Numer. Math. (2003)
Digital Object Identifier (DOI) 10.1007/s00211-003-0471-z

Numerische Mathematik

Finite termination of a dual Newton method for convex best C^1 interpolation and smoothing*

Houduo Qi[1], **Liqun Qi**[2]

[1] School of Mathematics, The University of New South Wales, Sydney, NSW 2052, Australia; e-mail: hdqi@maths.unsw.edu.au

[2] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong; e-mail: maqilq@polyu.edu.hk

Received May 2, 2002 / Revised version received February 1, 2003 /
Published online –

Summary. Given the data $(x_i, y_i) \in \Re^2$, $i = 0, 1, \ldots, n$ which are in convex position, the problem is to choose the convex best C^1 interpolant with the smallest mean square second derivative among all admissible cubic C^1-splines on the grid. This problem can be efficiently solved by its dual program, developed by Schmdit and his collaborators in a series of papers. The Newton method remains the core of their suggested numerical scheme. It is observed through numerical experiments that the method terminates in a small number of steps and its total computational complexity is only of $O(n)$. The purpose of this paper is to establish theoretical justification for the Newton method. In fact, we are able to prove its finite termination under a mild condition, and on the other hand, we illustrate that the Newton method may fail if the condition is violated, consistent with what is numerically observed for the Newton method. Corresponding results are also obtained for convex smoothing.

Mathematics Subject Classification (2000): 41A29, 65D15, 49J52, 90C25

1 Introduction

The given data $(x_i, y_i) \in \Re^2$, $i = 0, 1, \ldots, n$ is said to be in convex position if

$$\tau_i \leq \tau_{i+1} \qquad i = 1, \ldots, n-1$$

* The work was supported by the Australian Research Council for the first author and by the Hong Kong Research Grant Council for the second author.

Correspondence to : H. Qi

2	1	1	0	4	7	1	B	Dispatch: 5/4/2003 Total pages: 21 Disk Received ☑ Disk Used ☑	Journal: Numer. Math. Not Used ☐ Corrupted ☐ Mismatch ☐
Jour. No			Ms. No.						

where $\tau_i := (y_i - y_{i-1})/(x_i - x_{i-1})$ and

$$\Delta: \ a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

The problem to be considered in this paper is to find a convex interpolant to the given data with the smallest mean square second derivative, which is often referred to as the convex best interpolation problem and described as follows:

$$\text{minimize} \ \ \|s''\|_2 \tag{1}$$

$$\text{subject to} \ \ s(x_i) = y_i, \quad i = 0, 1, \cdots, n, \tag{2}$$

$$s \text{ is convex on } [a, b], \quad s \in V[a, b], \tag{3}$$

where constraint (2) is the *interpolation condition* and constraint (3) is the *convexity restriction* on the admissible functions from some function space $V[a, b]$ defined on $[a, b]$, giving meaningful definition of the objective (1).

Now we choose the admissible function space $V[a, b] := \mathrm{Sp}\,(3, \Delta)$, the set of cubic C^1 splines on Δ. Then for $s \in \mathrm{Sp}\,(3, \Delta)$, we have for $x \in [x_{i-1}, x_i]$,

$$\begin{aligned} s(x) = y_{i-1} + m_{i-1}(x - x_{i-1}) + (3\tau_i - 2m_{i-1} - m_i)\,\frac{(x - x_{i-1})^2}{h_i} \\ + (m_{i-1} + m_i - 2\tau_i)\,\frac{(x - x_{i-1})^3}{h_i^2} \end{aligned} \tag{4}$$

with $h_i := x_i - x_{i-1}$, $i = 1, \ldots, n$. It follows that

$$s(x_i) = y_i, \qquad s'(x_i) = m_i, \qquad i = 0, 1, \ldots, n. \tag{5}$$

Further, s is easily proved to be convex on $[a, b]$ if and only if

$$2m_{i-1} + m_i \le 3\tau_i \le m_{i-1} + 2m_i, \qquad i = 1, \ldots, n; \tag{6}$$

see [19]. Also it is easy to see that s is twice differentiable except, perhaps, at the nodes $\{x_i\}_0^n$. Hence the Lebesgue norm in (1) makes sense with respect to s'', meaning minimizing the mean square second derivative. Thus we arrive at the following minimization problem:

$$\begin{aligned} &\min \ \ \phi(m_0, \ldots, m_n) := \int_a^b s''(x)^2 dx = \sum_{i=1}^n \phi_i(m_{i-1}, m_i) \\ &\text{s. t.} \ \ 2m_{i-1} + m_i \le 3\tau_i \le m_{i-1} + 2m_i, \quad i = 1, \ldots, n \end{aligned} \tag{7}$$

where

$$\phi_i(x, y) := \frac{4}{h_i}\left\{x^2 + xy + y^2 - 3\tau_i(x + y) + 3\tau_i^2\right\}. \tag{8}$$

This is a quadratic programming problem of special structure. The Hessian of ϕ is symmetric positive definite. Hence problem (7) is uniquely solvable.

Although it can be directly solved by standard constrained optimization methods [11,20], problem (7) can be more effectively solved by its (unconstrained) dual program. Let $q : \Re^2 \to \Re$ be the continuously differentiable piecewise quadratic function defined by

$$q(a,b) := \begin{cases} (a^2+ab+b^2) & \text{for } a \le 0,\ b \le 0 \\ (\frac{1}{2}a+b)^2 & \text{for } a \ge 0,\ a+2b \le 0 \\ (a+\frac{1}{2}b)^2 & \text{for } b \ge 0,\ 2a+b \le 0 \\ 0 & \text{for } a+2b \ge 0,\ 2a+b \ge 0. \end{cases}$$

The dual problem of (7) is the following unconstrained optimization problem, which can be derived either by the Fenchel conjugate theory [4,26], or by the Kuhn-Tucker optimality theorem [2].

$$\max_{p \in \Re^{n-1}} \ -\sum_{i=1}^{n} \frac{h_i}{12} q(p_i, p_{i-1}) - \sum_{i=1}^{n-1} p_i(\tau_{i+1} - \tau_i) \tag{9}$$

with $p_0 = p_n = 0$. We write (9) as a minimization problem:

$$\min_{p \in \Re^{n-1}} \ L(p) := \sum_{i=1}^{n} \frac{h_i}{12} q(p_i, p_{i-1}) + \sum_{i=1}^{n-1} p_i(\tau_{i+1} - \tau_i) \tag{10}$$

with $p_0 = p_n = 0$. Since $L(\cdot)$ is convex, the optimality condition of (10) has the form of nonlinear equations

$$F(p) = -d \tag{11}$$

where $d \in \Re^{n-1}$ with $d_i = 12(\tau_{i+1} - \tau_i)$ and $F : \Re^{n-1} \to \Re^{n-1}$ is given by

$$F_i(p) = h_{i+1}\partial_2 q(p_{i+1}, p_i) + h_i \partial_1 q(p_i, p_{i-1}), \quad i = 1, \ldots, n-1. \tag{12}$$

Once a solution of (11), say $\bar{p}$, is obtained, the solution of (7), denoted by $\bar{m}$, can be calculated via the explicit formula [2, (28)]:

$$\bar{m}_{i-1} = \tau_i - \frac{h_i}{12}\left(\bar{p}_i + \frac{1}{2}\bar{p}_{i-1}^{+} - 2\bar{p}_{i-1}^{-}\right)^{-},$$
$$\bar{m}_i = \tau_i + \frac{h_i}{12}\left(\bar{p}_{i-1} + \frac{1}{2}\bar{p}_i^{+} - 2\bar{p}_i^{-}\right)^{-},$$

for $i = 1, 2, \ldots, n$, where $a^+ = \max(0, a)$ and $a^- = -\min(0, a)$ for $a \in \Re$. Then the convex best C^1 interpolant can be constructed by (4). Based on those theoretical results, it is suggested that the (ordinary) Newton method is applied to (10) or equivalently to (11). Numerical experiments [2,23–25] show that the Newton method terminates in a small number of steps (averaging $3-5$ steps) if a sufficiently good starting point is used, which is often provided by the steepest descent method. The finite termination is even observed when starting from rough points. It is also pointed out that the Newton method alone sometimes failed to find a solution [23,26]. However, it has been unknown when the Newton method has the finite termination property or when it fails. The difficulty is partially due to the combinatorial nature of the generalized Hessian, which defines the Newton method, see (13) and (16). This nature is the direct consequence of separability (L is the sum of n piecewise quadratic functions of dimension 2 and a linear term) and twice non-differentiability of the function L in (10), which in turn makes it extremely difficult in proving nonsingularity of the generalized Hessian.

The purpose of this paper is to establish theoretical justification for the Newton method. On the one hand, we show the finite termination of the Newton method under a mild condition by describing the accurate structure of the generalized Hessian, see Lemma 3.2; and on the other hand, we illustrate with an example that it may fail to find a solution if the condition is violated. These results are consistent with what is numerically observed for the Newton method. Corresponding results for convex smoothing are also obtained. Other contribution in this regard includes deriving the explicit formula (53) of the generalized Hessian. The problem of convex smoothing is to determine a smooth function s such that $s(x_i)$ is an approximation to y_i (instead of interpolating y_i) and s is convex. Convex smoothing is particularly useful when the feasible set (6) is empty, i.e., convex C^1 interpolation (4) does not exist, see [23,26]. See also [3] for general comments for smoothing.

Another optimization problem of (1) is of the convex L_2 interpolation, where the admissible function space $V[a, b]$ is $W^{2,2}[a, b]$, the Sobolev space of functions with absolutely continuous first derivative and second derivative in $L^2[a, b]$. This problem has been treated in [18,14,1,9]. The solution turns out to be a cubic spline, but with nodes in general unknown. The quadratic convergence of the Newton method for this problem, conjectured in [14], has been recently settled in [6–8], which also inspired our investigation to problems considered in this paper.

An outline of the paper follows. Since the nonsmooth equation (11) is piecewise linear and we will consider the convergence of the Newton method for solving it, it is necessary to formulate the Newton method formally for such a nonlinear system of equations, which is done in the next section. It is shown in Section 3 that the Newton method has the finite termination property under a mild condition, and violating this condition may result in failure of the method. In Section 4, we extend these results to the problem of convex smoothing. Conclusions are drawn in Section 5.

2 The newton method for systems of PC^1 equations

Since the purpose of this paper is to analyze the convergence property of the Newton method, it is necessary to present the method in a formal context, see (13). We also describe it in a different way using the B-differential, see (16). Both versions will benefit our convergence analysis in the next section.

Although the Newton method we will treat in this paper is for systems of piecewise linear equations, we found it extremely suitable to describe it under a more general framework for systems of PC^1 (piecewise continuously differentiable) equations, mainly due to the following two reasons: First, the result of quadratic convergence of the Newton method for PC^1 equations will be used in our convergence analysis. Hence there is need to present the Newton method for PC^1 equations separately; Second, the definition of the Newton method for PC^1 equations reveals a way of calculating the Hessian matrix, which has a good use in our key result Lemma 3.2. We begin with a formal definition of a PC^1 mapping.

Definition 2.1 *[16] Let $G : \Re^\ell \to \Re^\ell$ be a continuous mapping. G is a PC^1 mapping if there exists a countable family $\{U_i : i \in \Lambda\}$ of closed subsets of $\Re^\ell$ such that*

(a) cl (int U_i) $= U_i$ for every $i \in \Lambda$,
(b) $\cup_{i\in\Lambda} U_i = \Re^\ell$,
(c) (int U_i) $\cap$ (int U_j) $= \emptyset$ whenever $i, j \in \Lambda$ and $i \neq j$,
(c) $\{U_i : i \in \Lambda\}$ has a locally finite property, i.e., for any $x \in \Re^\ell$, there exists an open neighborhood $\mathcal{N}$ of x such that $\{i : \mathcal{N} \cap U_i\}$ is finite,
(e) for each $i \in \Lambda$ the restriction $G|U_i$ of the mapping to each U_i is a C^1 mapping. More precisely, there exists C^1 mapping G^i from an open neighborhood of U_i into $\Re^\ell$ such that $G(x) = G^i(x)$ for any $x \in U_i$.

We call the family $\{U_i : i \in \Lambda\}$ a subdivision of $\Re^\ell$, and each U_i (equivalently G^i) a piece. So we say that G is PC^1 on a subdivision $\{U_i : i \in \Lambda\}$ of $\Re^\ell$. For simplicity of discussions, we shall assume for the moment that G^i is defined on the whole space $\Re^\ell$ for each $i \in \Lambda$.

Let $G : \Re^\ell \to \Re^\ell$ be a PC^1 mapping with pieces U_i and G^i. Then the k-th step of the Newton method for the PC^1 equations: $G(p) = 0$ is as follows: Given $p^k \in \Re^\ell$, choose a piece U_i that contains p^k; then find

$$p^{k+1} = p^k - (\nabla G^i(p^k))^{-1} G(p^k). \tag{13}$$

Another way of stating the Newton method (13) is using the B-differential, denoted by ∂_B, for nonsmooth Lipschitz mappings, introduced by Qi in [22]. Suppose for a while that the mapping $G : \Re^\ell \to \Re^\ell$ is locally Lipchitzian, let D_G be the set where G is differentiable and $\nabla G(p)$ denote its Jacobian at $p \in D_G$. Then $\partial_B G(\cdot)$ at a point p is defined by

$$\partial_B G(p) := \left\{ \lim_{\substack{p^r \to p \\ p^r \in D_G}} \nabla G(p^r) \right\}. \tag{14}$$

It is easy to see that $\partial_B G(\cdot)$ is well defined everywhere and is bounded. Now we come back to the case that G is a PC^1 mapping. Let

$$I(p) = \{i \in \Lambda : \ p \in U_i\} \quad \text{for each } p \in \Re^\ell.$$

Then we have

$$\partial_B G(p) = \{\nabla G^i(p) : \ i \in I(p)\}. \tag{15}$$

Using the B-differential, the k-th step of the Newton method (13) has the following form: Choose $V_k \in \partial_B G(p^k)$; then find

$$p^{k+1} = p^k - V_k^{-1} G(p^k). \tag{16}$$

If $G(\cdot)$ is the gradient mapping of some continuously differentiable function $\theta : \Re^\ell \to \Re$, then $\partial_B G(p)$ is usually called the generalized Hessian of $\theta(\cdot)$ at p, see [13] for general treatment of the generalized Hessian with C^{L_1} data. The following local convergence result was established in [16, Theorem 1], restated in the B-differential:

Theorem 2.2 *Let p^* be a solution of the system of PC^1 equations $G(p) = 0$. Suppose that*
(i) all $V \in \partial_B G(p^*)$ are nonsingular matrices and
(ii) for each $i \in I(p^*)$, ∇G^i is locally Lipschitz continuous at p^*.
Then every sequence generated by the method (16) is quadratically convergent to p^* provided that the starting point p^0 is sufficiently close to p^*.

Before going to the next section, we would like to make two remarks on the calculation of $\partial_B G$: (i) If G is continuously differentiable at p, then $\partial_B G(p) = \{\nabla G(p)\}$. Differentiability only is not enough for $\partial_B G$ being a singleton. (ii) If G is the sum of two Lipschitz functions, say

$$G(p) = G^1(p) + G^2(p),$$

with both G^1 and G^2 being directionally differentiable, we have the relation in general [21]

$$\partial_B G(p) \subseteq \partial_B G^1(p) + \partial_B G^2(p).$$

If one of G^i, $i = 1, 2$ is continuously differentiable at p, equality holds. To our special function F defined in (12), even if neither of $\partial_B G^i(p)$, $i = 1, 2$ is singleton, it is still possible to detect elements in $\partial_B G^1(p)$ and in $\partial_B G^2(p)$ which make up elements in $\partial_B G(p)$. It is this possibility that allows us to prove nonsingularity of the generalized Hessian. See (19) for an illustration of this remark.

3 Finite termination

Let F be defined by (12). Then the Newton method for the equation (11) is as follows:

$$(17) \quad p^{k+1} = p^k - V_k^{-1}(F(p^k) + d) \qquad \text{for some } V_k \in \partial_B F(p^k).$$

In this section, we shall study the convergence properties of the Newton method (17). We first characterize when the condition (i) in Theorem 2.2 is satisfied for F, with which local quadratic convergence follows from Theorem 2.2 as the second condition (ii) is automatically met with any piecewise linear mappings. Then we show that the finite termination of the Newton method follows when the iterate p^k is sufficiently close to a solution p^* and it lies in the same piece U_i as p^* does. The finite termination property explains the numerically observed behavior of the Newton method that it terminates after a small number of steps when starting from a sufficiently good point [26,2,23,25].

To make it easy to calculate $\partial_B F(p)$, we further define two functions $f, g : \Re^2 \to \Re$ by

$$(18) \qquad f(a,b) := \partial_2 q(a,b) \quad \text{and} \quad g(a,b) := \partial_1 q(a,b).$$

Then

$$F_i(p) = h_{i+1} f(p_{i+1}, p_i) + h_i g(p_i, p_{i-1}), \quad i = 1, \ldots, n-1$$
$$\text{with } p_0 = p_n = 0$$

Hence when calculating elements in $\partial_B F_i(p)$, we need to calculate elements in $\partial_B f(p_{i+1}, p_i)$ and elements in $\partial_B g(p_i, p_{i-1})$ respectively. Take $\partial_B f(p_{i+1}, p_i)$ for example, the calculation is made with respect to all variables $(p_1, p_2, \ldots, p_{n-1})$. Since $f(p_{i+1}, p_i)$ is only dependent on p_{i+1} and p_i (independent of others), each element in $\partial_B f(p_{i+1}, p_i)$ must have the form of $\alpha e_i + \gamma e_{i+1}$, $\alpha, \gamma \in \Re$. Here e_i denotes the i-th unit vector in $\Re^{n-1}$. We also let e_0 and e_n be the zero vector in $\Re^{n-1}$. The task is to calculate α and γ. Likewise, each element in $\partial_B g(p_i, p_{i-1})$ has the form of $\omega e_{i-1} + \beta e_i$ with ω and β to be determined. Hence each element in $\partial_B F_i(p)$ has the form of

$$\omega e_{i-1} + (\alpha + \beta) e_i + \gamma e_{i+1} \quad \text{for some } \omega, \alpha, \beta, \gamma \in \Re.$$

Our key Lemma 3.2 characterizes when the condition (i) in Theorem 2.2 is satisfied. To give a clue on how to prove it, we first take a look at an example which gives rise to a four dimensional dual problem. The calculation process for elements in $\partial_B F(p)$ for some point $p \in \Re^4$ captures all ingredients for proving the lemma. Further, the example reveals more information at its solution, see Remark (iii) after the proof of Lemma 3.2

Example 3.1 *The given data is:* $(x_0, y_0) = (0, 0)$, $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2, 3)$, $(x_3, y_3) = (3, 7)$, $(x_4, y_4) = (4, 12)$ *and* $(x_5, y_5) = (5, 27)$.

The data is in convex position. We evaluate $\partial_B F(p)$ at $\bar{p} = (-1, 0, -1, -1)^T$. Let $\bar{p}_0 = \bar{p}_5 = 0$. We have four steps to calculate elements in $\partial_B F(p)$.

Step 1. $i = 1$. It then follows

$$g(p_1, p_0) = 2p_1 \quad \text{and} \quad f(p_2, p_1) = 2p_1 + p_2,$$

when p is near $\bar{p}$, resulting

$$F_1(p) = 2(h_1 + h_2)p_1 + h_2 p_2$$

for all p near $\bar{p}$ and hence

$$\partial_B F_1(\bar{p}) = \{2(h_1 + h_2)e_1 + h_2 e_2\}.$$

Step 2. $i = 2$. $g(p_2, p_1)$ is given by two pieces $p_2/2 + p_1$ and $2p_2 + p_1$ when p is near $\bar{p}$. To calculate $\partial_B g(\bar{p}_2, \bar{p}_1)$, we take a sequence $\{p^r\} \to \bar{p}$ with $p_2^r > 0$. Then $g(p_2^r, p_1^r) = p_2^r/2 + p_1^r$ for all r sufficiently large, yielding $\frac{1}{2}e_2 + e_1 \in \partial_B g(\bar{p}_2, \bar{p}_1)$. On the other hand, we take a sequence $\{p^r\} \to p$ with $p_2^r < 0$, then $g(p_2^r, p_1^r) = 2p_2^r + p_1^r$ for all r sufficiently large, yielding $2e_2 + e_1 \in \partial_B g(p_2, p_1)$. Hence

$$\partial_B g(\bar{p}_2, \bar{p}_1) = \left\{\frac{1}{2}e_2 + e_1,\ 2e_2 + e_1\right\}.$$

The function $f(p_3, p_2)$ near $\bar{p}$ is also given by two pieces: $p_2/2 + p_3$ and $2p_2 + p_3$. Like the calculation process above, we first take a sequence $\{p^r\} \to \bar{p}$ with $p_2^r > 0$, then $f(p_3^r, p_2^r) = p_2^r/2 + p_3^r$ for all r sufficiently large, resulting $\frac{1}{2}e_2 + e_3 \in \partial_B f(\bar{p}_3, \bar{p}_2)$. On the other hand, take $p_2^r < 0$, then $f(p_3^r, p_2^r) = 2p_2^r + p_3^r$ for all r sufficiently large, resulting $2e_2 + e_3 \in \partial_B f(\bar{p}_3, \bar{p}_2)$. Hence

$$\partial_B f(\bar{p}_3, \bar{p}_2) = \left\{\frac{1}{2}e_2 + e_3,\ 2e_2 + e_3\right\}.$$

Matching the cases of the positive sequence (i.e, $p_2^r > 0$) together as well as the cases of the negative sequence (i.e., $p_2^r < 0$) together, we have

$$\partial_B F_2(\bar{p}) = \left\{h_2 e_1 + \frac{1}{2}(h_2 + h_3)e_2 + h_3 e_3,\ h_2 e_1 + 2(h_2 + h_3)e_2 + h_3 e_3\right\}. \tag{19}$$

Step 3. $i = 3$. $g(p_3, p_2)$ is always given by $2p_3 + p_2$ for all p near $\bar{p}$, yielding

$$\partial_B g(\bar{p}_3, \bar{p}_2) = \{2e_3 + e_2\}.$$

$f(p_4, p_3)$ is given by $2p_3 + p_4$ for all p near $\bar{p}$, yielding

$$\partial_B f(\bar{p}_4, \bar{p}_3) = \{2e_3 + e_4\}.$$

Hence

$$\partial_B F_3(\bar{p}) = \{h_3 e_2 + 2(h_3 + h_4)e_3 + h_4 e_4\}.$$

Step 4. $i = 4$. Like Step 1, it is easy to see

$$\partial_B F_4(\bar{p}) = \{h_4 e_3 + 2(h_4 + h_5)e_4)\}\,.$$

Hence $\partial_B F(\bar{p})$ constitutes two elements, they are:

$$\begin{pmatrix} 2(h_1+h_2) & h_2 & & \\ h_2 & \frac{1}{2}(h_2+h_3) & h_3 & \\ & h_3 & 2(h_3+h_4) & h_4 \\ & & h_4 & 2(h_4+h_5) \end{pmatrix} \tag{20}$$

and

$$\begin{pmatrix} 2(h_1+h_2) & h_2 & & \\ h_2 & 2(h_2+h_3) & h_3 & \\ & h_3 & 2(h_3+h_4) & h_4 \\ & & h_4 & 2(h_4+h_5) \end{pmatrix}. \tag{21}$$

Matrix (21) is positive definite as it is diagonally dominant. Although matrix (20) is not diagonally dominant, it is positive definite too. A typical feature leading to this is that the diagonal element value $\frac{1}{2}(h_i + h_{i+1})$ does not occur in a consecutive way; That is, it cannot happen that any two adjacent diagonal elements can take values of the kind $\frac{1}{2}(h_i + h_{i+1})$ simultaneously. This feature is not accidental. It holds uniformly for points satisfying some condition, which is stated in the following lemma.

Lemma 3.2 *Let $\bar{p} \in \Re^{n-1}$ be given and $\bar{p}_0 = \bar{p}_n = 0$. Suppose*

$$(\bar{p}_i, \bar{p}_{i-1}) \in W \quad \textit{for all} \;\; i = 1, \ldots, n \tag{22}$$

where

$$W := \{(a, b) \mid 2a + b < 0 \;\textit{or}\; a + 2b < 0\}. \tag{23}$$

Then each element $V \in \partial_B F(\bar{p})$ has the following tridiagonal structure

$$V = \begin{pmatrix} V_{11} & h_2 & & \\ h_2 & \ddots & \ddots & \\ & \ddots & \ddots & h_{n-1} \\ & & h_{n-1} & V_{n-1,n-1} \end{pmatrix}$$

with V_{ii}, $i = 1, \ldots, n-1$ satisfying the following conditions
(i) $V_{11} = 2(h_1 + h_2)$ and $V_{n-1,n-1} = 2(h_{n-1} + h_n)$,
(ii) for $i = 2, \ldots, n-1$, $V_{ii} = 2(h_i + h_{i+1})$ or $V_{ii} = \frac{1}{2}(h_i + h_{i+1})$, and
(iii) it cannot occur that any two adjacent diagonal elements, say V_{ii} and $V_{i+1,i+1}$ take the values $\frac{1}{2}(h_j + h_{j+1})$, $j = i, i+1$ simultaneously.
Consequently, every element in $\partial_B F(\bar{p})$ is positive definite.

Proof. For any point $p \in \Re^{n-1}$ at which $F(p)$ is differentiable, $\nabla F(p)$ is tridiagonal. It then follows from the definition of ∂_B in (14) that every element in $\partial_B F(\bar{p})$ must be tridiagonal too. Now let $V \in \partial_B F(\bar{p})$.

(i) The first row of V must belong to $\partial_B F_1(\bar{p})$. We recall that

$$F_1(p) = h_2 f(p_2, p_1) + h_1 g(p_1, p_0).$$

Since $\bar{p}_0 = 0$ and $(\bar{p}_1, \bar{p}_0) \in W$, we must have $\bar{p}_1 < 0$. It then holds

$$q(p_1, p_0) = p_1^2 \quad \text{and} \quad g(p_1, p_0) = 2p_1$$

for all p sufficiently close to $\bar{p}$. Hence

$$\partial_B g(\bar{p}_1, \bar{p}_0) = \{2e_1\}.$$

We also have

$$q(p_2, p_1) = \begin{cases} (\frac{1}{2}p_2 + p_1)^2 & \text{if } p_1 < 0,\ p_2 \geq 0,\ p_2 + 2p_1 \leq 0 \\ p_2^2 + p_2 p_1 + p_1^2 & \text{if } p_1 < 0,\ p_2 \leq 0 \end{cases}$$

and $f(p_2, p_1)$ is given by one piece, i.e., $f(p_2, p_1) = 2p_1 + p_2$ for all $p_1 < 0$. Hence, we have

$$\partial_B f(\bar{p}_2, \bar{p}_1) = \{2e_1 + e_2\}.$$

It holds that

$$\partial_B F_1(\bar{p}) = h_2 \partial_B f(\bar{p}_2, \bar{p}_1) + h_1 \partial_B g(\bar{p}_1, \bar{p}_0) = \{2(h_1 + h_2)e_1 + h_2 e_2\}.$$

That is

$$V_{11} = 2(h_1 + h_2) \qquad \text{and} \qquad V_{12} = h_2.$$

Similarly, we can prove that

$$M_{n-1,n-2} = h_{n-1} \qquad \text{and} \qquad V_{n-1,n-1} = 2(h_{n-1} + h_n).$$

This finishes the proof for (i). We now prove (ii) and (iii) together.

(ii) and (iii). We consider $F_i, F_{i+1}, i \in \{2, \ldots, n-2\}$ together. There are four cases to be considered depending on which region the pair $(\bar{p}_i, \bar{p}_{i-1})$ falls in, which are depicted in Figure 1 for easy reference.

Case 1. $\bar{p}_i > 0$ and $\bar{p}_i + 2\bar{p}_{i-1} < 0$.

It then holds that

$$q(p_i, p_{i-1}) = (\frac{1}{2}p_i + p_{i-1})^2 \quad \text{and} \quad g(p_i, p_{i-1}) = \frac{1}{2}p_i + p_{i-1}$$

for all p sufficiently close to $\bar{p}$, yielding

$$\partial_B g(\bar{p}_i, \bar{p}_{i-1}) = \{\frac{1}{2}e_i + e_{i-1}\}. \tag{24}$$

Since $\bar{p}_i > 0$ and $(\bar{p}_{i+1}, \bar{p}_i) \in W$, we must have $\bar{p}_{i+1} < 0$, implying

$$q(p_{i+1}, p_i) = (p_{i+1} + \frac{1}{2}p_i)^2$$

and

$$f(p_{i+1}, p_i) = p_{i+1} + \frac{1}{2}p_i \quad \text{and} \quad g(p_{i+1}, p_i) = 2p_{i+1} + p_i$$

for all p sufficiently close to $\bar{p}$. Hence

$$\partial_B f(\bar{p}_{i+1}, \bar{p}_i) = \{e_{i+1} + \frac{1}{2}e_i\} \quad \text{and} \quad \partial_B g(\bar{p}_{i+1}, \bar{p}_i) = \{2e_{i+1} + e_i\}. \tag{25}$$

Now since $\bar{p}_{i+1} < 0$ and $(\bar{p}_{i+2}, \bar{p}_{i+1}) \in W$, we have for all p sufficiently close to $\bar{p}$

$$q(p_{i+2}, p_{i+1}) = \begin{cases} (\frac{1}{2}p_{i+2} + p_{i+1})^2 & \text{if } p_{i+2} \geq 0 \\ p_{i+2}^2 + p_{i+2}p_{i+1} + p_{i+1}^2 & \text{if } p_{i+2} \leq 0. \end{cases}$$

It is easy to see for those p, $f(p_{i+2}, p_{i+1})$ is given by one piece, that is

$$f(p_{i+2}, p_{i+1}) = 2p_{i+1} + p_{i+2},$$

resulting in

$$\partial_B f(\bar{p}_{i+2}, \bar{p}_{i+1}) = \{2e_{i+1} + e_{i+2}\}. \tag{26}$$

Hence we have from (24)–(26) that

$$\begin{aligned} \partial_B F_i(\bar{p}) &= h_{i+1}\partial_B f(\bar{p}_{i+1}, \bar{p}_i) + h_i \partial_B g(\bar{p}_i, \bar{p}_{i-1}) \\ &= \left\{ h_i e_{i-1} + \frac{1}{2}(h_i + h_{i+1})e_i + h_{i+1}e_{i+1} \right\} \end{aligned} \tag{27}$$

$$\begin{aligned} \partial_B F_{i+1}(\bar{p}) &= h_{i+2}\partial_B f(\bar{p}_{i+2}, \bar{p}_{i+1}) + h_{i+1}\partial_B g(\bar{p}_{i+1}, \bar{p}_i) \\ &= \{h_{i+1}e_i + 2(h_{i+1} + h_{i+2})e_{i+1} + h_{i+2}e_{i+2}\}. \end{aligned} \tag{28}$$

Case 2. $\bar{p}_i < 0$ and $\bar{p}_{i-1} < 0$.

It then holds that

$$q(p_i, p_{i-1}) = p_i^2 + p_i p_{i-1} + p_{i-1}^2$$

for all p sufficiently close to $\bar{p}$, implying

$$g(p_i, p_{i-1}) = 2p_i + p_{i-1}$$

and hence

$$\partial_B g(\bar{p}_i, \bar{p}_{i-1}) = \{2e_i + e_{i-1}\}. \tag{29}$$

Noticing $\bar{p}_i < 0$ and $(\bar{p}_{i+1}, \bar{p}_i) \in W$, we have

$$q(p_{i+1}, p_i) = \begin{cases} (\frac{1}{2}p_{i+1} + p_i)^2 & \text{if } p_{i+1} \geq 0 \\ (p_{i+1}^2 + p_{i+1}p_i + p_i^2) & \text{if } p_{i+1} \leq 0 \end{cases}$$

for all p sufficiently close to $\bar{p}$. It is easy to see for those p that

$$f(p_{i+1}, p_i) = 2p_i + p_{i+1}$$

and

$$g(p_{i+1}, p_i) = \begin{cases} \frac{1}{2}p_{i+1} + p_i & \text{if } p_{i+1} \geq 0 \\ 2p_{i+1} + p_i & \text{if } p_{i+1} \leq 0 \end{cases} \tag{30}$$

Hence

$$\partial_B f(\bar{p}_{i+1}, \bar{p}_i) = \{2e_i + e_{i+1}\}. \tag{31}$$

Also we note that for $(p_{i+2}, p_{i+1}) \in W$,

$$q(p_{i+2}, p_{i+1}) = \begin{cases} (p_{i+2} + \frac{1}{2}p_{i+1})^2 & \text{if } p_{i+1} \geq 0 \ \& \ p_{i+2} \leq 0 \\ (p_{i+2}^2 + p_{i+2}p_{i+1} + p_{i+1}^2) & \text{if } p_{i+1} \leq 0 \ \& \ p_{i+2} \leq 0 \\ (\frac{1}{2}p_{i+2} + p_{i+1})^2 & \text{if } p_{i+1} \leq 0 \ \& \ p_{i+2} \geq 0. \end{cases}$$

Thus we have for those p that

$$f(p_{i+2}, p_{i+1}) = \begin{cases} \frac{1}{2}p_{i+1} + p_{i+2} & \text{if } p_{i+1} \geq 0 \\ 2p_{i+1} + p_{i+2} & \text{if } p_{i+1} \leq 0. \end{cases} \tag{32}$$

It then follows from (30) and (32) that $\partial_B g(\bar{p}_{i+1}, \bar{p}_i)$ and $\partial_B f(\bar{p}_{i+2}, \bar{p}_{i+1})$ depending on the value $\bar{p}_{i+1}$ might take:
Case 2.1. If $\bar{p}_{i+1} > 0$, then we have

$$\partial_B g(\bar{p}_{i+1}, \bar{p}_i) = \{\frac{1}{2}e_{i+1} + e_i\} \text{ and } \partial_B f(\bar{p}_{i+2}, \bar{p}_{i+1}) = \{\frac{1}{2}e_{i+1} + e_{i+2}\}. \tag{33}$$

Case 2.2. If $\bar{p}_{i+1} < 0$, then we have

$$\partial_B g(\bar{p}_{i+1}, \bar{p}_i) = \{2e_{i+1} + e_i\} \text{ and } \partial_B f(\bar{p}_{i+2}, \bar{p}_{i+1}) = \{2e_{i+1} + e_{i+2}\}. \tag{34}$$

We will consider the remaining case $\bar{p}_{i+1} = 0$ later. It follows from (29), (31), (33) and (34) that for $\bar{p}_{i+1} > 0$

$$\partial_B F_i(\bar{p}) = \{h_i e_{i-1} + 2(h_i + h_{i+1})e_i + h_{i+1}e_{i+1}\} \tag{35}$$

$$\partial_B F_{i+1}(\bar{p}) = \left\{h_{i+1}e_i + \frac{1}{2}(h_{i+1} + h_{i+2})e_i + h_{i+2}e_{i+2}\right\} \tag{36}$$

and for $\bar{p}_{i+1} < 0$

$$\partial_B F_i(\bar{p}) = \{h_i e_{i-1} + 2(h_i + h_{i+1})e_i + h_{i+1}e_{i+1}\} \tag{37}$$

$$\partial_B F_{i+1}(\bar{p}) = \{h_{i+1}e_i + 2(h_{i+1} + h_{i+2})e_i + h_{i+2}e_{i+2}\}. \tag{38}$$

Now we consider the case $\bar{p}_{i+1} = 0$. Let a sequence $\{p^r\}$ converge to $\bar{p}$ with $p^r_{i+1} \neq 0$. Then $\partial_B F_i(p^r)$ and $\partial_B F_{i+1}(p^r)$ can be calculated by (35) and (36) for $p^r_{i+1} > 0$ and by (37) and (38) for $p^r_{i+1} < 0$. Hence by the definition of ∂_B in (14), we have

$$\partial_B F_i(\bar{p}) = \{h_i e_{i-1} + 2(h_i + h_{i+1})e_i + h_{i+1}e_{i+1}\} \tag{39}$$

$$\partial_B F_{i+1}(\bar{p}) = \left\{ \begin{array}{l} h_{i+1}e_i + \frac{1}{2}(h_{i+1} + h_{i+2})e_i + h_{i+2}e_{i+2} \\ h_{i+1}e_i + 2(h_{i+1} + h_{i+2})e_i + h_{i+2}e_{i+2} \end{array} \right\}. \tag{40}$$

Case 3. $(\bar{p}_i, \bar{p}_{i-1}) \in W$ is such that $\bar{p}_i = 0$.

Then we must have $\bar{p}_{i-1} < 0$. We say that Case 3 is the limit case of Case 1 and Case 2 in the sense that there exists a sequence $\{p^r\}$ with $p^r_i \neq 0$ converging to $\bar{p}$ with only two possibilities: either $p^r_i > 0$ or $p^r_i < 0$. When $p^r_i > 0$, $\partial_B F_j(p^r)$, $j = i, i+1$ can be calculated as for Case 1; while for $p^r_i < 0$, they can be calculated as for Case 2. In the end, elements in $\partial_B F_j(\bar{p})$, $j = i, i+1$ can be calculated according to (27) and (28), or (35) and (36), or (37) and (38), or (39) and (40).

Case 4. $\bar{p}_i < 0$ and $2\bar{p}_i + \bar{p}_{i-1} < 0$.

It then holds that

$$q(p_i, p_{i-1}) = \begin{cases} (p_i + \frac{1}{2}p_{i-1})^2 & \text{if } p_{i-1} \geq 0 \\ (p_i^2 + p_i p_{i-1} + p_{i-1}^2) & \text{if } p_{i-1} \leq 0 \end{cases}$$

for all p sufficiently close to $\bar{p}$, implying

$$g(p_i, p_{i-1}) = 2p_i + p_{i-1}$$

for all those p. Hence, we have

$$\partial_B g(\bar{p}_i, \bar{p}_{i-1}) = \{2e_i + e_{i-1}\}.$$

Noticing $\bar{p}_i < 0$ and $(\bar{p}_{i+1}, \bar{p}_i) \in W$, the situation becomes the same as in Case 2 from (29) and below it. Hence, elements in $\partial_B F_j(\bar{p})$, $j = i, i+1$ can be calculated according to (35) and (36), or (37) and (38), or (39) and (40).

In summary, elements in $\partial_B F_j(\bar{p})$, $j = i, i+1$, under the condition of (22), can be calculated by one of the pairs: (27) and (28), (35) and (36), (37) and (38), and (39) and (40). Observations in (ii) and (iii) hold for those pairs. This completes the proof for (ii) and (iii).

Finally, we prove the positive definiteness of every element in $\partial_B F(\bar{p})$. Let $V \in \partial_B F(\bar{p})$ and $0 \neq u \in \Re^{n-1}$. We then have

$$
\begin{aligned}
u^T V u &= \sum_{i=1}^{n-1} V_{ii}^2 u_i^2 + \sum_{i=2}^{n-1} 2h_i u_{i-1} u_i \\
&= 2h_1 u_1^2 + 2h_2 u_1^2 + \sum_{i=2}^{n-2} V_{ii}^2 u_i^2 \\
&\quad + \sum_{i=2}^{n-1} 2h_i u_{i-1} u_i + 2h_{n-1} u_{n-1}^2 + 2h_n u_{n-1}^2 \\
&\geq 2h_1 u_1^2 + \sum_{i=2}^{n-1} h_i \left\{ \min\{2u_{i-1}^2 + \frac{1}{2} u_i^2,\ \frac{1}{2} u_{i-1}^2 + 2u_i^2\} + 2u_{i-1} u_i \right\} \\
&\qquad + 2h_n u_{n-1}^2 \\
&= 2h_1 u_1^2 + \frac{1}{2} \sum_{i=2}^{n-1} h_i \min\{4u_{i-1}^2 + 4u_{i-1} u_i + u_i^2,\ u_{i-1}^2 \\
&\quad + 4u_{i-1} u_i + 4u_i^2\} + 2h_n u_{n-1}^2 \\
&= 2h_1 u_1^2 + \frac{1}{2} \sum_{i=2}^{n-1} h_i \min\{(2u_{i-1} + u_i)^2,\ (u_{i-1} + 2u_i)^2\} + 2h_n u_{n-1}^2 \\
&> 0. \qquad (41)
\end{aligned}
$$

Hence V is positive definite. The first inequality (41) used the fact of (iii). □

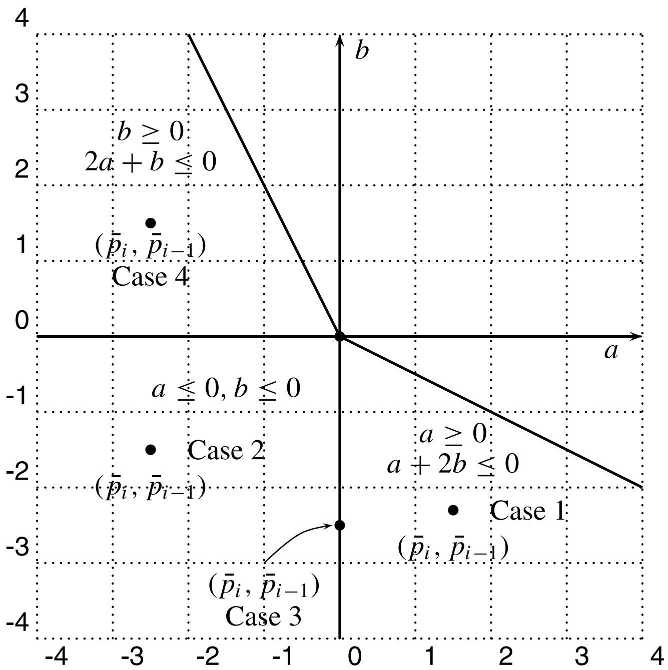

Fig. 1. Illustration of all four cases in the proof of Lemma 3.2

Lemma 3.2 allows us to establish finite termination of the Newton method for the equation (11).

Theorem 3.3 *Let $p^* \in \Re^{n-1}$ be a solution of the piecewise linear equation (11). Let $p_0^* = p_n^* = 0$. Suppose that*

$$(p_i^*, p_{i-1}^*) \in W \qquad \text{for all } i = 1, \ldots, n \tag{42}$$

where W is defined by (23). Then the Newton method finds the solution in a finite number of steps provided the starting point p^0 is sufficiently close to p^*.

Proof. Since every element in $\partial_B F(p^*)$ is nonsingular (Lemma 3.2), the iterates $\{p^k\}$ generated by the Newton method converge to p^* quadratically provided that p^0 is sufficiently close to p^* (Theorem 2.2). Suppose now p^k is so close to p^* that p^k and p^* fall in the same piece, say U_i, of F. Hence a linear function $F^i : \Re^{n-1} \to \Re^{n-1}$ exists and satisfies $F^i(U_i) = F|U_i$. Necessarily, $\nabla F^i(p^*) \in \partial_B F(p^*)$, see (15). That is, $\nabla F^i(p^*)$ is nonsingular. Noticing that $F^i(p^*) = F(p^*) = -d$, we have

$$\begin{aligned}
p^{k+1} &= p^k - (\nabla F^i(p^k))^{-1}(F(p^k) + d) \\
&= p^k - (\nabla F^i(p^k))^{-1}(F^i(p^k) - F^i(p^*) + F^i(p^*) + d) \\
&= p^k - (\nabla F^i(p^k))^{-1}(F^i(p^k) - F^i(p^*)) \\
&= p^k - (\nabla F^i(p^k))^{-1}\nabla F^i(p^k)(p^k - p^*) \\
&= p^k - (p^k - p^*) = p^*.
\end{aligned}$$

Since the above derivation is valid for any piece U_i which contains p^k and p^*, the Newton method finds the solution p^* and terminates at p^{k+1}. □

Remarks (i) The key idea of proving the finite termination of the Newton method is that when the iterate is sufficiently close to the solution, and both the iterate and the solution fall in one piece on which the underlying function is linear, the Newton method finds the solution in one step. This idea is not new and has been used in [10,15,17,27] in showing the finite termination of various Newton methods for a number of problems, which can be reformulated as piecewise linear equations.

(ii) Under the condition of (42), every element in $\partial_B F(p^*)$ is nonsingular. This fact implies that p^* is an isolated solution of (11), see [22, Proposition 2.5]. Since the solution set of the equation (11) is convex, p^* is the only solution. It then follows from [12, Proposition 3.2.5] that the level set of the unconstrained minimization problem (10) is bounded, i.e., the set

$$\{p \in \Re^{n-1} | \, L(p) \leq \alpha\}$$

is bounded for any $\alpha \in \Re$ if nonempty. Hence the steepest descent method can be used at an early stage to provide a sufficiently good starting point for the Newton method, and the Newton method will stop in a few more steps, according to Theorem 3.3. This numerical scheme is exactly what suggested in a series of papers by Schmidt and his collaborators [2,23,24,4,25,26], and their numerical observation is consistent with the convergence theory proved in this paper.

(iii) Remarks (i) and (ii) may be invalid if the condition (42) is violated. Let us illustrate this possibility by Example 3.1. It is easy to verify that the unique solution for Example 3.1 is $p^* = (2, -13, 60, -42)$. Obviously, $(p_1^*, p_0^*) = (2, 0) \notin W$, violating the condition (42). $\partial_B F(p^*)$ contains only one element (noticing all $h_i = 1$):

$$\begin{pmatrix} 1/2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1/2 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

which is singular. Hence Newton' method fails since the Newton equation is not guaranteed to admit a solution at all when p^k is sufficiently close to p^*.

4 Convex smoothing

In this section, we obtain the finite termination of the Newton method for the problem of convex smoothing, which is to construct the best approximation to the given data (often categorized as data fitting).

Given the data $(x_i, y_i) \in \Re^2$, $i = 0, 1, \ldots, n$, the convex smoothing considered in this section is to determine a convex C^1 spline s on the grid with $s(x_i) = z_i$ being approximation to y_i at x_i. Hence, it follows (4) that the function has the expression for $x \in [x_{i-1}, x_i]$

$$\begin{aligned} s(x) &= z_{i-1} + m_{i-1}(x - x_{i-1}) \\ &\quad + \left(3\frac{z_i - z_{i-1}}{h_i} - 2m_{i-1} - m_i\right)\frac{(x - x_{i-1})^2}{h_i} \\ &\quad + \left(m_{i-1} + m_i - 2\frac{z_i - z_{i-1}}{h_i}\right)\frac{(x - x_{i-1})^3}{h_i^2} \end{aligned} \tag{43}$$

with $s'(x_i) = m_i$, $i = 0, 1, \ldots, n$. Furthermore, s is convex if and only if

$$2m_{i-1} + m_i \le 3\frac{z_i - z_{i-1}}{h_i} \le m_{i-1} + 2m_i, \qquad i = 1, \ldots, n; \tag{44}$$

see (6). Then the best C^1 spline of this kind minimizes

$$\Phi(s) := \sum_{i=1}^{n}\left\{w_i \ell \int_{x_{i-1}}^{x_i} s''(x)^2 dx + r_{i-1}(s(x_{i-1}) - y_{i-1})^2 + r_i(s(x_i) - y_i)^2\right\} \tag{45}$$

subject to the constraints that

$$\text{(46)} \quad s \text{ has the form (43) on each } [x_{i-1}, x_i] \text{ and is convex on } [x_0, x_n]$$

where w_i, r_i and ℓ are positive parameters to be determined a priori [3,23].

With some labor of calculation, problem (45) with (46) reads

$$\min_{z,m\in\Re^{n+1}} \sum_{i=1}^{n} \Phi_i(z_{i-1}, m_{i-1}, z_i, m_i) \tag{47}$$

$$\text{s.t. } (z_{i-1}, m_{i-1}, z_i, m_i) \text{ satisfies (44) for } i = 1, \ldots, n \tag{48}$$

where

$$\Phi_i(a,b,c,d) := \frac{4w_i\ell}{h_i}\left\{\left(b - \frac{c-a}{h_i}\right)^2 + \left(b - \frac{c-a}{h_i}\right)\left(d - \frac{c-a}{h_i}\right) + \left(d - \frac{c-a}{h_i}\right)^2\right\} + r_{i-1}(a - y_{i-1})^2 + r_i(c - y_i)^2.$$

It follows from [23, Theorem 2] that problem (47)–(48) is uniquely solvable and can be effectively solved by its (unconstrained) dual program (in the form of minimization)

$$\min_{u,v\in\Re^{n-1}} \psi(u,v) := \sum_{i=1}^{n} H_i(u_{i-1}, v_{i-1}, u_i, v_i) \tag{49}$$

with $u_0 = u_n = v_0 = v_n = 0$ and

$$\begin{aligned} H_i(\rho,\xi,\sigma,\eta) := {} & y_{i-1}\rho - y_i\sigma + \frac{y_i - y_{i-1}}{h_i}(\xi - \eta) + \frac{h_i}{12w_i\ell}q(\xi,\eta) \\ & + \frac{h_i^2\rho^2 - 2h_i\rho(\xi-\eta) + (\xi-\eta)^2}{4r_{i-1}h_i^2} \\ & + \frac{h_i^2\sigma^2 - 2h_i\sigma(\xi-\eta) + (\xi-\eta)^2}{4r_{i-1}h_i^2}. \end{aligned}$$

Once a solution of (49) is obtained, the solution of the primal problem (47)–(48) can be calculated via the formula in [23, Theorem 3], and the convex function s minimizing (45) is constructed via (43). Note that we have made sign change in function $H_i(\cdot,\cdot,\cdot,\cdot)$, compared with [23] where $-\sigma$ and $-\eta$ are used. This sign change allows use of the same piecewise quadratic function $q(\cdot,\cdot)$, instead of using one of its variants [23, Proposition 4].

For simplicity, we assume $w_i = 1$ and $\ell = 1$, extension to arbitrary $w_i > 0$ and $\ell > 0$ is straight forward. Since $\psi(u,v)$ is piecewise continuously differentiable quadratic function, the piecewise linear system of equations obtained by setting $\nabla\psi(u, v) = 0$ (optimality condition) are

$$\begin{aligned}\nabla_{u_i}\psi(u, v) &= \frac{1}{2}\left(\frac{1}{r_{i-1}} + \frac{1}{r_i}\right)u_i - \frac{1}{2r_{i-1}h_i}v_{i-1}\\ &\quad + \frac{1}{2}\left(\frac{1}{r_{i-1}h_i} - \frac{1}{r_ih_{i+1}}\right)v_i + \frac{1}{2r_ih_{i+1}}v_{i+1} = 0 \qquad (50)\end{aligned}$$

and

$$\begin{aligned}\nabla_{v_i}\psi(u, v) &= \tau_{i+1} - \tau_i + \frac{1}{2r_{i-1}h_i}u_{i-1} + \frac{1}{2}\left(\frac{1}{r_{i-1}h_i} - \frac{1}{r_ih_{i+1}}\right)u_i\\ &\quad - \frac{1}{2r_ih_{i+1}}u_{i+1} - \frac{1}{r_{i-1}h_i^2}v_{i-1} + \left(\frac{1}{r_{i-1}h_i^2} + \frac{1}{r_ih_{i+1}^2}\right)v_i\\ &\quad - \frac{1}{r_ih_{i+1}^2}v_{i+1} + \frac{1}{12}F_i(v) = 0 \qquad (51)\end{aligned}$$

for $i = 1, \ldots, n-1$, where F_i is defined as in (12). For simplicity of notation, let $\Psi : \Re^{n-1} \times \Re^{n-1} \to \Re^{2(n-1)}$ be defined by

$$\Psi_i(u, v) := \nabla_{u_i}\psi(u, v) \text{ and } \Psi_{n-1+i}(u, v) := \nabla_{v_i}\psi(u, v),\ i = 1, \ldots, n-1.$$

Then the k-th step of the Newton method at (u^k, v^k) for the dual problem (49) (or equivalently for the equations (50) and (51)) is: Choose $V_k \in \partial_B\Psi(u^k, v^k)$ and let

$$(u^{k+1}, v^{k+1}) = (u^k, v^k) - V_k^{-1}\Psi(u^k, v^k). \qquad (52)$$

The remaining task is to work out the structure of $\partial_B\Psi(u, v)$. Let $D := \mathrm{diag}(1/r_0 + 1/r_1, \ldots, 1/r_{n-2} + 1/r_{n-1})$, and

$$A := \begin{pmatrix} \frac{1}{r_0h_1} - \frac{1}{r_1h_2} & \frac{1}{r_1h_2} & & \\ -\frac{1}{r_1h_2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{r_{n-2}h_{n-1}} \\ & & -\frac{1}{r_{n-2}h_{n-1}} & \frac{1}{r_{n-2}h_{n-1}} - \frac{1}{r_{n-1}h_n} \end{pmatrix}$$

and

$$B := \begin{pmatrix} \frac{1}{r_0h_1^2} + \frac{1}{r_1h_2^2} & -\frac{1}{r_1h_2^2} & & \\ -\frac{1}{r_1h_2^2} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{1}{r_{n-2}h_{n-1}^2} \\ & & -\frac{1}{r_{n-2}h_{n-1}^2} & \frac{1}{r_{n-2}h_{n-1}^2} + \frac{1}{r_{n-1}h_n^2} \end{pmatrix}.$$

Noticing that the only term which is nondifferentiable in $\nabla_{v_i}\psi(u, v)$ is $F_i(v)$, with some labor of calculation, we have

$$\partial_B\Psi(u, v) = \left\{\frac{1}{2}\begin{pmatrix} D & A \\ A^T & 2B + \frac{1}{6}M \end{pmatrix} \ \middle|\ M \in \partial_B F(v)\right\}. \qquad (53)$$

Using the evaluation above, we can prove a result parallel to Theorem 3.3 of the finite termination of the Newton method (52).

Theorem 4.1 *Let* $(u^*, v^*) \in \Re^{n-1} \times \Re^{n-1}$ *be a solution of the dual problem (49). Let* $v_0^* = v_n^* = 0$. *Suppose that*

$$(v_i^*, v_{i-1}^*) \in W \qquad \text{for all } i = 1, \ldots, n \tag{54}$$

where W *is defined by (23). Then the Newton method (52) finds the solution in a finite number of steps provided that the starting point* (u^0, v^0) *is sufficiently close to* (u^*, v^*).

Proof. We only need to prove that every element in $\partial_B \Psi(u^*, v^*)$ is positive definite. The remaining proof argument is similar to the corresponding part in the proof of Theorem 3.3.

We first note that, under the current assumption, Lemma 3.2 implies that each element in $\partial_B F(v^*)$ is positive definite. Let V be an element in $\partial_B \Psi(u^*, v^*)$, then there exists a matrix M in $\partial_B F(v^*)$ such that V has the form of (53). Let $(\bar{u}, \bar{v}) \in \Re^{n-1} \times \Re^{n-1}$ be given and let $\bar{u}_0 = \bar{u}_n = \bar{v}_0 = \bar{v}_n = 0$, then

$$\begin{aligned}
&\kappa(\bar{u}, \bar{v}) \\
&\quad := (\bar{u}^T, \bar{v}^T) V \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \\
&\quad = \frac{1}{2}\bar{u}^T D \bar{u} + \bar{u}^T A \bar{v} + \bar{v}^T B \bar{v} + \frac{1}{12}\bar{v}^T M \bar{v} \\
&\quad = \sum_{i=1}^{n} \left(\frac{(h_i u_{i-1} - (\bar{v}_{i-1} - \bar{v}_i))^2}{4 r_{i-1} h_i^2} + \frac{(h_i u_i - (\bar{v}_{i-1} - \bar{v}_i))^2}{4 r_i h_i^2} \right) + \frac{1}{12}\bar{v}^T M \bar{v}.
\end{aligned}$$

Hence $\kappa(\bar{u}, \bar{v}) = 0$ only if $\bar{v} M \bar{v}^T = 0$, which in turn implies $\bar{v} = 0$ (by the positive definiteness of M). Then

$$\kappa(\bar{u}, \bar{v}) = \sum_{i=1}^{n} \left(\frac{u_{i-1}^2}{4 r_{i-1}} + \frac{u_i^2}{4 r_i} \right).$$

Therefore, $\kappa(\bar{u}, \bar{v}) = 0$ implies $\bar{u} = \bar{v} = 0$. Moreover, if $(\bar{u}, \bar{v}) \neq 0$, then $\kappa(\bar{u}, \bar{v}) > 0$. This proves the positive definiteness of V. □

Remark The theoretical result on the finite termination verifies the numerical experience reported in [23]. On the other hand, the Newton method may fail to find a solution if the condition (54) is violated.

5 Conclusions

The Newton method has long been known to be numerically effective for solving the convex best C^1 interpolation problem and its smoothing. However, no theoretical justification was available in the literature for its effectiveness. The paper fills this gap by showing that the Newton method has the finite termination property under a mild condition, and violation of this condition may force the method fails. The convergence analysis relies on accurate estimation of the generalized Hessian. It would be very interesting if the finite termination domain can be enlarged beyond the region defined by (23). We also hope that our proof technique can be extended to some more general problems as shape-preserving interpolation with obstacles.

References

1. Andersson, L.-E., Elfving, T.: An algorithm for constrained interpolation. SIAM J. Sci. Statist. Comput. **8**, 1012–1025 (1987)
2. Burmeister, W., Hess, W., Schmidt, J.W.: Convex spline interpolants with minimal curvature. Computing **35**, 219–229 (1985)
3. de Boor, C.: A Practical Guide to Splines. Springer-Verlag, New York, 1978
4. Dietze, S., Schmidt, J.W.: Determination of shape preserving spline interpolants with minimal curvature via dual programs. J. Approx. Theory **52**, 43–57 (1988)
5. Dontchev, A.L., Kalchev, B.D.: Duality and well-posedness in convex interpolation. Numer. Funct. Anal. and Optim. **10**, 673–689 (1989)
6. Dontchev, A.L., Qi, H.-D., Qi, L.: Convergence of Newton's method for convex best interpolation. Numer. Math. **87**, 435–456 (2001)
7. Dontchev, A.L., Qi, H.-D., Qi, L.: Quadratic convergence of Newton's method for convex interpolation and smoothing. Constr. Approx. **19**, 123–143 (2003)
8. Dontchev, A.L., Qi, H.-D., Qi, L., Yin, H.: A Newton method for shape-preserving spline interpolation. SIAM J. Optim. **13**, 588–602 (2002)
9. Elfving, T., Andersson, L.-E.: An algorithm for computing constrained smoothing spline functions. Numer. Math. **52**, 583–595 (1988)
10. Fischer, A., Kanzow, C.: On finite termination of an iterative method for linear complementarity problems. Math. Programming **74**, 279–292 (1996)
11. Fletcher, R.: Practical Methods of Optimization, Vol. 2: Constrained Optimization. John Wiley 1981
12. Hiriart-Urruty, J-B., Lemaréchal, C.: Convex Analysis and Minimization Algorithms I. Springer-Verlag 1993
13. Hiriart-Urruty, J-B., Strodiot, J.J., Nguyen, V.H.: Generalized Hessian matrix and second-order optimality conditions for problems with C^{L_1} data. Appl. Math. Optim. **11**, 43–56 (1984)
14. Irvine, L.D., Marin, S.P., Smith, P.W.: Constrained interpolation and smoothing. Constr. Approx. **2**, 129–151 (1986)
15. Kanzow, C., Qi, H.-D., Qi, L.: On the minimum norm solution of linear programs. J. Optim. Theory and Appl. **116**, 333–345 (2003)
16. Kojima, M., Shindo, S.: Extension of Newton and quasi-Newton methods for systems of PC^1 equations. J. Oper. Res. Soc. Japan **29**, 352–373 (1986)

17. Mangasarian, O.L.: Finite Newton method for classification problems. Optim. Method and Software **17**, 913–929 (2002)
18. Micchelli, C.A., Smith, P.W., Swetits, J., Ward, J.D.: Constrained L_p approximation. Constr. Approx. **1**, 93–102 (1985)
19. Neuman, E.: Uniform approximation by some Hermite interpolation splines. J. Comput. Appl. Math. **4**, 7–9 (1978)
20. Nocedal, J., Wright, S.J.: Numerical Optimization. Springer-Verlag, 1999
21. Pang, J.-S., Sun, D., Sun, J.: Semismooth homeomorphisms and strong stability of semidefinite and Lorentz complementarity problems. to appear in: Math. Oper. Res.
22. Qi, L.: Convergence analysis of some algorithms for solving nonsmooth equations. Math. Oper. Res. **18**, 227–244 (1993)
23. Schmidt, J.W.: An unconstrained dual program for computing convex C^1-spline approximants. Computing **39**, 133–140 (1987)
24. Schmidt, J.W.: On tridiagonal linear complementarity problems. Numer. Math. **51**, 11–21 (1987)
25. Schmidt, J.W., Hess, W.: Spline interpolation under two-sided restrictions on the derivatives. Z. Angew. Math. Mech. **69**, 353–365 (1989)
26. Schmidt, J.W.: Dual algorithms for solving convex partially separable optimization problems. Jber. d. Dt. Math.-Verein. **94**, 40–62 (1992)
27. Sun, D., Han, J., Zhao, Y.: The finite termination of the damped Newton algorithm for linear complementarity problems. (Chinese) Acta Math. Appl. Sinica **21**, 148–154 (1998)