MATH3085/6143 Survival Models – Worksheet 1 Solutions

1. i) Determine a period of investigation running from time 0 to time t (recorded in days for convenience). Observe all (or a random sample) of patients who undergo the operation during the period of investigation. Then record, for each patient i:

   - The time of the operation, o_i.
   - The time of discharge from hospital (if discharged from the hospital during the period of investigation), D_i.
   - The time of death (if death occurred prior to the end of the period of investigation), d_i.
   - Whether or not the patient was discharged from hospital during the period of investigation, δ_i.

   Persons still in hospital at the end of the period of the investigation are assumed to be right-censored at that point. While persons who die prior to discharge are assumed to be right-censored by their death.

   These data will allow us to create, for each person i observed, the pair $\min[T_i, C_i]$, δ_i, where $\min[T_i, C_i]$ is the time until discharge from hospital, for those who were discharged during the period of the investigation, and the time until censoring for those who were not, and δ_i is an indicator variable which takes value of 1 if person i was discharged, and 0 otherwise. The data can be summarized in a table:

	$\min[T_i, C_i]$	δ_i
Discharged before end of period of investigation	$D_i - o_i$	1
Died during period of investigation, and before discharged	$d_i - o_i$	0
Still in hospital at end of period of investigation	$t - o_i$	0

   ii)
   - Deaths prior to discharge during the period of investigation.
   - The end of period of investigation.
   - Transfer to another hospital.
   - The patient undergoing second operation.

   iii) We treat patients who die as censored at the point of death. By assuming that censoring is non-informative, we assume the distribution of times to discharge among those who died would, had they not died, have been the same as the observed distribution among those who did not die.

   - But patients who die prior to discharge might be expected to have a poorer prognosis following the operation, and therefore to be kept longer than the average, thus violating the assumption.
   - The same applies to patients who undergo second operation. The second operation could be undertaken to correct complication arising from the first operation, again suggesting slower recovery than normal.

2. Since

$$S_T(t) = \exp\left(-\int_0^t h(s)\mathrm{d}s\right),$$

with the specified hazard, we have

$$S_T(t) = \exp\left(-\int_0^t (\alpha + \beta s)\mathrm{d}s\right) = \exp\left(-\left[\alpha s + \frac{\beta s^2}{2}\right]_0^t\right) = \exp\left(-\left[\alpha t + \frac{\beta t^2}{2}\right]\right).$$

Recall that $h_T(t) = \frac{f_T(t)}{S_T(t)}$, thus

$$f_T(t) = h_T(t)S_T(t) = (\alpha + \beta t)\exp\left(-\left[\alpha t + \frac{\beta t^2}{2}\right]\right).$$

As mentioned in lecture notes, hazard function MUST be non-negative, thus

$$h_T(t) = \alpha + \beta t \geq 0 \text{ for all } t > 0.$$

Note that $\alpha + \beta t$ is merely a straight line with intercept, α and slope, β. So first, the intercept, $\alpha \geq 0$ to ensure that the hazard starts off with positive values for small t. In addition, the slope has to be non-negative, $\beta \geq 0$ (either increasing or constant line) for the domain $(0, \infty)$ because otherwise the line would eventually cross the x-axis to become negative, for any value of α . Therefore, we require $\alpha \geq 0$ and $\beta \geq 0$ for a proper hazard function. Note that $\alpha = \beta = 0$ corresponds to zero hazard function ($h_T(t) = 0$), which is possible but not particularly interesting.

3.

$$\begin{aligned} S_T(t) &= \int_t^\infty f_T(s)\mathrm{d}s \\ &= \int_t^\infty \frac{\alpha\lambda s^{\alpha-1}}{(1+\lambda s^\alpha)^2}\mathrm{d}s \\ &= \left[-\frac{1}{1+\lambda s^\alpha}\right]_t^\infty \\ &= \frac{1}{1+\lambda t^\alpha}. \end{aligned}$$

Again, recall that $h_T(t) = \frac{f_T(t)}{S_T(t)}$, we have

$$\begin{aligned} h_T(t) &= \frac{\frac{\alpha\lambda t^{\alpha-1}}{(1+\lambda t^\alpha)^2}}{\frac{1}{1+\lambda t^\alpha}} \\ &= \frac{\alpha\lambda t^{\alpha-1}}{1+\lambda t^\alpha}. \end{aligned}$$

4. The survival function is

$$\begin{aligned} S_T(t) &= \int_t^\infty f_T(u)\mathrm{d}u \\ &= \int_t^\infty \alpha \exp\left(\beta u - \frac{\alpha}{\beta}(e^{\beta u} - 1)\right) \mathrm{d}u. \end{aligned}$$

Using the substitution $v = \frac{\alpha}{\beta}(e^{\beta u} - 1)$, then

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \alpha e^{\beta u} \qquad \text{and} \qquad \mathrm{d}u = \frac{\mathrm{d}v}{\alpha e^{\beta u}}.$$

Now

$$\begin{aligned} S_T(t) &= \int_{\frac{\alpha}{\beta}(e^{\beta t}-1)}^\infty \exp(-v)\mathrm{d}v \\ &= [-\exp(-v)]_{\frac{\alpha}{\beta}(e^{\beta t}-1)}^\infty \\ &= \exp\left(-\frac{\alpha}{\beta}(e^{\beta u} - 1)\right). \end{aligned}$$

The hazard function is

$$\begin{aligned} h_T(t) &= \frac{f_T(t)}{S_T(t)} \\ &= \frac{\alpha \exp\left(\beta u - \frac{\alpha}{\beta}(e^{\beta u} - 1)\right)}{\exp\left(\beta u - \frac{\alpha}{\beta}(e^{\beta u} - 1)\right)} \\ &= \alpha \exp\left(\beta t\right). \end{aligned}$$

5.

$$\begin{aligned} S_T(t) &= \exp\left(-\int_0^t h_T(s)\mathrm{d}s\right) \\ &= \exp\left[-\int_0^t [\lambda + \alpha\exp(\beta s)]\mathrm{d}s\right] \\ &= \exp\left[-\left[\lambda s + \frac{\alpha}{\beta}\exp(\beta s)\right]_0^t\right] \\ &= \exp\left[-\left(\lambda t + \frac{\alpha}{\beta}\exp(\beta t) - \frac{\alpha}{\beta}\right)\right]. \end{aligned}$$

$$\begin{aligned} f_T(t) &= h_T(t)S_T(t) \\ &= [\lambda + \alpha\exp(\beta t)]\exp\left[-\left(\lambda t + \frac{\alpha}{\beta}\exp(\beta t) - \frac{\alpha}{\beta}\right)\right]. \end{aligned}$$

The extra term λ in the Makeham model captures a constant hazard which might be thought to be due to accidental death (time/age-independent)