MATH3085/6143 Survival Models – Worksheet 2

1. Suppose that we consider that a particular survival variable T has a constant hazard λ_1 over the initial period $(0, t')$ and then a potentially different constant hazard λ_2 thereafter, so

$$h_T(t) = \begin{cases} \lambda_1 & 0 < t \le t' \\ \lambda_2 & t' < t \end{cases}$$

   i) Show that the survival function is given by

$$S_T(t) = \begin{cases} \exp(-\lambda_1 t) & 0 < t \le t' \\ \exp[-\lambda_1 t' - \lambda_2(t - t')] & t' < t \end{cases}$$

   ii) Hence find the density function $f_T(t)$

   Now suppose that we have observed survival times $t_1, \ldots t_n$ with corresponding censoring indicators $d_1, \ldots, d_n$. Let n_1 and n_2 be the numbers of t_i values in the ranges $[0, t')$ and $[t', \infty)$ respectively, and let m_1 and m_2 be the corresponding numbers of observed failures (t_i values with $d_i = 1$).

   iii) Show that the likelihood is given by

$$L(\lambda_1, \lambda_2) = \lambda_1^{m_1} \lambda_2^{m_2} \exp\left[-\lambda_1 \left(n_2 t' + \sum_{i:t_i<t'} t_i\right)\right] \exp\left[-\lambda_2 \sum_{i:t_i \ge t'} (t_i - t')\right]$$

   where m is the number of failure times (observed or censored) at times greater than or equal to t'.

   iv) Hence find the maximum likelihood estimators $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

2. An electronics company developed a revolutionary new battery which it believed would make enormous profits. It commissioned a sub-contractor to estimate the survival function of battery life for the first 12 prototypes. The sub-contractor inserted each prototype battery into an identical electrical device at the same time and measured the duration elapsing between the time each device was switched on and the time its battery ran out. The sub-contractor was instructed to terminate the test immediately after the failure of the 8^{th} battery, and to return all 12 batteries to the company. When the test was complete, the sub-contractor reported that he had terminated the test after 150 days. He further reported that:

   - Two batteries had failed after 97 days.
   - Three further batteries had failed after 120 days.
   - Two further batteries had failed after 141 days.
   - One further battery had failed after 150 days.

   However, he reported that he was only able to return 11 batteries, as one had exploded after 110 days, and he had treated this battery as censored at that duration when working out the Kaplan-Meier estimate of the survival function.

   i) Calculate the Kaplan-Meier estimate of the survival function based on the information supplied by the sub-contractor.

   ii) Calculate the standard errors for the estimates in (i) and hence 95% confidence intervals.

   iii) Calculate the Nelson-Aalen estimate for the cumulative hazard function.

   iv) Transform your answer in (iii) to an estimate of the survival function and compare your answer to the one you obtained in (i).

[Based on Q9 in the Institute of Actuaries and Faculty of Actuaries CT4 exam, September 2009.]

3. A certain town runs a training course for traffic wardens each year. The course lasts for 30 days, but the examination which enables someone to qualify as a traffic warden can be sat any day during the course. In 2011 there were 13 participants who started the training course. The following table has been compiled to show the day each candidate qualified or the day each candidate who did not qualify left the course.

Candidate	Day Qualified	Day Left Without Qualifying
A		30
B	5	
C		21
D	19	
E	12	
F		30
G	1	
H		19
I	12	
J		30
K	15	
L		10
M	24	

i) Explain whether the following types of censoring are present:
   - Interval censoring.
   - Right censoring.
   - Informative censoring.

ii) Calculate the Kaplan-Meier estimate of the non-qualification function.

iii) Sketch a graph of the Kaplan-Meier estimate, labelling the axes.

[Q9 in the Institute and Faculty of Actuaries CT4 exam, September 2012.]

4. An engineer investigating failure times of a particular type of motor proposes to model failure times as observations of a random variable T, and he proposes

$$S_T(t) = \frac{1}{(1+t)^\beta}$$

for some parameter $\beta > 0$, as the survival function for T.

(a) Giving your reasons, describe why this function is valid as a survival function.

(b) Find the density and hazard functions for T.

A sample of n such motors are tested over a period of 1000 hours, during which time m of the motors were observed to fail. The remaining $n - m$ motors had not failed at 1000 hours when observation ended. For each failed motor, $i = 1, \ldots, m$, the failure time is denoted by t_i.

(c) Show that the log-likelihood $\ell(\beta)$ is given by

$$\ell(\beta) = m \log \beta - (\beta + 1) \sum_{i=1}^{m} \log(1 + t_i) - (n - m)\beta \log(1001)$$

(d) Find a general expression for the maximum likelihood estimator $\hat{\beta}$ of β, and its standard error and calculate both of these when $n = 12$, $m = 3$, $t_1 = 200$, $t_2 = 800$, $t_3 = 800$.

(e) Giving your reasons, state whether the censoring in this example is informative or non-informative.