

# Dynamics of nearly-degenerate multimode lasers

G.D'Alessandro<sup>♣</sup>, F.Papoff<sup>♡</sup>, D.R.J.Chillingworth<sup>♣</sup>

<sup>♣</sup> Department of Mathematics, University of Southampton, Southampton SO17 1BJ, England, UK  
<sup>♡</sup> Department of Physics and Applied Physics, University of Strathclyde, Glasgow G4 0NG, Scotland, UK



## Abstract

Most laser models consist of partial differential equations and assume circular symmetry and large pumps. As a consequence of these assumptions, the cavity modes are grouped in families of degenerate modes that have the same resonant frequency and the same first threshold. However, perturbations usually reduce the symmetry and lift this degeneracy at least partially. Here, we reduce a complex model of a two level laser, derived from first principles, to a simple system of ordinary differential equations (normal form equations) under the assumption that the perturbation is weak. This approach allows us to take full advantage of the symmetry of the problem and, at the same time, to keep track of the physical parameters of the original model. As an example, we apply the equations to the case of a two mode laser whose axial symmetry is broken by astigmatism. To highlight the advantages and limitations of the normal form equations we compare their predictions with the results of numerical simulations.

## Introduction

Many laser models assume that the laser is “ideal”, in the sense that it has a high degree of symmetry. For example, it is usually assumed that the laser cavity has no optical imperfections like astigmatism that break the axial symmetry of the cavity (see Figure 1). A more subtle symmetry-breaking perturbation is due to the interaction between metallic apertures and polarisers [1,2,3]: the cross-polarised component created by the aperture is killed by the polariser, thus breaking the axial symmetry of the laser.

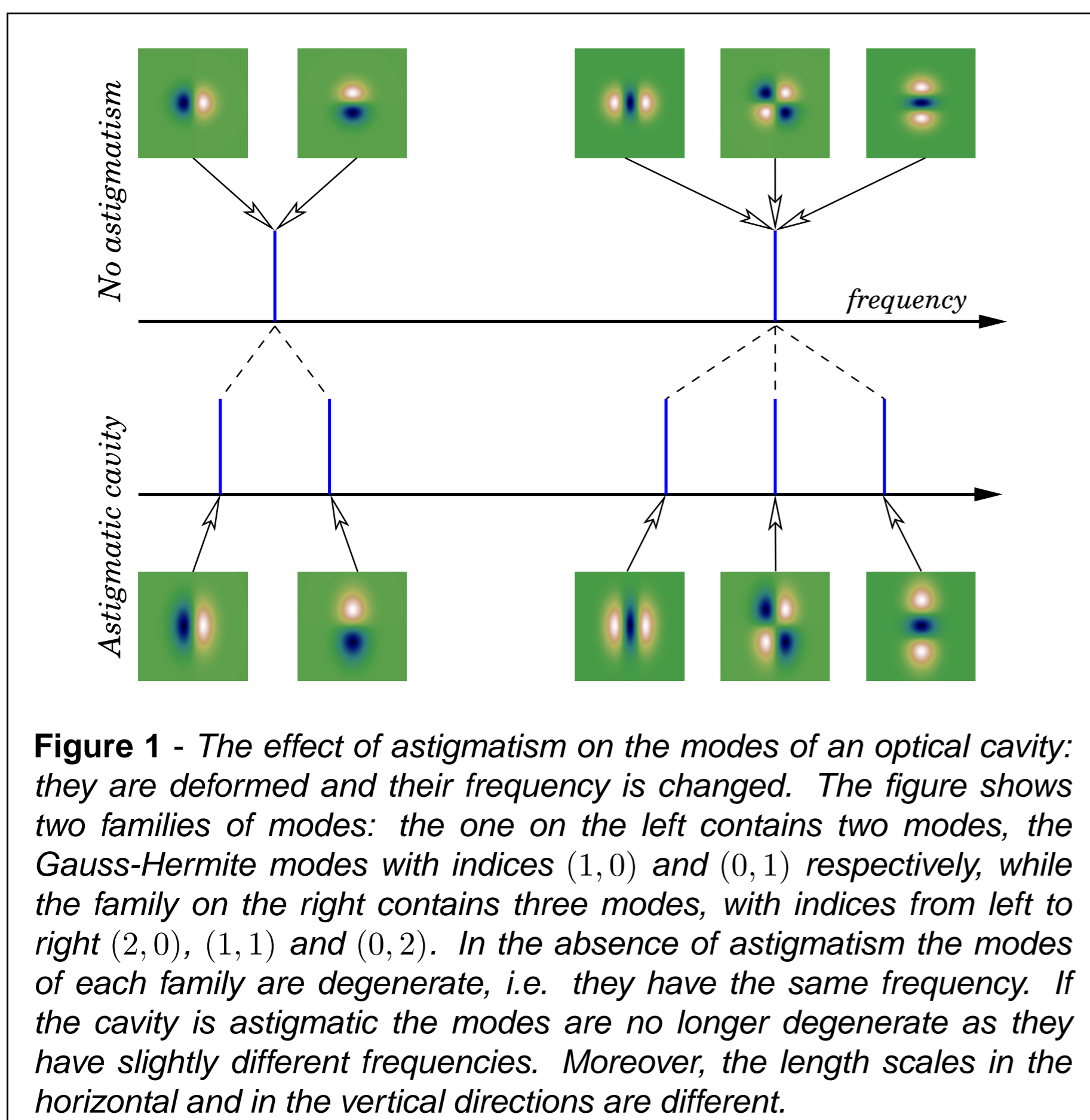
Both these examples of symmetry-breaking can be seen in a wider framework: the “ideal” system is degenerate, in the sense that there are eigenvalues of the linearised equations that have more than one eigenvector (mode), while small perturbations lift this degeneracy at least partially and may have deep effects on the observed dynamics. This observation raises the question of what is the best method to describe a non-ideal system. The aim of the work summarised in this poster is to derive a set of ordinary differential equations that describe the dynamics of a non ideal laser close to threshold [4].

The first two hypotheses are essential (note, however, that the second one is not at all restrictive). The last three are convenient, because they considerably reduce the amount of algebra involved in obtaining the equations.

Under these hypotheses centre manifold theory shows that only a small set  $\mathcal{J}$  of modes plays an active rôle in the dynamics (*active modes*). Their amplitudes  $f_n$  satisfy the set of ordinary differential equations:

$$\frac{d}{dt}f_n = \lambda_n f_n - \sum_{p,j,k \in \mathcal{J}} \gamma_{npj} \bar{f}_j \bar{f}_k f_p, \quad n \in \mathcal{J} \quad (1)$$

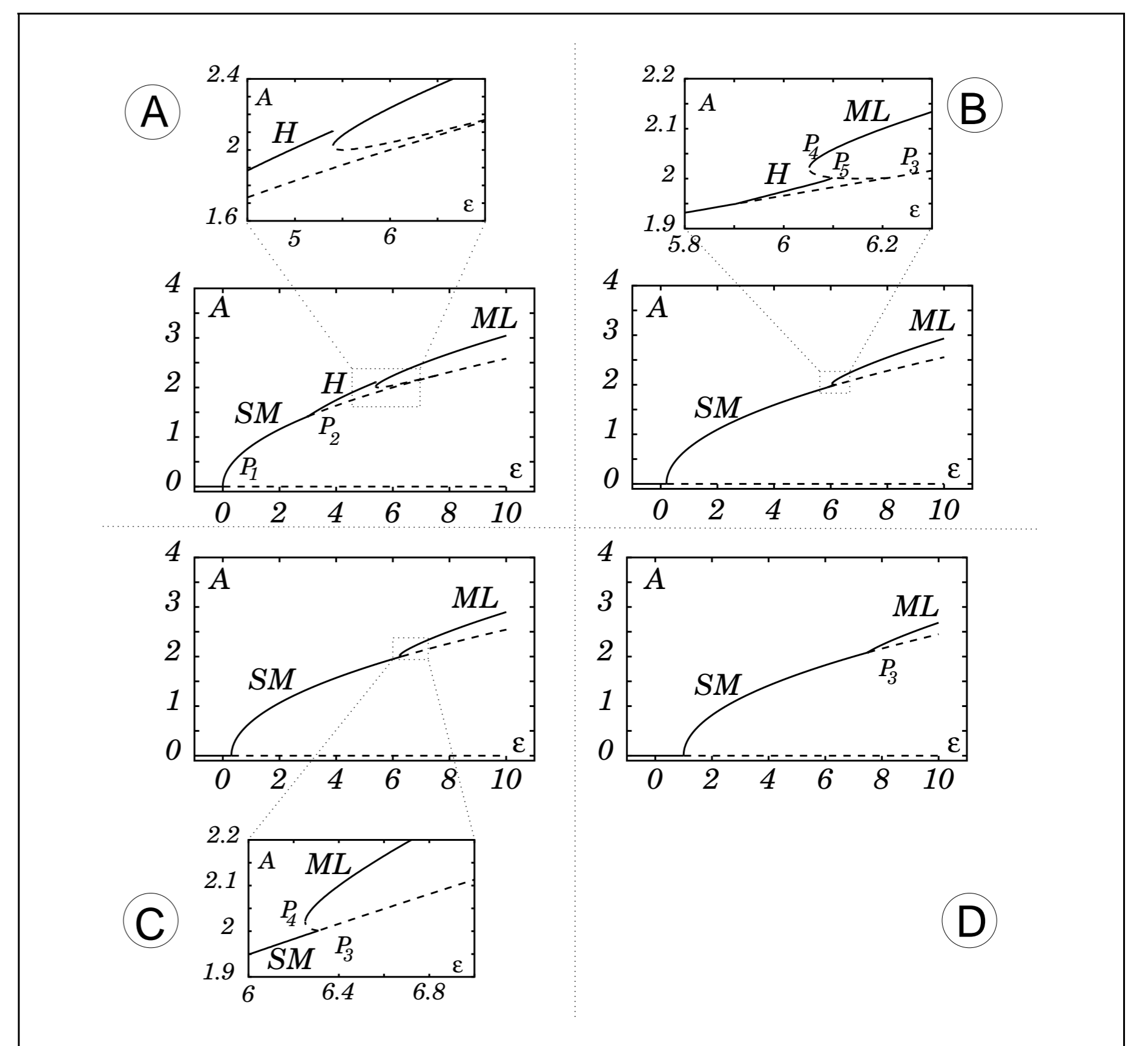
where the over-bar symbol means complex conjugate and  $\lambda_n$  and  $\gamma_{npj}$  are constants that depend on the laser parameters.



## The equation for the active modes

We assume that:

1. The light amplitude is small, i.e. the parameter region of interest is close to the laser's first threshold.
2. The laser field can be expanded in suitable cavity modes. We make no assumptions on the nature of these modes and of the cavity.
3. The lasing medium can be described by a two-level model.
4. The “ideal” laser is described by a set of partial differential equations in time and in the coordinates transverse to the direction of propagation. The longitudinal coordinate has been eliminated by taking the mean field limit.
5. The “pump” is flat, i.e. the energy source of the laser is constant over the transverse plane.



## A two mode astigmatic cavity

As an example of the application of equation (1) we consider the case of an astigmatic laser with only two active modes, i.e. such that the set  $\mathcal{J}$  contains only two indices. In the absence of astigmatism the two modes are degenerate, i.e. they have the same frequency (see the top left hand part of Figure 1). The astigmatism removes the degeneracy: the frequencies and the shapes of the modes are different (see bottom left hand part of Figure 1).

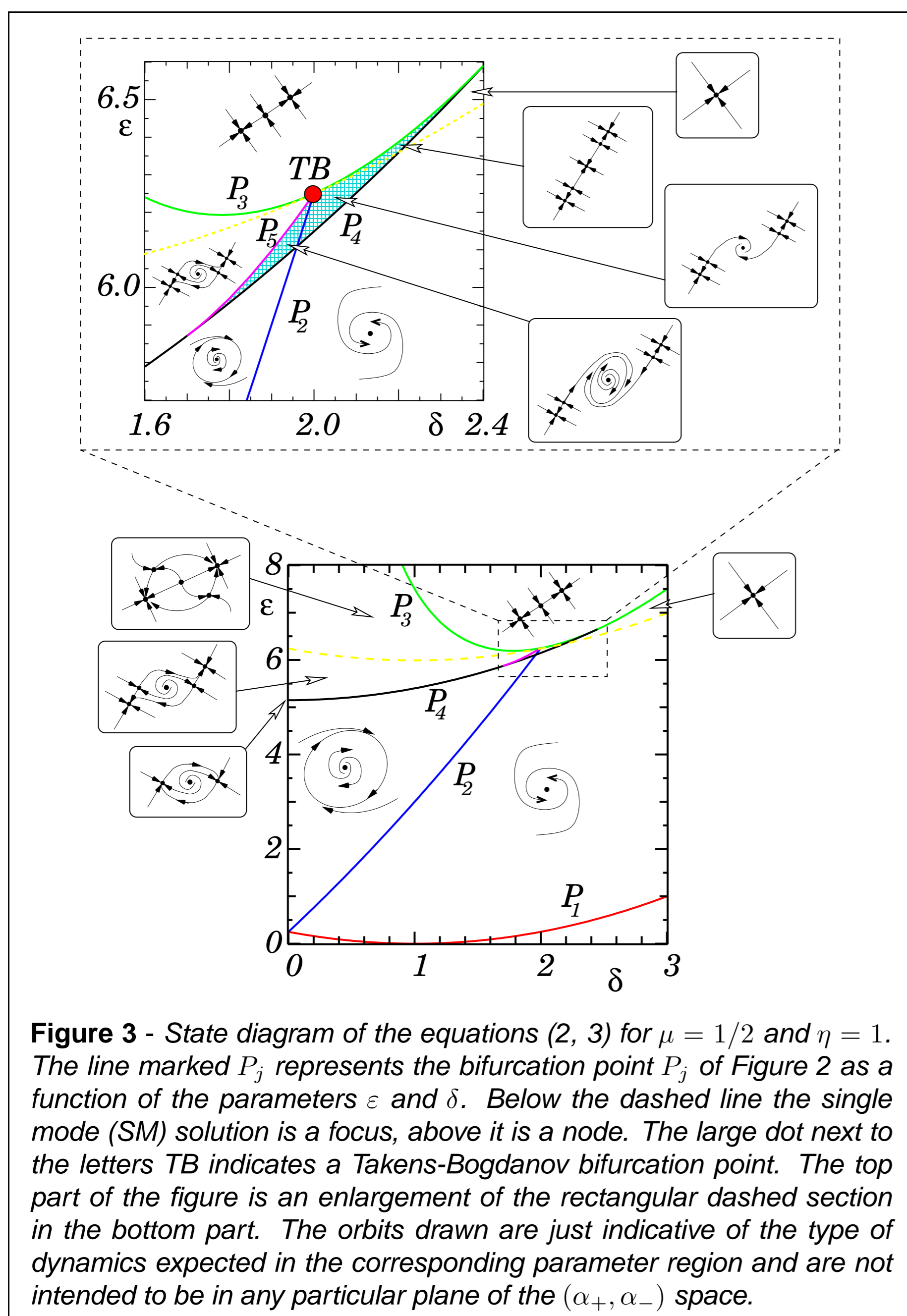
We call  $\alpha_+$  and  $\alpha_-$  the amplitudes of the two modes. They satisfy the equations:

$$\frac{d\alpha_+}{d\tau} = \left[ \varepsilon - \mu^2 (\delta - \eta)^2 - i(\delta - \eta) \right] \alpha_+ - (3|\alpha_+|^2 + 2|\alpha_-|^2) \alpha_+ - \bar{\alpha}_+ \alpha_-^2, \quad (2)$$

$$\frac{d\alpha_-}{d\tau} = \left[ \varepsilon - \mu^2 (\delta + \eta)^2 - i(\delta + \eta) \right] \alpha_- - (3|\alpha_-|^2 + 2|\alpha_+|^2) \alpha_- - \bar{\alpha}_- \alpha_+^2, \quad (3)$$

where  $\eta$  is the symmetry-breaking parameter: the interval between the frequencies of the two modes is  $2\eta$  and if  $\eta = 0$  the system is perfectly symmetric. The other constants are related to the laser parameters. In particular,  $\varepsilon$  is proportional to the pump intensity and measures how much energy is provided to the laser: as  $\varepsilon$  is increased the laser passes from the “off” state  $\alpha_{\pm} = 0$  to the “on” state, where at least one of the amplitudes is different from zero. The parameter  $\delta$ , instead, is proportional to the frequency difference between the atomic transition and the unperturbed mode frequency (*detuning*). This parameter determines which mode becomes active first: if, for example,  $\delta = \eta$  then the active medium is resonant with the mode at frequency  $\eta$  and this will be the first mode to become different from zero as  $\varepsilon$  is increased. Finally, the parameter  $\mu$  is related to the electric field decay rate and ranges from 0 (good cavity limit) to 1 (bad cavity limit).

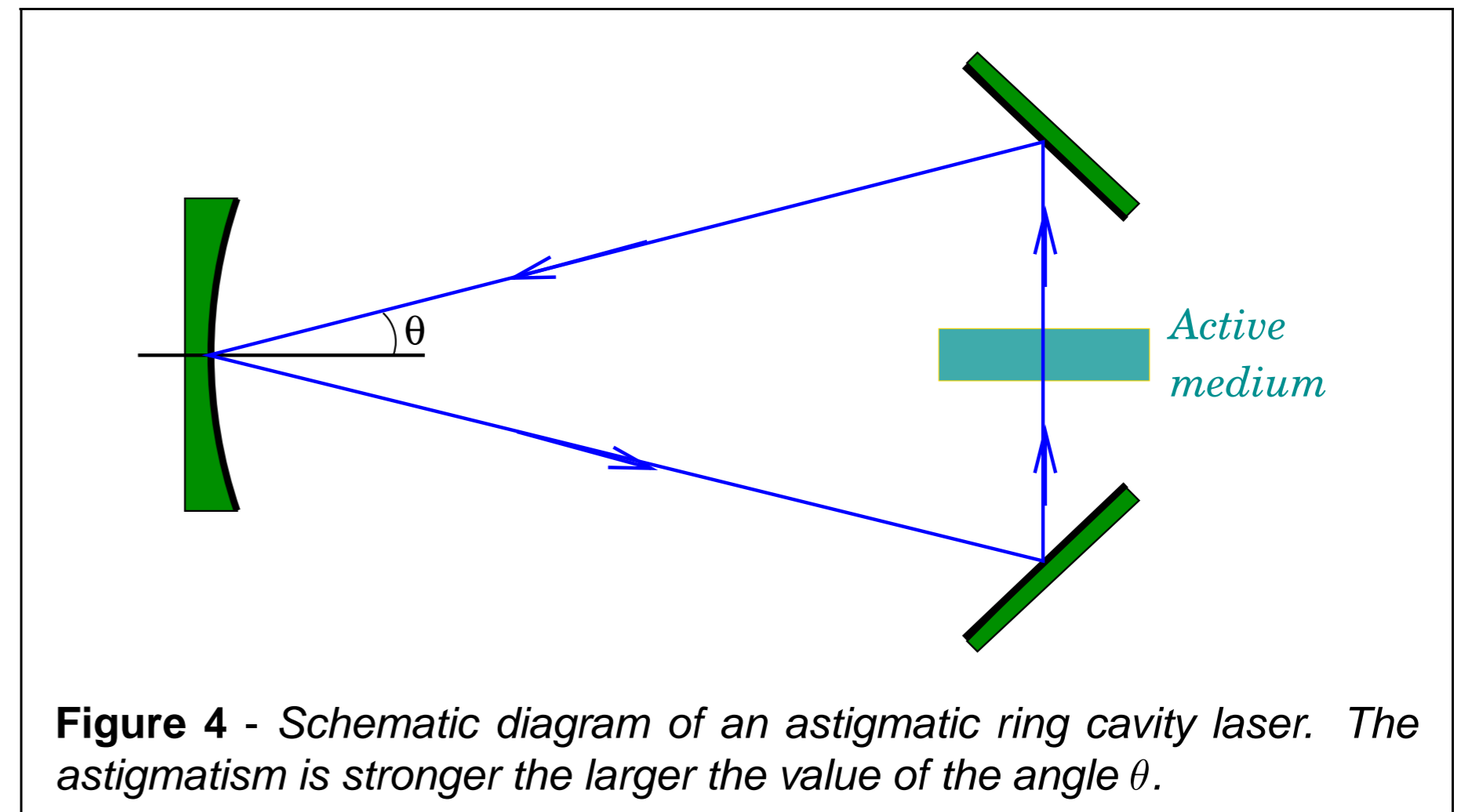
Equations (2,3) have a complicated bifurcation structure as a function of the two parameters  $\varepsilon$  and  $\delta$ . Many of these bifurcations have been studied in various contexts [4,5,6,7,8], but it is a novel feature of this model to incorporate them all. They are illustrated in the diagrams in Figures 2 and 3.



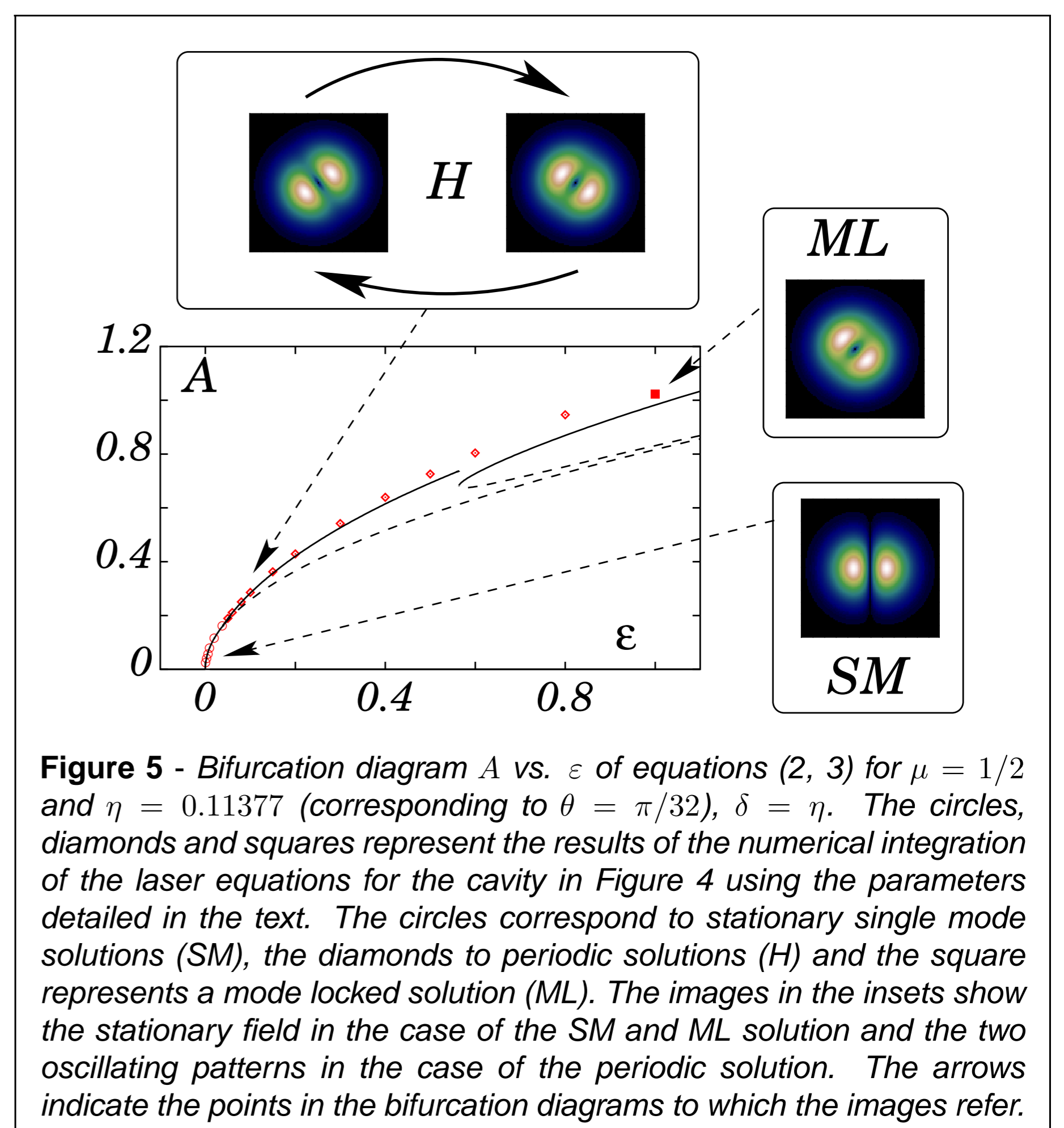
**Figure 3** - State diagram of the equations (2, 3) for  $\mu = 1/2$  and  $\eta = 1$ . The line marked  $P_j$  represents the bifurcation point  $P_j$  of Figure 2 as a function of the parameters  $\varepsilon$  and  $\delta$ . Below the dashed line the single mode (SM) solution is a focus, above it is a node. The large dot next to the letters TB indicates a Takens-Bogdanov bifurcation point. The top part of the figure is an enlargement of the rectangular dashed section in the bottom part. The orbits drawn are just indicative of the type of dynamics expected in the corresponding parameter region and are not intended to be in any particular plane of the  $(\alpha_+, \alpha_-)$  space.

The equations (1) and (2,3) are derived under the assumption that the amplitudes of the modes are small and that the “ideal” model is only weakly perturbed. It is important, therefore, to ascertain to what degree they represent the behaviour of the original model. In order to do this we have written a numerical code to integrate the full model equations, without any approximation.

We have considered the case of the astigmatic cavity represented schematically in Figure 4. The astigmatism of this cavity is proportional to the angle  $\theta$ . The results of the numerical simulations and the comparison with the results of equations (2,3) are summarised in Figure 5. From this we can see that the agreement between the two approaches is excellent for small values of the amplitude, but gets progressively worse as the pump parameter is increased. In particular the periodic solution loses its stability for a higher value of the pump parameter than predicted using equations (2,3).



**Figure 4** - Schematic diagram of an astigmatic ring cavity laser. The astigmatism is stronger the larger the value of the angle  $\theta$ .



**Figure 5** - Bifurcation diagram  $A$  vs.  $\varepsilon$  of equations (2, 3) for  $\mu = 1/2$  and  $\eta = 0.11377$  (corresponding to  $\theta = \pi/32$ ),  $\delta = \eta$ . The circles, diamonds and squares represent the results of the numerical integration of the laser equations for the cavity in Figure 4 using the parameters detailed in the text. The circles correspond to stationary single mode solutions (SM), the diamonds to periodic solutions (H) and the square represents a mode locked solution (ML). The images in the insets show the stationary field in the case of the SM and ML solution and the two oscillating patterns in the case of the periodic solution. The arrows indicate the points in the bifurcation diagrams to which the images refer.

## Conclusions

We have shown that the normal form equations derived using centre manifold theory reproduce accurately the laser dynamics near the first threshold. The technique used is general and could be extended to other laser models. Its application to the case of a two-mode laser has allowed us to perform a complete and thorough bifurcation analysis of this system. The bifurcation structure observed is determined by the symmetry of the laser and of the perturbation and does not depend on the detailed features of the laser. On the other hand, the precise location of the bifurcation points is model-dependent.

## References

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