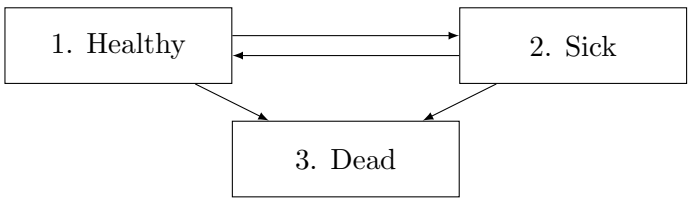
MATH3085/6143 Survival Models – Worksheet 4

1. The Healthy-Sick-Dead model used in sickness insurance investigations has three states, Healthy, Sick and Dead, and four possible transitions, as shown below.



Let the intensity of the transition between states i and j at age $x+t$ be μ_{x+t}^{ij} and the probability that a person aged x in state i will be in state j at age $x+t$ be ${}_tp_x^{ij}$. [This is an alternative notation to the one used in lectures, which you may see in other sources.]

i) Use Kolmogorov's forward equations to derive an expression for $\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x^{23}$ in terms of only the transition intensities μ_{x+t}^{ij} $(i \neq j)$ and the ${}_tp_x^{ij}$.

ii) Explain in words what each term in the expression you have derived in part (i) means.

iii) Use Kolmogorov's forward equations to derive an expression for $\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x^{21}$ in terms of only the transition intensities μ_{x+t}^{ij} $(i \neq j)$ and the ${}_tp_x^{ij}$.

2. A government has introduced a two-tier driving test system. Once someone applies for a provisional licence they are considered a Learner driver. Learner drivers who score 90% or more on the primary examination (which can be taken at any time) become Qualified. Those who score between 50% and 90% are obliged to sit a secondary examination and are given driving status Restricted. Those who score 50% or below on the primary examination remain as Learners. Restricted drivers who pass the secondary examination become Qualified, but those who fail revert back to Learner status and are obliged to start again.

i) Sketch a diagram showing the possible transitions between the states.

ii) Write down the likelihood of the data, assuming transition rates between states are constant over time, clearly defining all the terms you use.

Figures over the first year of the new system based on those who applied for a provisional licence during that time in one area showed the following:

Person-months in Learner State	1161
Person-months in Restricted State	1940
Number of transitions from Learner to Restricted	382
Number of transitions from Restricted to Learner	230
Number of transitions from Restricted to Qualified	110
Number of transitions from Learner to Qualified	217

iii) (a) Derive the maximum likelihood estimator of the transition rate from Learner to Restricted.

(b) Estimate the constant transition rate from Learner to Restricted.

(c) Calculate a 95% confidence interval around the estimate made in (iii)(b).

3. A certain non-fatal medical condition affects adults. Adults with the condition suffer frequent episodes of blurred vision. A study was carried out among a group of adults known to have the condition. The study lasted one year, and each participant in the study was asked to record the duration of each episode of blurred vision. All participants remained under observation for the entire year. The data from the study were analysed using a two-state Markov model with states:

   1. Not suffering from blurred vision.
   2. Suffering from blurred vision.

   The results of the study were as follows:

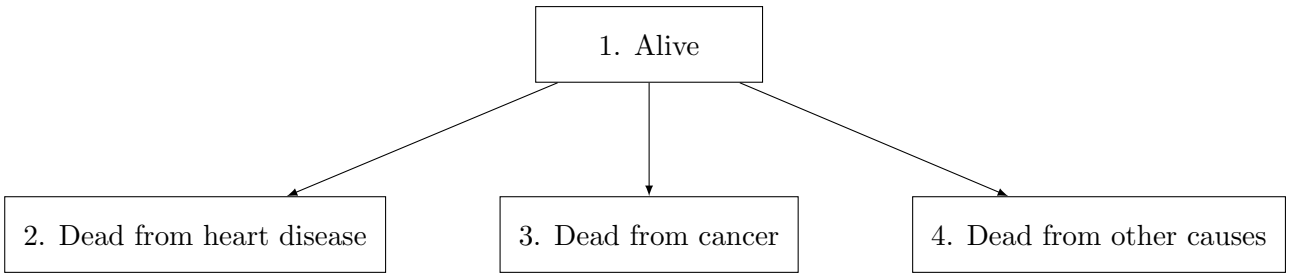
Participant-days in state 1	21650
Participant-days in state 2	5200
Number of transitions from state 1 to state 2	4330
Number of transitions from state 2 to state 1	4160

   Assume that the transition intensities are constant over time.

   i) Calculate the maximum likelihood estimates of the transition intensities from state 1 to state 2 and from state 2 to state 1.

   ii) Use Kolmogorov's forward equation to derive an expression for $p_{12}(x,t)$, the probability that an adult with the condition who is not suffering from blurred vision at time x will be suffering from blurred vision at time $x+t$, in this time-homogeneous model. Hence, estimate the probability that an adult with the condition who is presently not suffering from blurred vision will be suffering from blurred vision in 3 days time.

4. An investigation into mortality by cause of death used the four-state Markov model shown below.



   The investigation was carried out separately for each year of age, and the transition intensities were assumed to be constant within each single year of age.

   i) Write down, defining all the terms you use, the likelihood for the transition intensities.

   ii) Derive the maximum likelihood estimator of the force of mortality from heart disease (transition intensity to death from heart disease) for any single year of age.