Giant noise amplification in synchronously pumped optical parametric oscillators

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Photonics Day
Outline

1 The Physics of SPOPOs
2 Pseudospectra
3 Pseudospectra of SPOPOs
1. The Physics of SPOPOs
2. Pseudospectra
3. Pseudospectra of SPOPOs
Synchronously pumped
Optical Parametric Oscillator

\[ \omega_2 \quad \omega_1 \quad \omega_2 \]
\[ z=0 \quad z=1 \]

A pump beam of frequency \( \omega_1 \) is down-converted in a signal and idler beams of frequency \( \omega_2 \) and \( \omega_3 \), respectively.

The pump pulse repetition time is \( T_R \). The signal pulses are reflected by the mirrors and return to the crystal after a time \( T_c \). In order for the conversion to take place we must have \( T_R \simeq T_c \). The difference \( \tau_c = T_R - T_c \) is a key control parameter.
The negative $\tau_c$ regime

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The negative $\tau_c$ regime

There are heavy losses at the mirrors so that only a small part of the signal field returns to the crystal. The pump pulse arrives at the crystal earlier than the signal pulse. The front tail of the pulse sees is amplified and depletes the pump. The rear part of the signal pulse is not amplified and is “killed off” at the mirrors. As a consequence the signal pulses may be shorter than the pump pulse and dominated by noise.
These results are confirmed by numerical simulations of the SPOPO equations. The signal is narrower than the pump (here the pump width is approx. 4ps) and its dynamics is irregular.

The result shown in the previous slide shows that noise is important in the dynamics of SPOPOs. In particular, it seems that the signal pulses are extremely sensitive to noise: simulations run on the different CPUs give different pulses.

We now want to quantify the role of noise and determine the magnitude of the noise amplification factor. This will allow us to tune the parameters of the SPOPO in order to minimise or maximise as needed the effect of noise on the dynamics. We use pseudospectra to measure the amplification factor and we show that this can be of the order of $10^9$ in standard operating conditions.
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We use pseudospectra to measure the amplification factor and we show that this can be of the order of $10^9$ in standard operating conditions.
Stability and perturbations

What is a stable state of a dynamical system?

A state is stable if small perturbations of it decay to zero. This is correct, provided that we also accept that the decay may only be asymptotic, i.e. that the perturbations may initially grow. In some systems this initial growth (transient growth) can be extremely large so that it dominates the dynamics and amplifies microscopic perturbations (noise) to macroscopic level (a non-zero time dependent signal field). This is the case of the SPOPO.
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A simple example

Consider the system of two linear differential equations

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \alpha x_2, \\
\dot{x}_2 &= -10 x_2.
\end{align*}
\]

This has a stable fixed point at \((x_1, x_2) = (0, 0)\). The stability properties of the fixed point are determined by the eigenvalues and eigenvectors of the matrix of the coefficients:

\[
M = \begin{pmatrix}
-1 & \alpha \\
0 & -10
\end{pmatrix} \implies \begin{cases}
\lambda_1 = -1, & \mathbf{v}_1 = (1, 0) \\
\lambda_2 = -10, & \mathbf{v}_2 = (-\sin \theta, \cos \theta),
\end{cases}
\]

with \(\alpha = \tan(\theta)\).
The general solution is

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-10t} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}.
\]

A perturbation in the direction of \( \mathbf{v}_1 \) decays as \( e^{-t} \).
A perturbation in the direction of \( \mathbf{v}_2 \) decays as \( e^{-10t} \).
What happens to a perturbation in an arbitrary direction?
Small $\theta$

The component along $\nu_2$ decays quickly and the perturbation aligns itself along $\nu_1$. The magnitude of the perturbation never grows.
If $\theta \simeq \pi/2$, i.e. if the two eigenvectors are nearly parallel, then even a small perturbation can be the sum of two very large vectors in the directions of $\mathbf{v}_1$ and $\mathbf{v}_2$.

As the component along $\mathbf{v}_2$ rapidly decays to zero, the magnitude of the perturbation can increase and we have *transient growth*.
Response to a modulation

The asymptotic amplitude $x_0$ of the response of a linear system

$$\dot{x} = Mx + f(t)$$

to a modulation $f(t) = f_0 e^{zt}$, $z \in \mathbb{C}$, e.g.

$$f(t) = f_0 e^{i\omega t},$$

is given by

$$x_0 = (z - M)^{-1}f_0.$$

The amplification factor is

$$\sup_{\|f_0\|} \frac{\|x_0\|}{\|f_0\|} = \|(z - M)^{-1}\| \equiv \|R(z, M)\|.$$

The operator $R(z, M)$ is called the \textit{resolvent} of the matrix (operator) $M$. This last equation tells us that the norm of the resolvent is the amplification factor of the system.
The pseudospectrum of a matrix (operator) $M$ is defined in terms of the resolvent. Its formal definition is

$$\Lambda_\epsilon(M) = \{ z \in \mathbb{C} : \| R(z, M) \| \geq \epsilon^{-1} \}, \quad \epsilon > 0.$$ 

Very roughly, the pseudospectrum is the set of contour levels of $\| R(z, M) \|$. If the exponent (frequency) of the perturbation is within the contour level corresponding to $\epsilon$ then we can expect that the system will amplify the perturbation by a factor approximately equal to $1/\epsilon$. 
Pseudospectrum for $\theta$ small

For $\theta$ small we do not expect to see any large response to a periodic modulation.
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The contour line of $\epsilon = 0.1$ crosses the imaginary axis. Hence we can expect that the largest amplification factor of a periodic modulation is of order $1/0.1 = 10$. 
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The SPOPO is below threshold. The pseudospectrum extends reasonably far outside the circle of radius 1, but this is not enough to give a sustained signal dynamics.

\[ \tau_c = 0.46\text{ps}, \text{ Pump } 87\% \text{ of threshold, Pump width } \simeq 4\text{ps} \]
The SPOPO is below threshold. The pseudospectrum extends so far outside the circle of radius 1, that there is a noise-sustained macroscopic signal dynamics.

\[ \tau_c = -0.96\text{ps}, \text{ Pump 87\% of threshold, Pump width } \approx 4\text{ps} \]
Conclusions

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- Mathematically, the SPOPO equations are non-normal, i.e. they have non-orthogonal eigenvectors. Their pseudospectrum is a convenient tool to measure the amplification factor.

In optics, noise amplification is usually called excess noise and is associated to the presence of non-orthogonal cavity modes. However, it can take place also in systems that do not satisfy this condition [Papoff et al, PRL 100, 123905 (2008)].

Noise amplification is not present only in optics. Its mathematical origin is in functional analysis and numerical methods. Its first applications were in atmospheric dynamics and in turbulence. [Trefethen]
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