*$. Therefore, $x^*$ is a $B$-stationary point of (1.1) by Table 2.

For (8), if $x^*$ is $B$-stationary point and $f$ is $2s$-restricted convex, then for any $x \in N(x^*, \delta) \cap S_+$ that implies $|\Gamma_{x^*}| = |\text{supp}(x) \cup \Gamma^*| = |\Gamma^*| = s \leq 2s$, we have

$$f(x) \geq f(x^*) + \langle \nabla f(x^*), x - x^* \rangle \geq f(x^*),$$

where the last inequality is from Pro. 2.2(i), which means $x^*$ is a local minimizer of (1.1). \qed

We would like to make some comments regarding the above theorem.

(R1) It follows from the relations (7) and (9) in Fig. 1 that a local minimizer must be a $B$- or $C$-stationary point. This means that the $B$- or $C$-stationarity forms a necessary condition for the sparse optimization (1.1). For the converse to be true, one must need some condition such as (c). In particular, the condition (a): $\|x^*\|_0 = s$ (i.e., the sparsity constraint is tight) is also part of the sufficient condition for a $C$-stationary point to be a local minimizer. Without this condition, a $C$-stationary point may fail to be a local minimizer even $f$ is assumed to be convex, as shown by the following example:

$$\min f(x) = (x_1 + 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2$$

s.t. $\|x\|_0 \leq 2$, $x \geq 0$.

The objective function $f$ is convex on $\mathbb{R}_+^3$ and its gradient $\nabla f(x) = 2(x_1 + 1, x_2 - 1, x_3 - 1)^\top$. It is obvious that $x^* = (0, 0, 1)^\top$ with $\nabla f(x^*) = (2, -2, 0)^\top$ is $C$-stationary point, but not a local minimizer because $f((0, \epsilon, 1)^\top) < f(x^*), 0 < \epsilon \leq 1$.

(R2) If one further assumes that $f$ has Lipschitzian gradient (not just being continuously differentiable), $\alpha$-stationarity becomes the $L$-stationarity introduced in [5]. Moreover, the $\alpha$-stationarity is also a necessary condition of $x^*$ being a local minimizer [5, Thm. 2.2]. Without the Lipschitzian property of the gradient function, the relation (6) shows that it is also a necessary condition provided that the sparse constraint is not tight.
2.2 Global properties

In this subsection, we collect several useful global properties of $B$- and $C$-stationary points under the restricted (strong) convexity. Our first result is a simple consequence of the results reported above. Recalling the variational inequality characterization of the $B$-stationary point in Prop. 2.2(ii), the relationships (11) and (3) in Fig. 1 establish the following important characterization of a global minimizer of (1.1).

**Corollary 2.1** Suppose $f$ is $2s$-RC and $x^* \in S_+$ with $\|x^*\|_0 < s$. The following are equivalent.

(i) $x^*$ is a global minimizer of (1.1).

(ii) $x^*$ is an $\alpha$-stationary point.

(iii) $x^*$ is a $B$-stationary point.

(iv) It holds that $\langle \nabla f(x^*), x - x^* \rangle \geq 0, \forall x \in S_+$.

The next result shows that a $B$-stationary point or a $C$-stationary point can be a global minimizer when restricted to certain subspace.

**Theorem 2.2** Suppose $f$ is $s$-RC. Let $x^* \in S_+$. Then the following hold.

(i) If $x^*$ is a $B$-stationary point, then it is a global minimizer on the subspace $\mathbb{R}^n_\Gamma$ for any $\Gamma \subseteq \{1, \cdots, n\}$ satisfies $\Gamma^* \subseteq \Upsilon$ and $|\Upsilon| = s$.

(ii) If $x^*$ is a $C$-stationary point, then it is a global minimizer on the subspace $\mathbb{R}^n_{\Gamma^*}$.

(iii) If $f$ is $s$-RSC, then the local minimizer of problem (1.1) on any $s$-dimensional subspace is unique. Furthermore, the number of the local minimizers is finite.

**Proof** (i) For any $x \in \mathbb{R}^n_\Upsilon \cap \mathbb{R}^n_{\Gamma^*}$, if $\|x^*\|_0 = s = |\Gamma^*|$ (and hence $\Upsilon = \Gamma^*$), we have $(x - x^*)_i = 0, \forall i \notin \Gamma^*$; if $\|x^*\|_0 < s$, $(x - x^*)_i = x_i \geq 0, \forall i \notin \Gamma^*$, which together with the fact of $f$ being $s$-restricted convex and the expression of $B$-stationary point in Table 2 yields that

$$
\begin{align*}
f(x) & \geq f(x^*) + \langle \nabla f(x^*), x - x^* \rangle \\
& = f(x^*) + \sum_{i \in \Gamma^*} \nabla_i f(x^*)(x_i - x^*_i) + \sum_{i \notin \Gamma^*} \nabla_i f(x^*)(x_i - x^*_i) \\
& \geq f(x^*).
\end{align*}
$$

Thus the conclusion is derived. The proof of (ii) is similar to (i), and thus its proof is omitted.

(iii) We note that under the assumption of $s$-restricted strong convexity of $f$, the inequality in the proof of (i) becomes strict. Therefore, there exists only one local minimizer on any $s$-dimensional subspace. We also note that from Table 2 (Relation (9)), any local minimizer of (1.1) is also a $C$-stationary point. However, according to (ii), any $C$-stationary point must be a unique minimizer on an $s$-dimensional subspace. Since the number of the subspaces whose dimension is no larger than $s$ is finite, we conclude that the number of the local minimizers of (1.1) is finite. □
The following example shows that there may exist multiple minimizers of (1.1) under the $s$-RSC. That is, one cannot establish the uniqueness of the global minimizer in Thm. 2.2(iii), unless stronger assumptions are in place.

$$\min \|x - 1\|, \text{ s.t. } \|x\|_0 \leq 1, \ x \geq 0,$$

where $1 = (1,1,\cdots,1)^\top$. The objective function is strongly convex on $\mathbb{R}^n$, and every $e_i, i = 1,\ldots,n$, is a global minimizer.

3 Improved IHT and Its Theoretical Analysis

In this section, we will present an improved iterative hard thresholding (IIHT) algorithm for (1.1) and then analyze its convergence properties utilizing the results reported above.

3.1 IIHT Algorithm

As reviewed in Introduction, in order for the generated iterates by the IHT algorithm to converge to a point satisfying certain optimality conditions (stationarities), a proper selection of stepsize seems necessary at each iteration. For example, for the classical linear compressed sensing, Blumensath and Davies introduced an adaptive stepsize rule based on the RIP to ensure a sufficient decrease in the objective per iteration. Recently, for the nonlinear sparse optimization problem, Lu [26] introduced a nonmonotone line search to a projection algorithm to ensure its convergence. In this paper, we choose to use the classical Armijo stepsize rule in the IHT, leading to what we call an Improved IHT (IIHT) algorithm. Another new element that we introduced in IIHT is a new stopping criterion that is motivated by the $C$-stationarity. The remaining part of IIHT just follows the original IHT and hence the framework of IIHT is very simple and is described as follows.

<table>
<thead>
<tr>
<th>Step 0</th>
<th>Initialize $x^0 = 0$, $0 &lt; \alpha_0 &lt; \frac{1}{L_{2s}}$, $\sigma &gt; 0$, $0 &lt; \beta &lt; 1$, $\epsilon &gt; 0$. Set $k \Leftarrow 0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>Compute $x^{k+1} = P_{S_+}(x^k - \alpha_k \nabla f(x^k))$, where $\alpha_k = \alpha_0 \beta^q_k$ and $q_k$ is the smallest nonnegative integer $q$ such that $f(x^k(\alpha_{k,0}^{0,q})) \leq f(x^k) - \frac{\sigma}{2} |x^k(\alpha_{k,0}^{0,q}) - x^k|^2$, and $x^k(\alpha) := P_{S_+}(x^k - \alpha \nabla f(x^k))$.</td>
</tr>
<tr>
<td>Step 2</td>
<td>If $|\nabla f(x^k)| \leq \epsilon$, then Stop; Otherwise, let $k \Leftarrow k + 1$ and go to Step 1.</td>
</tr>
</tbody>
</table>

The stopping criterion used will be justified by Thm. 3.2(iii). We emphasize that the major computation $P_{S_+}(\cdot)$ is very easy to obtain via (2.1). The following result shows that the Armijo stepsize is well defined under some condition.

Lemma 3.1 Let $f$ be $2s$-RSS and $x^k \in S_+$ be given. Then it holds

$$f(x^k(\alpha)) \leq f(x^k) - \frac{\sigma}{2} \|x^k(\alpha) - x^k\|^2 \quad \text{for} \quad 0 < \alpha \leq \frac{1}{L_{2s} + \sigma}. \quad (3.1)$$

Therefore $\alpha_k$ in the algorithm is well defined.
Proof According to the computation of $x^k(\alpha)$ in Step 1, we have

$$x^k(\alpha) \in \text{argmin}\{\|x - x^k + \alpha \nabla f(x^k)\|^2 : x \in S_+\},$$

which implies that $\|x^k(\alpha) - x^k + \alpha \nabla f(x^k)\|^2 \leq \|\alpha \nabla f(x^k)\|^2$ by $x^k \in S_+$. This leads to

$$\|x^k(\alpha) - x^k\|^2 \leq -2\alpha \langle \nabla f(x^k), x^k(\alpha) - x^k \rangle.$$ 

It follows from the property of $2s$-RSS of $f$ and (3.2) that

$$f(x^k(\alpha)) \leq f(x^k) + \langle \nabla f(x^k), x^k(\alpha) - x^k \rangle + \frac{L_{2s}^2}{2} \|(x^k(\alpha) - x^k)\|^2$$

$$\leq f(x^k) - \frac{1}{2\alpha} \|x^k(\alpha) - x^k\|^2 + \frac{L_{2s}^2}{2} \|(x^k(\alpha) - x^k)\|^2$$

$$= f(x^k) - \frac{1}{2}(1/\alpha - L_{2s}) \|x^k(\alpha) - x^k\|^2.$$

By restricting $\alpha \in (0, \frac{1}{L_{2s} + \sigma}]$, we obtain the desired result. The proof is completed. 

3.2 Convergence Analysis

Combining the restricted strong convexity and smoothness of $f$, the convergence of IIHT can be established in this subsection. We first present a technical result.

Lemma 3.2 Suppose that the function $f$ is $s$-RC and $s$-RSS with parameter $L_s$. Then for any $x, y \in \mathbb{R}^n$ satisfying $|\Gamma_{xy}| \leq s$, we have

$$\|\nabla f(y) - \nabla f(x)\| \leq L_s\|y - x\|.$$ 

Proof Let us fix $x \in S$ and define the following function of variable $y$ at point $x$:

$$\phi_x(y) := f(y) - \langle \nabla f(x), y - x \rangle.$$ 

Due to the $s$-RC of $f(\cdot)$, the point $x$ is a minimizer of $\phi_x(y)$ over all $y$ satisfying $|\Gamma_{xy}| \leq s$. This is because

$$(3.3) \quad \phi_x(y) - \phi_x(x) = f(y) - \langle \nabla f(x), y - x \rangle - f(x) \geq 0 \quad \forall y \text{ such that } |\Gamma_{xy}| \leq s.$$ 

We note that function $\phi_x(\cdot)$ has the same properties of $s$-restricted strong smoothness as $f(\cdot)$. Define $d \in \mathbb{R}^n$ by

$$d_i := \begin{cases} \frac{1}{L_s}(\nabla \phi_x(y))_i, & \text{if } i \in \Gamma_{xy} \\ 0, & \text{otherwise}. \end{cases}$$

We have

$$\|y - d\|_0 \leq |\Gamma_{xy}| \leq s \quad \text{and} \quad \langle \nabla \phi_x(y), d \rangle = \frac{1}{L_s}\|(\nabla \phi_x(y))\|_{\Gamma_{xy}}^2.$$ 

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which, together with (3.3) and the s-RSS of $\phi_x(\cdot)$, imply

\[
\phi_x(x) \leq \phi_x(y - d) \\
\leq \phi_x(y) + \langle \nabla \phi_x(y), -d \rangle + \frac{L_s}{2} \| \nabla \phi_x(y) \Gamma_{xy} \|^2 \\
= \phi_x(y) - \frac{1}{2L_s} \| \nabla \phi_x(y) \Gamma_{xy} \|^2.
\]

Rewrite (3.4) as

\[
(3.5) \quad f(x) \leq f(y) - \langle \nabla f(x), y - x \rangle - \frac{1}{2L_s} \| \nabla f(y) - \nabla f(x) \Gamma_{xy} \|^2.
\]

By interchanging $x$ and $y$ in (3.5) and adding the resulting inequality to (3.5), we get

\[
(3.6) \quad \| \nabla f(y) - \nabla f(x) \Gamma_{xy} \|^2 \leq L_s \langle \nabla f(y) - \nabla f(x), y - x \rangle.
\]

The desired results then follows from applying the Cauchy-Schwarz inequality to (3.6). \hfill \square

We report our first convergence result below.

**Theorem 3.1** Let the sequence $\{x^k\}$ be generated by IIHT. Suppose $f$ is $2s$-RSS. Then the following hold.

(i) $\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0$ and $\inf_{k \geq 0} \{\alpha_k\} > 0$;

(ii) Any accumulation point of $\{x^k\}$ is an $\alpha$-stationary point of (1.1).

Moreover, if $f$ is $2s$-RC, then the following hold.

(iii) The sequence of projected gradients converges to zero, i.e.,

\[
\lim_{k \to \infty} \| \nabla \Gamma_k f(x^k) \| = 0.
\]

(iv) Any accumulation point of $\{x^k\}$ is a local minimizer of (1.1).

**Proof** (i) As required in IIHT, we have $f(x^k) - f(x^{k+1}) \geq \frac{\sigma}{2} \| x^{k+1} - x^k \|^2$. Then

\[
\sum_{k=0}^{\infty} \| x^{k+1} - x^k \|^2 \leq \frac{2}{\sigma} \sum_{k=0}^{\infty} (f(x^k) - f(x^{k+1})) < \frac{2}{\sigma} \left( f(x^0) - \lim_{k \to +\infty} f(x^k) \right) < +\infty,
\]

where the last inequality is due to $f$ being bounded from below. Hence $\lim_{k \to \infty} \| x^{k+1} - x^k \| = 0$.

Armijo-type stepsize rule and Lemma 3.1 imply that $\{\alpha_k\}$ is bounded from below by a positive constant. In fact,

\[
(3.7) \quad \inf_{k \geq 0} \{\alpha_k\} \geq \frac{\beta}{\sigma + L_{2s}} := \alpha > 0.
\]
(ii) Suppose that \( x^* \) is an accumulation point of the sequence \( \{ x^k \} \). There exists a subsequence \( \{ x^{k_j} \} \) converging to \( x^* \). It follows from (i) that

\[
\lim_{j \to \infty} x^{k_j+1} = \lim_{j \to \infty} x^{k_j} = x^*.
\]

Based on the update

\[
x^{k_j+1} = P_{S_+}(x^{k_j} - \alpha_{k_j} \nabla f(x^{k_j}))
\]

in Step 1 of the IIHT algorithm, we consider two cases.

Case 1. For \( i \in \Gamma^* \). There must exist a sufficiently large index \( n_1 \) and a positive constant \( c_0 \) such that

\[
\min \{ x_i^{k_j}, x_i^{k_j+1} \} \geq c_0 \quad \forall \ j \geq n_1.
\]

This together with the projection formula of \( P_{S_+}(\cdot) \) in (2.1) and (3.9) implies

\[
x_i^{k_j+1} = x_i^{k_j} - \alpha_{k_j} \nabla_i f(x^{k_j}).
\]

Therefore, the positive lower bound in (3.7) and the limit in (3.8) yield

\[
\nabla_i f(x^*) = 0, \quad \forall \ i \in \Gamma^*.
\]

Case 2. For \( i \notin \Gamma^* \). Without loss of any generality, we may assume \( \lim_{j \to \infty} \alpha_{k_j} = c_1 > 0 \) on the subsequence \( \{ k_j \} \). We consider two subcases. Subcase 2.1: \( \| x^* \|_0 = s \). Then we have

\[
0 = \lim_{j \to \infty} x_i^{k_j+1} = \lim_{j \to \infty} \left( P_S \left( P_{\mathbb{R}^n_+} (x^{k_j} - \alpha_{k_j} \nabla f(x^{k_j})) \right) \right)_i.
\]

Due to the property of the projections \( P_S(\cdot) \) and \( P_{\mathbb{R}^n_+}(\cdot) \), we must have

\[
\max \{ x_i^{k_j} - \alpha_{k_j} \nabla_i f(x^{k_j}), \ 0 \} \leq \left( x_i^{k_j+1} \right)_i^{\frac{1}{s}}.
\]

Taking limits on both sides, we obtain

\[
\nabla_i f(x^*) \geq -\frac{1}{c_1} (x^*)^{\frac{1}{s}}.
\]

Subcase 2.2: \( \| x^* \|_0 < s \). Suppose \( \nabla_i f(x^*) < 0 \). We then have

\[
\lim_{j \to \infty} \left( x_i^{k_j} - \alpha_{k_j} \nabla_i f(x^{k_j}) \right) = -c_1 \nabla_i f(x^*) > 0,
\]

leading to

\[
\left( P_{\mathbb{R}^n_+} (x^{k_j} - \alpha_{k_j} \nabla f(x^{k_j})) \right)_i \geq -\frac{1}{2} c_1 \nabla_i f(x^*)
\]

for all sufficiently large \( j \). Since \( \| x^* \|_0 < s \), we must have for \( j \) sufficiently large

\[
x_i^{k_j+1} = \left( P_S \left( P_{\mathbb{R}^n_+} (x^{k_j} - \alpha_{k_j} \nabla f(x^{k_j})) \right) \right)_i = \left( P_{\mathbb{R}^n_+} (x^{k_j} - \alpha_{k_j} \nabla f(x^{k_j})) \right)_i \geq -\frac{1}{2} c_1 \nabla_i f(x^*) > 0.
\]

This contradicts the assumption \( i \notin \Gamma^* \) (which in turn implies \( \lim_{j \to \infty} x_i^{k_j+1} = 0 \)). Therefore, we must have \( \nabla_i f(x^*) \geq 0 \) for Subcase 2.2.
Summarizing the above two cases, we obtained

\[
\nabla_i f(x^*) \left\{ \begin{array}{ll}
0, & \text{if } i \in \Gamma^*, \\
\in [-\frac{1}{c_1}(x^*)^{\frac{1}{2}}, \infty), & \text{if } i \notin \Gamma^*,
\end{array} \right.
\]

which means that \( x^* \) is an \( \alpha \)-stationary point of (1.1) by Table 2.

(iii) Notice that \( T^C_{S+}(x^k) = \mathbb{R}^n_{1k} \) is a subspace. The projection of the negative gradient \( (-\nabla f(x^k)) \) to this subspace has the following property due to [13, Lemma 3.1]:

\[
\|P_{\mathbb{R}^n_{1k}}(-\nabla f(x^k))\| = \max \{\langle -\nabla f(x^k), v \rangle : v \in \mathbb{R}^n_{1k}, \|v\| \leq 1 \} = \|\nabla_{\Gamma^*} f(x^k)\|.
\]

Moreover, the maximum takes place at the boundary of \( \|v\| = 1 \). Therefore, for any given \( \epsilon > 0 \), there exists \( v^k \in \mathbb{R}^n_{1k} \) with \( \|v^k\| = 1 \) such that

\[
(3.11) \quad \|\nabla_{\Gamma^*} f(x^k)\| \leq -\langle \nabla f(x^k), v^k \rangle + \epsilon.
\]

It follows from \( x^{k+1} = P_{S+}(x^k - \alpha_k \nabla f(x^k)) \) and the property in (2.2) that

\[
x^{k+1}_{\Gamma^* + 1} = \left( x^k - \alpha_k \nabla f(x^k) \right)_{\Gamma^* + 1}.
\]

In other words, the vector \( (x^{k+1} - (x^k - \alpha_k \nabla f(x^k))) \) is orthogonal to \( \mathbb{R}^n_{\Gamma^* + 1} \). This yields that

\[
\langle x^{k+1} - (x^k - \alpha_k \nabla f(x^k)), w^{k+1} - x^{k+1} \rangle = 0, \quad \forall w^{k+1} \in \mathbb{R}^n_{\Gamma^* + 1}.
\]

Choose a particular \( w^{k+1} \) by \( w^{k+1} := x^{k+1} + v^{k+1} \). The Cauchy-Schwartz inequality implies

\[
(3.12) \quad -\langle \nabla f(x^k), v^{k+1} \rangle \leq \frac{\|x^{k+1} - x^k\|}{\alpha_k}.
\]

Since \( f \) is 2s-RSS, we have from Lemma 3.2 and (3.12) that

\[
-\langle \nabla f(x^{k+1}), v^{k+1} \rangle = -\langle \nabla f(x^{k+1}) - \nabla f(x^k), v^{k+1} \rangle - \langle \nabla f(x^k), v^{k+1} \rangle \\
\leq L_{2s}\|x^{k+1} - x^k\| - \frac{\|x^{k+1} - x^k\|}{\alpha_k}.
\]

Taking limits on both sides and using the facts established in (i), we have

\[
\limsup_{k \to \infty} -\langle \nabla f(x^{k+1}), v^{k+1} \rangle \leq 0.
\]

From (3.11) and the arbitrariness of \( \epsilon \), we proved \( \lim_{k \to \infty} \|\nabla_{\Gamma^*} f(x^k)\| = 0 \).

(iv) The convergence to a local minimizer of (1.1) provided that \( f \) is 2s-restricted convex follows directly from Theorem 2.1 (Relation (5)).\)

The following result further characterizes when the whole sequence converges to a local minimizer and when the local minimizer becomes a global one.

**Theorem 3.2** Assume \( f \) is both 2s-RSS and 2s-RSC. Then the whole sequence \( \{x^k\} \) converges to a local minimizer \( x^* \) of (1.1). Furthermore, depending on whether the sparse constraint is tight or not at \( x^* \), we have the following detailed characterization of \( x^* \).
(i) If the sparse constraint is tight at $x^*$ (i.e., $\|x^*\|_0 = s$), then

$$\Gamma^k \equiv \Gamma^* \quad \text{for all sufficiently large } k.$$ 

(ii) If the sparse constraint is not tight at $x^*$ (i.e., $\|x^*\|_0 < s$), then $x^*$ is a global minimizer of (1.1).

**Proof** From Thm. 2.2(iii), the number of the local minimizers of (1.1) is finite and from Theorem 3.1(iv), every accumulation point of $\{x^k\}$ is a local minimizer of (1.1). Hence, the number of accumulation points of sequence $\{x^k\}$ is finite and every accumulation point, which is also a local minimizer, is isolated. Since $f$ is 2s-RSC, the sequence $\{x^k\}$ is bounded. Theorem 3.1(i) has established that the whole sequence $\{x^k\}$ satisfies $\|x^{k+1} - x^k\| \to 0$. It follows from [29, Lemma 4.10] (which is restated as [22, Prop. 7], which is more relevant to our current setting) that the whole sequence must converge to a local minimizer. We now prove the remaining two claims.

(i) If $\|x^*\|_0 = s$, since $x^k \to x^*$, we have $\|x^k - x^*\| < \delta$ where $0 < \delta < \min\{x^*_i: i \in \Gamma^*\}$ for all sufficiently large $k$. Then following the same reasoning as proving (2.4), we have $\Gamma^k \equiv \Gamma^*$ for all sufficiently large $k$.

(ii) If $\|x^*\|_0 < s$, the conclusion can be derived immediately due to $f$ being 2s-RSC and Theorem 2.1 (Relation (2)).

What we have proved in the above theorem is that when the sparse constraint $\|x\|_0 \leq s$ is tight at $x^*$, we can only claim that the whole sequence converges to a local minimizer, whereas when it is not tight, the whole sequence converges to the global minimizer. We note that a similar result has also been recently proved by Lu [26]. If the sparse constraint is tight, then the sequence generated in [26] only converges to a local minimizer under the assumptions that $f$ has Lipschitz gradient and is convex. If the sparse constraint is not tight, then the sequence generated in [26] converges to the global minimizer provided that $f$ further satisfies Assumption (3) in [26]. Here we assumed 2s-RSS and 2s-RSC. Our result in (i) states that the active index set can be correctly identified in the case of tight constraint. This is the crucial property that allows us to establish the Q-linear convergence rate (Thm. 3.4) below.

### 3.3 Sub-linear and Q-linear convergence rate

In this subsection, we will show the linear convergence rate both in terms of functional value sequence $\{f(x^k)\}$ and the sequence itself $\{x^k\}$. From the view of point in Theorem 3.2, we need the assumptions of both 2s-RSC and 2s-RSS. Consequently, the whole sequence $\{x^k\}$ converges to a local minimizer $x^*$.

First we make an easy observation. For $x^k$, denote

$$F_k(x) := \|x - x^k + \alpha_k \nabla f(x^k)\|^2.$$ 

Then it is obvious that

$$x^{k+1} = P_{S_+}(x^k - \alpha_k \nabla f(x^k)) = \arg \min_{x \in S_+} F_k(x).$$
We claim that it holds

\[ \langle \nabla F_k(x^{k+1}), x^* - x^{k+1} \rangle \geq 0 \quad \text{for any } k \text{ such that } \Gamma^* \subseteq \Gamma^{k+1}. \]

We prove above inequality by considering two cases. Case 1: \( \|x^{k+1}\|_0 < s \) and Case 2: \( \|x^{k+1}\|_0 = s \). For Case 1, apply Cor. 2.1(iv) to \( F_k \) (instead of \( f \) therein) to get (3.13) because \( F_k(\cdot) \) is 2s-RC due to \( F_k(\cdot) \) being strongly convex. For Case 2, \( x^{k+1} \) is the global minimizer of \( F_k(x) \) and thus a \( B \)-stationary point, which implies \( \nabla_{\Gamma^{k+1}} F_k(x^{k+1}) = 0 \). Then by \( \Gamma^* \subseteq \Gamma^{k+1} \) when \( k \) is sufficiently large, we must have

\[ \langle \nabla F_k(x^{k+1}), x^* - x^{k+1} \rangle = \sum_{i \in \Gamma^{k+1}} \nabla_i F_k(x^{k+1})(x_i^* - x_i^{k+1}) = 0. \]

Hence (3.13) holds and it leads to the following linear rate convergence.

**Theorem 3.3** Assume \( f \) is 2s-RSS and 2s-RSC. Let \( \{x^k\} \) be generated by IIHT and be convergent to a local minimizer \( x^* \) of (1.1) (the convergence is guaranteed by Thm. 3.2). Then for any \( k > k_0 \), the following inequality holds:

\[ f(x^k) - f(x^*) \leq \frac{1}{(k - k_0)\alpha_2s} \left(f(x^{k_0}) - f(x^*)\right), \]

where \( \alpha := \frac{\beta}{L_{2s} + \sigma} \) and \( k_0 \) is the smallest positive integer such that \( \Gamma^* \subseteq \Gamma^k \) for any \( k > k_0 \).

**Proof** Since \( x^k \to x^* \), there exists \( k_0 \) such that \( \Gamma^* \subseteq \Gamma^k, \forall k > k_0 \). Therefore, (3.13) holds for any \( k > k_0 \). Since \( f \) is convex on any 2s-dimensional subspace and 2s-RSS with \( 0 < \alpha_k < \frac{1}{L_{2s}} \), it follows that

\[ f(x^*) \geq f(x^k) + \langle \nabla f(x^k), x^* - x^k \rangle, \]

\[ f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2\alpha_k} \|x^{k+1} - x^k\|^2. \]

By the function \( F_k(x) \) being strongly convex with modulus 2 and (3.13), we have

\[ F_k(x^*) = F_k(x^{k+1}) + \langle \nabla F_k(x^{k+1}), x^* - x^{k+1} \rangle + \|x^* - x^{k+1}\|^2 \geq F_k(x^{k+1}) + \|x^* - x^{k+1}\|^2. \]

Substituting the definition of \( F_k(x) \) into the above inequality and simplifying lead to

\[ \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2\alpha_k} \|x^{k+1} - x^k\|^2 \leq \langle \nabla f(x^k), x^* - x^k \rangle + \frac{1}{2\alpha_k} (\|x^* - x^k\|^2 - \|x^* - x^{k+1}\|^2). \]

Combining (3.16), (3.17) and (3.15), it holds that

\[ f(x^{k+1}) + \frac{1}{2\alpha_k} \|x^* - x^{k+1}\|^2 \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2\alpha_k} \|x^{k+1} - x^k\|^2 + \frac{1}{2\alpha_k} \|x^* - x^{k+1}\|^2 \quad \text{(by (3.16))} \]

\[ f(x^k) + \langle \nabla f(x^k), x^* - x^k \rangle + \frac{1}{2\alpha_k} \|x^* - x^k\|^2 \quad \text{(by (3.17))} \]

\[ f(x^*) + \frac{1}{2\alpha_k} \|x^* - x^k\|^2, \quad \text{(by (3.15))} \]
which amounts to
\[
f(x^{k+1}) - f(x^*) \leq \frac{1}{2\alpha_k} \left( \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \right)
\]
\[
\leq \frac{1}{2\alpha} \left( \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \right),
\]
where we have used the fact $\alpha_k \geq \frac{\beta}{\sigma + l_{2s}} = \alpha$ proved in Theorem 3.1(i) (3.7). For any positive integer $j$, using this inequality and the monotonically decreasing property of $\{f(x^k)\}$, we have
\[
j(f(x^{k+j}) - f(x^*)) \leq \sum_{i=k}^{k+j-1} (f(x^{i+1}) - f(x^*)) \leq \frac{1}{2\alpha} \left( \|x^k - x^*\|^2 - \|x^{k+j} - x^*\|^2 \right).
\]
We thus have
\[\tag{3.18} f(x^{k+j}) - f(x^*) \leq \frac{1}{2j\alpha} \|x^k - x^*\|^2. \]

In addition, since $x^*$ is a local minimizer and thus a $B$-stationary point. By Prop. 2.2(i), it holds $\langle \nabla f(x^*), x^k - x^* \rangle \geq 0$ because $x^k$ is in a neighborhood of $x^*$. This and $f$ being $2s$-RSC yield
\[
f(x^k) - f(x^*) \geq \langle \nabla f(x^*), x^k - x^* \rangle + \frac{l_{2s}}{2} \|x^k - x^*\|^2 \geq \frac{l_{2s}}{2} \|x^k - x^*\|^2,
\]
which together with (3.18) contributes to
\[
f(x^{k+j}) - f(x^*) \leq \frac{1}{j\alpha l_{2s}} (f(x^k) - f(x^*)).
\]
Therefore, for any $k > k_0$, it holds that
\[
f(x^k) - f(x^*) \leq \frac{1}{(k - k_0)\alpha l_{2s}} (f(x^{k_0}) - f(x^*)),
\]
which completes the proof. \qed

We now show the Q-linear convergence rate of the iterative points sequence of IIHT under assumption $\|x^*\|_0 = s$.

**Theorem 3.4** Assume $f$ is $2s$-RSS and $2s$-RSC. Let $x^*$ be the limit of the sequence $\{x^k\}$ generated by IIHT that satisfies $\|x^*\|_0 = s$. Then for any sufficiently large $k$, it holds,
\[\tag{3.19} \|x^{k+1} - x^*\|^2 \leq \rho \|x^k - x^*\|^2, \quad 0 < \rho < 1,
\]
where $\rho := 1 - 2l_{2s}^2 \alpha / L_{2s} + l_{2s}^2 \alpha^2$ with $\alpha$ being defined in Theorem 3.3.

**Proof** As already used, the convergence of $\{x^k\}$ to $x^*$ is guaranteed by Thm. 3.2. Since $f$ is $2s$-restricted strongly convex with parameters $l_{2s}$ in (2.7), we can easily obtain that
\[
\| (\nabla f(x) - \nabla f(y))\Gamma_{xy} \| \geq l_{2s} \| x - y \| \quad \forall |\Gamma_{xy}| \leq 2s.
\]

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This together with Lemma 3.2 and Thm. 3.2(i) (proving $\Gamma^k \equiv \Gamma^*$ for all sufficiently large $k$) yields that for any sufficiently large $k$,

$$\|x^{k+1} - x^*\|^2 = \|x^k - \alpha_k \nabla_{\Gamma^*} f(x^k) - x^* + \alpha_k \nabla_{\Gamma^*} f(x^*)\|^2$$

$$= \|x^k - x^*\|^2 - 2\alpha_k \langle x^k - x^*, \nabla f(x^k) - \nabla f(x^*) \rangle + \alpha_k^2 \|\nabla f(x^k) - \nabla f(x^*)\|_{\Gamma^*}^2$$

$$\leq \|x^k - x^*\|^2 - (2\alpha_k / L_{2s} - \alpha_k^2) \|\nabla f(x^k) - \nabla f(x^*)\|_{\Gamma^*}^2$$

$$\leq (1 - 2l_{2s}^2 \alpha_k / L_{2s} + l_{2s}^2 \alpha_k^2) \|x^k - x^*\|^2,$$

where $\nabla_{\Gamma^*} f(x^*) = 0$ in the first equality holds due to Thm. 3.1(iii), namely, $\nabla_{\Gamma^*} f(x^*) = \lim_{k \to \infty} \nabla_{\Gamma^*} f(x^k) = 0$. It follows from $\alpha \leq \alpha_k < 1 / L_{2s}$ that

$$1 - 2l_{2s}^2 \alpha_k / L_{2s} + l_{2s}^2 \alpha_k^2 = 1 + l_{2s}^2 (\alpha_k - 1 / L_{2s})^2 - l_{2s}^2 / L_{2s}^2$$

$$\leq 1 + l_{2s}^2 (\alpha - 1 / L_{2s})^2 - 1 / L_{2s}^2$$

$$= 1 - 2l_{2s}^2 \alpha / L_{2s} + l_{2s}^2 \alpha^2 = \rho.$$

Moreover, $\rho = l_{2s}^2 (\alpha - 1 / L_{2s})^2 + 1 - l_{2s}^2 / L_{2s}^2 > 0$ and $\rho = 1 - l_{2s}^2 \alpha \left( \frac{2}{L_{2s}} - \frac{\beta}{L_{2s}^2 + \sigma} \right) < 1$. The proof is completed. \qed

We note that convergence result of the type (3.19) is known to be $Q$ linear rate in optimization. We are only able to establish this result for the special case when the sparse constraint is tight. The key reason is that we were able to correctly identify the active index set for this case.

## 4 Numerical experiments

In this section, we report our numerical experiments of IIHT on three classes of problems: Linear compressed sensing under nonnegativity constraints, Sparse logistic regression and Phase retrieval. Our stopping criterion is set as

$$\text{number of iterations} \leq \text{Maxiter} \quad \text{or} \quad \|\nabla f(x^k)\|_{\Gamma^*} \leq \epsilon,$$

where we stop our algorithm whenever the number of iterations exceeds $\text{Maxiter}$ or the projected gradient becomes less than $\epsilon$. We will set a different level for $\epsilon$ and $\text{Maxiter}$ for each class of test problems. The CPU time reported here does not include the time for data initialization. All those simulations are carried out on a CPU 3.2GHz, RAM 4.0GB desktop.

### 4.1 Compressed Sensing

We first test the classical linear CS problem under the nonnegativity constraint with $f(x) = f_A(x) := \|Ax - b\|^2$, where $A \in \mathbb{R}^{m \times n}$ is a linear measurement matrix satisfying

$$b = Ax + \xi.$$

We will test two scenarios. One is the exact recovery where $\xi \equiv 0$ and the other is the so-called stable recovery where $\xi$ follows the normal distribution. More specifically, three types of sensing
 matrices of $A$ will be generated, namely, random Gaussian matrix, random partial Discrete Cosine Transform (pDCT) matrix, and over-sampled partial DCT matrix (denoted as opDCT):

\[
\begin{align*}
\text{Gaussian:} & \quad A_j \sim \mathcal{N}(0, I/m), \\
\text{pDCT:} & \quad A_{ij} = m^{-1/2} \cos(2\pi(j - 1)\psi_i), \quad i = 1, 2, \ldots, m, \ j \in \{1, 2, \ldots, n\}, \\
\text{opDCT:} & \quad A_{ij} = m^{-1/2} \cos(2\pi(j - 1)\psi_i/H), \quad i = 1, 2, \ldots, m, \ j \in \{1, 2, \ldots, n\},
\end{align*}
\]

where $A_j$ denotes the $j$th column of $A$, $\psi_i$, $i = 1, \ldots, m$ are uniformly and independently sampled from $[0, 1]$, and the positive integer $H$ is the so-called refinement factor fixed as $H = 10$ for simplicity. After generating them, we orthogonalize them to satisfy $AA^\top = I$.

For Gaussian and pDCT matrices, we generate the ‘true’ original signal $x_{\text{orig}}$ as follow: first produce an index set $T$ with $s$ indices randomly selected from $\{1, \ldots, n\}$; then for each element of $x_{\text{orig}}$ with index in $T$, uniformly choose them from $[-10, 10]$. The data are generated as follows (in Matlab format):

\[
\begin{align*}
x_{\text{orig}} &= \text{zeros}(n, 1); \quad T = \text{randperm}(n); \\
x_{\text{orig}}(T(1 : s)) &= -10 + 20 * \text{rand}(s, 1); \\
b &= A * x_{\text{orig}} + \sigma_0 * \text{randn}(m, 1).
\end{align*}
\]

For the nonnegative ‘true’ original signal, we generate it by $x_{\text{orig}}(T(1 : s)) = 1 + 9 * \text{rand}(s, 1)$. Clearly, the case $\sigma_0 = 0$ is the exact recovery. For stable recovery, we take $\sigma_0 = 0.01$.

For opDCT matrix, the ‘true’ signal is generated to satisfy $\min \{i, j \in \text{supp}(x_{\text{orig}}) : |i - j| \geq 2H\}$:

\[
\begin{align*}
T &= \text{randsample}(n - 2 \star H * (s - 1), s); \quad T = \text{sort}(T); \\
T &= T + (0 : s - 1)' * 2 * H; \\
x_{\text{orig}} &= \text{zeros}(n, 1); \quad x_{\text{orig}}(T) = -10 + 20 * \text{rand}(s, 1);
\end{align*}
\]

In our implementation, we set $\text{Maxiter} = 1000, \epsilon = m \times 10^{-8}, \beta = 0.8$ and $\sigma = 10^{-5}$ for simplicity. Instead of fixing $\alpha_0$ for each step in IIHT, we update it according to [11] to accelerate the computational speed, namely,

\[
\alpha^k_0 = \frac{\|A_{\Gamma k}^\top (b - A x^k)\|^2}{\|A_{\Gamma k} A_{\Gamma k}^\top (b - A x^k)\|^2}.
\]

We begin with running 100 independent trials for each type of matrix and recording the corresponding success rate which is defined as the percentage of successful recovery of 100 trials. If the relative error is smaller than $10^{-2}$, i.e.,

\[
\text{Relative Error} := \frac{\|x - x_{\text{orig}}\|}{\|x\|} < 10^{-2},
\]

the recovery is regarded as a success. Here $x$ denotes our computed solution by IIHT. For Gaussian and pDCT matrices, we generate them with $m = 64$ and $n = 256$ and the ‘true’ solutions with sparsity level from 15 to 35. For opDCT matrices, we generate them with $m = 100$ and $n = 2000$ and the ‘true’ solutions with sparsity level from 1 to 20. Figure 2 shows the performance of IIHT under nonnegative constraint with different types of matrices.
Next, in order to see the accuracy and speed of IIHT, we run 40 trials with slightly high dimensions for Gaussian and pDCT matrices. We let \( n \) increase from 1000 to 9000, while keeping \( m = n/4 \) and \( s = 1\% n \) or \( s = 5\% n \). The average relative error and CPU time are respectively recoded to illustrate the accuracy of the solutions and speed of IIHT, see Figure 3. For exact recovery (\( \sigma_0 = 0 \)), the relative error is pretty small. The CPU time is around two seconds for all trails, which demonstrates the efficiency of IIHT.

Finally, we run 40 trials for Gaussian and pDCT matrices with \( n = 5000, m = n/4 \) and
\( s = 1\% n \) or \( s = 5\% n \) for exact and stable recovery to see the decreasing of objective function at each iteration. Results recorded in Figure 4 show that only 5 (15) iterations are needed to get the desirable solutions when \( s = 1\% n \) (\( s = 5\% n \)) for both exact and stable recovery, which shows that the gain in decreasing the objective function per iteration is sufficient.

![Figure 4: Objective function value at each iteration.](image)

### 4.2 Sparse Logistic Regression Problem

The logistic regression model plays an important role in two-class classification method that has been used widely in many applications ranging from data mining, machine learning, computer vision, to bioinformatics. Specifically, given data \( z \in \mathbb{R}^n \) and weights \((v, w)\), it assumes the following probability model

\[
P(b = \pm 1|v, w) = \frac{1}{1 + \exp(-b(v + w^\top z))},
\]

where \( b \) is the class label. If \( z^i \in \mathbb{R}^n, i = 1, \cdots, m \) are \( m \) given samples with \( n \) features and \( b_i \in \{1, -1\}, i = 1, \cdots, m \) are given \( m \) binary outcomes or labels, one estimates \((v, w)\) by minimizing the negative log-likelihood:

\[
\min_{v,w} L(v, w) := \sum_{i=1}^{m} \log \left( 1 + \exp(-b_i(v + w^\top z^i)) \right)
\]

Recently, sparse logistic regression is attractive in many applications involving high-dimensional data, seen [23, 27] and references therein. The corresponding optimization model is

\[
(4.2) \quad \min_{v,w} L(v, w), \quad \text{s.t. } \|w\|_0 \leq s.
\]

Letting \( x = (v; w) \in \mathbb{R}^{n+1} \) and \( p^i = (1; z^i) \in \mathbb{R}^{n+1} \), denote the so-called *logistic loss* as

\[
f(x) := \frac{1}{m} \sum_{i=1}^{m} \log \left( 1 + \exp(-b_i v \cdot x^\top p^i) \right),
\]

We select two popular methods for numerical comparison. One is the penalty method of Lu and Zhang [27] proposed a penalty decomposition (PD) method. The other is the first-order method
SLEP of [25]. To compare the solution quality of the three methods, we adopt the criterion, error rate, from [27], which is defined by

\[
\text{Error Rate} := \frac{1}{m} \sum_{i=1}^{m} |\text{sign}(x^T p^i) - b_i|,
\]

where \(x\) is the solution obtained by methods and \(\text{sign}(a)\) is the sign function, i.e., \(\text{sign}(a) = 1\) if \(a > 0\); \(\text{sign}(a) = -1\) if \(a < 0\); \(\text{sign}(a) = 0\), otherwise.

(a) Parameter setting. We will test two kinds of data sets: real data sets and random data sets to be described below. For the PD method, we set \(\text{eps} = 10^{-3}, \text{maxit} = 1000\), and the rest of its parameters are set by default. For SLEP method, we set \(\text{opts.mFlag} = 1, \text{opts.lFlag} = 1, \text{opts.tFlag} = 2\), and fix \(\rho = 0.05\) for the random data sets, where \(\rho\) corresponds to \(l_1\) norm penalty parameter \(\lambda\). However, \(\rho\) is appropriately adjusted for the real data sets. The rest of its parameters are set by default. For our IIHT, we use

<table>
<thead>
<tr>
<th>Table 4: Parameters for IIHT.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real data</td>
</tr>
<tr>
<td>Random data</td>
</tr>
</tbody>
</table>

Here, \(A := [p^1, \cdots, p^m]\) and \(r := \max\{m, n\}/\min\{m, n\}\), \(x^0 = (v^0; w^0)\). We always start with \(w^0\) as a zero vector, and initialize \(v^0 = 10\) for real data sets but \(v^0 = 1\) for random data sets.

(b) Comparison on real data. In our first experiment, we test three real data sets. The first data set is the colon tumor gene expression data\(^1\) with 62 samples and 2000 features. The second data is the ionosphere\(^2\) data with 351 samples and 34 features. The third one is the German Credit data\(^3\) with 1000 samples and 24 features. The first and third data sets are from the UCI machine learning bench market repository [32]. We standardize each data set so that the sample mean is zero and the sample variance is one. We first apply SLEP to (4.2) with a sequence of suitably chosen \(\rho\) to obtain solutions \(\hat{w}\) with an increasing sparsity sequence such as \(\|\hat{w}\|_0 = 1, 2, \cdots, 20\). We then set \(s\) being same as \(\|\hat{w}\|_0\) for PD and IIHT, so that the solutions of these three methods are of the same sparsity.

Results for the first two data sets are recorded in Figure 5. In terms of CPU time, PD performs poorly for both data sets, while SLEP and IIHT run very fast. For Colon data, SLEP basically gets lowest logistic loss and error rate, and PD produces the highest ones. For Ionosphere data, there is no big difference for logistic loss between PD and IIHT. Both are better than SLEP. In terms of error rate, IIHT behaves the best, followed by PD and SLEP.

In fact, the error rate is often used to evaluate the quality of a model vector, which is taken the sum over the testing samples instead of the training samples in (4.3). It is well known that when the ratio between the number of training samples and the number of features is small, namely, \(m/n\), the error rate is usually high for most of models. Thus, Colon and Ionosphere data sets may not be appropriate for evaluating the error rate. Based on this, we chose German Credit data to estimate it. Specifically, we simply divide this data into two parts: the first 900 samples being training data and the rest 100 samples being testing data. Then we apply three

3. German Credit data: https://archive.ics.uci.edu/ml/datasets/Statlog+(German+Credit+Data).
methods in the way as above. The results are shown in Figure 6. For training data, PD returns the best results in terms of logistic loss and error rate, followed by IIHT and SLEP. However, IIHT basically outperforms SLEP and PD for testing data, as it generates lowest logistic loss for most cases.

(c) Comparison on random data. Now we compare the three methods on the random data sets, where the samples \( \{z^1, \ldots, z^m\} \) and the corresponding outcomes \( \{b^1, \ldots, b^m\} \) are generated in the same manner as [27]. In detail, for each instance we choose equal number of positive and negative samples, that is, \( m_+ = m_- = m/2 \), where \( m_+ \) (resp., \( m_- \)) is the number of samples with outcome +1 (resp., -1). The features of positive (resp., negative) samples are independent and identically distributed, drawn from a normal distribution \( \mathcal{N}(\phi, 1) \), where \( \phi \) is in turn drawn from a uniform distribution on \( [0, 1] \) (resp., \( [-1, 0] \)). Corresponding pseudo
MTALAB codes are:

\[
T = \text{randperm}(m); \quad b = \text{ones}(m,1); \quad b(T(1:m/2)) = -1;
\]
\[
z^i = b^i * \text{rand} + \text{randn}(n,1), \quad i = 1, \cdots, m.
\]

Data of different sizes are generated. For each size, we randomly generate the data set consisting of 40 trials. For each trial, let \( \hat{w} \) be the approximate optimal solution obtained by \text{SLEP}. We then apply our PD and \text{IIHT} methods to solve problem (4.2) with \( s = \| \hat{w} \|_0 \) so that the resulting approximate optimal solutions are at least as sparse as \( \hat{w} \). The results of the three methods for the these randomly generated instances are presented in Table 5. Clearly, \text{IIHT} obtains the best results with lowest logistic loss and least CPU time. PD outperforms \text{SLEP} in terms of logistic loss but takes the most time.

<p>| Table 5: Average results generated by three methods. |
|----------------|----------------|----------------|----------------|</p>
<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( s )</th>
<th>\text{Logistic Loss}</th>
<th>\text{Time}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>\text{IIHT}</td>
<td>\text{PD}</td>
</tr>
<tr>
<td>1000</td>
<td>1000</td>
<td>127.2</td>
<td>2.42e-04</td>
<td>2.46e-04</td>
</tr>
<tr>
<td>3000</td>
<td>145.3</td>
<td>4.90e-05</td>
<td>2.26e-04</td>
<td>1.55e-01</td>
</tr>
<tr>
<td>5000</td>
<td>165.2</td>
<td>3.40e-05</td>
<td>1.89e-04</td>
<td>1.50e-01</td>
</tr>
<tr>
<td>3000</td>
<td>1000</td>
<td>195.4</td>
<td>9.94e-05</td>
<td>4.10e-04</td>
</tr>
<tr>
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<td>233.3</td>
<td>5.31e-05</td>
<td>4.10e-04</td>
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<tr>
<td>5000</td>
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</tr>
<tr>
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<td>1000</td>
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<td>3.06e-05</td>
<td>5.72e-04</td>
</tr>
<tr>
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<td>304.5</td>
<td>1.58e-05</td>
<td>3.84e-04</td>
<td>1.87e-01</td>
</tr>
<tr>
<td>5000</td>
<td>326.3</td>
<td>2.14e-05</td>
<td>2.96e-04</td>
<td>1.86e-01</td>
</tr>
</tbody>
</table>

4.3 Phase retrieval problem

Phase retrieval is the problem that aims at recovering a signal from the magnitude of its Fourier transform. Namely, it is to find a real-valued discrete time signal \( x \in \mathbb{R}^N \) from its magnitude-squared of an \( N \) point discrete Fourier transform (DFT):

\[
b_j = \left\| \sum_{k=1}^{n} x_k e^{-2\pi i (j-1)(k-1)/N} \right\|^2, \quad j = 1, \cdots, N.
\]

Here \( x \) is constructed as \( x = (x_1, \cdots, x_n, 0, \cdots, 0)^\top \in \mathbb{R}^N \). If we denote \( F \) the DFT matrix, then each elements \( F_{jk} = e^{-2\pi i (j-1)(k-1)/N} \) and \( b = |Fx|^2 \), where \( | \cdot |^2 \) denotes the element-wise absolute-squared value. Therefore, phase retrieval of sparse signals actually can be reformulated as the following model (see [35] for details).

\[
\min_{x \in \mathbb{R}^N} \sum_{i=1}^{N} (|F_i x|^2 - b_i)^2,
\]
\[
\text{s.t.} \quad \|x\|_0 \leq s, \quad \text{supp}(x) \subseteq \{1, \cdots, n\}.
\]

where \( F_i \) is the \( i \)-th row of \( F \). Actually, phase retrieval of sparse signals is a special case of the more general quadratic compressed sensing (QCS) problem [5, 36]. For our \text{IIHT}, we set
Maxiter = 2000, α₀ = 0.001, ε = 10⁻² and β = 0.1, σ = 10⁻⁴. We generate $y \in \mathbb{R}^n$ with sparsity $s$ as in (4.1), and then get $x_{\text{orig}}$ and $b$ by the pseudo MATLAB codes:

$$x_{\text{orig}} = [y; \text{zeros}(N - n, 1)];$$
$$b = \text{abs}(\text{fft}(x_{\text{orig}})).^2 + \sigma_0 * \text{randn}(N, 1);$$

In order to evaluate the performance of IIHT, we compare it with GESPAR proposed in [35]. Its parameters are set by default.

![Figure 7: Success rates with three types of matrices.](image)

By fixing $n = 64$ and $N = 64, 128$ under $\sigma_0 = 0.01$ and $\sigma_0 = 0.1$, we test 100 trials for these two methods with different sparsity level $s$. The corresponding success rate which is defined as before and CPU time are taken into consideration in illustrating their performance. Results shown in Figure 7 demonstrate that IIHT outperforms GESPAR when $N = 64$, while performs worse than GESPAR when $N = 128$, regardless of noise level.

To see the accuracy of the solutions and the speed of these two methods, we now run 40 trials with slightly higher dimensions $n$ increasing from 500 to 3000 and keeping $N = 2n$, $s = 1\%n$. We also test them under two noise levels $\sigma_0 = 0.01$ and $\sigma_0 = 0.1$. We only report results associated with successful recovery, i.e., Relative Error $< 0.01$. Such results are recorded in Table 6, in which IIHT outperforms GESPAR in terms of both average CPU time and average relative error when $n \geq 1500$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\sigma_0 = 0.01$</th>
<th>$\sigma_0 = 0.1$</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>Time</td>
<td>Relative Error</td>
</tr>
<tr>
<td></td>
<td>IIHT</td>
<td>GESPAR</td>
</tr>
<tr>
<td>500</td>
<td>1.82</td>
<td>0.82</td>
</tr>
<tr>
<td>1000</td>
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<td>98.53</td>
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<td>9.07</td>
<td>284.07</td>
</tr>
<tr>
<td>3000</td>
<td>12.45</td>
<td>490.90</td>
</tr>
</tbody>
</table>

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5 Conclusion

In this paper, we studied an improved version of the popular Iterated Hard-Thresholding (IHT) algorithm, for the sparsity and nonnegativity constrained optimization, from the viewpoint of optimization. We try to answer the questions that are common in optimization. Those questions include towards what stationary point that the IHT would converge to and at what speed. In order to answer those questions, we studied the relationships among the three stationary points ($\alpha$-, $B$- and $C$-stationary points) and local (global) minimizers of (1.1). Moreover, we established some results on convergence and linear convergence rates of IHT by including the Armijo line search in IHT. Numerical experiments demonstrated the efficiency of the improved IHT on three widely tested problems.

Two immediate questions arise from this research. One is to assess whether the nonmonotone line search strategy used by Lu [26] would lead to more efficient performance of IHT and lead to stronger convergence results. The second question is whether we can establish convergence to the global minimizer for the case that the sparse constraint is not tight.

References


