Solutions to MATH3091 problem sheet 1

10 Feb 2021

1. (a) The p.f. of each $Y_i \sim \text{Poisson}(\lambda)$ is

$$f_Y(y;\lambda) = \frac{\lambda^y e^{-\lambda}}{y!},$$

so the likelihood is

$$L(\lambda) = \prod_{i=1}^n f_Y(y_i;\lambda) = \prod_{i=1}^n \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n y_i}}{\prod_{i=1}^n y_i!}.$$

The log-likelihood is

$$\ell(\lambda) = \log L(\lambda) = -n\lambda + \left(\sum_{i=1}^n y_i\right)\log\lambda - \sum_{i=1}^n \log(y_i!).$$

(b) The p.d.f of each $Y_i \sim \text{Unif}(a,b)$ is

$$f_Y(y;\, a,b) = \frac{1}{b-a}, \quad a \le y \le b$$

so the likelihood is

$$L(a,b) = \prod_{i=1}^n f_Y(y_i;a,b) = \prod_{i=1}^n \frac{1}{b-a} = \frac{1}{(b-a)^n}, \quad a \le y_i \le b,\ i = 1,\dots n$$

The log-likelihood is

$$\ell(a,b) = \log L(a,b) = -n\log(b-a), \quad a \le y_i \le b,\ i=1,\dots n.$$

(c) The p.f. of each $\boldsymbol{Y}_i = (Y_{i1}, Y_{i2}, Y_{i3})^T \sim \text{Multinomial}(1,\theta,2\theta,1-3\theta)$ is

$$f_Y(y;\theta) = \frac{1}{y_1!y_2!y_3!}\theta^{y_1}(2\theta)^{y_2}(1-3\theta)^{y_3},$$

so the likelihood is

$$L(\theta) = \prod_{i=1}^n f_Y(y_i;\theta) = \frac{\theta^{\sum_{i=1}^n y_{i1}}(2\theta)^{\sum_{i=1}^n y_{i2}}(1-3\theta)^{\sum_{i=1}^n y_{i1}}}{\prod_{i=1}^n y_{i3}!\prod_{i=1}^n y_{i2}!\prod_{i=1}^n y_{i3}!}.$$

The log-likelihood is

$$\begin{aligned}\ell(\theta) &= \log L(\theta)\\ &= \left(\sum_{i=1}^n y_{i1}\right)\log\theta + \left(\sum_{i=1}^n y_{i2}\right)\log(2\theta) + \left(\sum_{i=1}^n y_{i3}\right)\log(1-3\theta) - \sum_{i=1}^n\sum_{j=1}^3 \log(y_{ij}!).\end{aligned}$$

(d) The p.d.f. of each $Y_i = \text{Weibull}(\alpha)$ is

$$f_Y(y;\alpha) = \alpha y^{\alpha-1} e^{-y^\alpha}$$

so the likelihood is

$$L(\alpha) = \prod_{i=1}^n f_Y(y_i;\alpha) = \alpha^n \prod_{i=1}^n y_i^{\alpha-1} e^{-\sum_{i=1}^n y_i^\alpha}.$$

The log-likelihood is

$$\ell(\alpha) = \log L(\alpha) = n\log(\alpha) + (\alpha-1)\sum_{i=1}^n \log(y_i) - \sum_{i=1}^n y_i^\alpha.$$

(e) The p.d.f. of each $Y_i = \text{Pareto}(\alpha)$ is

$$f_Y(y;\alpha) = \alpha(1+y)^{-\alpha-1}, \quad y > 0$$

so the likelihood is

$$L(\alpha) = \prod_{i=1}^n f_Y(y_i;\alpha) = \alpha^n \prod_{i=1}^n (1+y_i)^{-\alpha-1}.$$

The log-likelihood is

$$\ell(\alpha) = \log L(\alpha) = n\log(\alpha) - (\alpha+1)\sum_{i=1}^n \log(1+y_i)$$

2. (a) Differentiating the $\ell(\lambda)$ from Question 1(a) with respect to λ gives

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^n y_i}{\lambda}.$$

So a stationary point $\hat{\lambda}$ solves

$$-n + \frac{\sum_{i=1}^n y_i}{\hat{\lambda}} = 0,$$

which gives

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^n y_i = \bar{y}.$$

We have

$$\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} = -\frac{\sum_{i=1}^n y_i}{\lambda^2},$$

which is ≤ 0 for all values of λ, so $\hat{\lambda} = \bar{y}$ is the MLE.

(b) Differentiating the $\ell(a,b)$ from Question 1(b) with respect to a and b respectively gives

$$\frac{\partial \ell(a,b)}{\partial a} = \frac{n}{b-a}, \qquad \frac{\partial \ell(a,b)}{\partial b} = -\frac{n}{b-a}$$

We cannot alter the value of a and b to let above scores to become exactly zero. However, note that the $\frac{\partial \ell(a,b)}{\partial a}$ is always >0, means $\ell(a,b)$ is increasing with a. Thus, to maximise $\ell(a,b)$ we just need to find the largest possible value of a.

Since we require $a \leq y_i \leq b,\ i = 1, \ldots n$, therefore $a \leq \min_i y_i$, which implies the MLE is just

$$\hat{a} = \min_i y_i$$

.

Similarly, we have

$$\hat{b} = \max_i y_i$$

(detail is omitted, please complete it by yourself).

(c) Differentiating the $\ell(\theta)$ from Question 1(c) with respect to θ gives

$$\frac{\partial \ell(\theta)}{\partial \theta} = \left(\sum_{i=1}^{n} y_{i1}\right)\frac{1}{\theta} + \left(\sum_{i=1}^{n} y_{i2}\right)\frac{1}{\theta} - \left(\sum_{i=1}^{n} y_{i3}\right)\frac{3}{1-3\theta}.$$

So a stationary point $\hat{\theta}$ solves

$$\left(\sum_{i=1}^{n} y_{i1}\right)\frac{1}{\hat{\theta}} + \left(\sum_{i=1}^{n} y_{i2}\right)\frac{1}{\hat{\theta}} - \left(\sum_{i=1}^{n} y_{i3}\right)\frac{3}{1-3\hat{\theta}} = 0,$$

Simplify the equation gives

$$\left(\sum_{i=1}^{n} y_{i1} + \sum_{i=1}^{n} y_{i2}\right)(1-3\hat{\theta}) - \left(\sum_{i=1}^{n} y_{i3}\right)3\hat{\theta} = 0$$

As a result,

$$\hat{\theta} = \frac{1}{3}\frac{\sum_{i=1}^{n} y_{i1} + \sum_{i=1}^{n} y_{i2}}{\sum_{i=1}^{n} y_{i1} + \sum_{i=1}^{n} y_{i2} + \sum_{i=1}^{n} y_{i3}} = \frac{\sum_{i=1}^{n} y_{i1} + \sum_{i=1}^{n} y_{i2}}{3n}.$$

(d) Differentiating the $\ell(\alpha)$ from Question 1(e) with respect to α gives

$$\frac{\partial \ell(\alpha)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \log(1+y_i).$$

So a stationary point $\hat{\alpha}$ solves

$$\frac{n}{\hat{\alpha}} - \sum_{i=1}^{n} \log(1+y_i)$$

which gives

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log(1 + y_i)}.$$

More over, we have

$$\frac{\partial^2 \ell(\alpha)}{\partial \alpha^2} = -\frac{n}{\alpha^2},$$

which is ≤ 0 for all values of α, so the obtained $\hat{\alpha}$ is the MLE.

3. (a) The likelihood is

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta) = \prod_{i=1}^{n} \theta \exp(-\theta y_i) = \theta^n \exp(-\theta \sum_{i=1}^{n} y_i).$$

The log-likelihood is

$$\ell(\theta) = \log L(\theta) = n \log \theta - \theta \sum_{i=1}^{n} y_i.$$

The score is

$$u(\theta) = \frac{\partial}{\partial \theta} \ell(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} y_i.$$

So a stationary point of the log-likelihood $\hat{\theta}$ solves

$$u(\hat{\theta}) = \frac{n}{\hat{\theta}} - \sum_{i=1}^{n} y_i = 0,$$

which gives

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} y_i} = \frac{1}{\bar{y}}.$$

The Hessian is

$$H(\theta) = \frac{\partial^2}{\partial \theta^2} \ell(\theta) = -\frac{n}{\theta^2} < 0$$

for all θ, so $\hat{\theta}$ is the MLE. The Fisher information is

$$\mathcal{I}(\theta) = E[-H(\theta)] = E\left[\frac{n}{\theta^2}\right] = \frac{n}{\theta^2}.$$

(b) The likelihood is

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta) = \prod_{i=1}^{n} \theta y_i^{\theta-1} = \theta^n \prod_{i=1}^{n} y_i^{\theta-1}.$$

The log-likelihood is

$$\ell(\theta) = \log L(\theta) = n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log y_i.$$

The score is

$$u(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \log y_i$$

So a stationary point of the log-likelihood $\hat{\theta}$ solves

$$u(\hat{\theta}) = \frac{n}{\hat{\theta}} + \sum_{i=1}^{n} \log y_i = 0,$$

which gives

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log y_i}.$$

The Hessian is

$$H(\theta) = -\frac{n}{\theta^2} < 0$$

for all θ, so $\hat{\theta}$ is the MLE. The Fisher information is

$$\mathcal{I}(\theta) = E[-H(\theta)] = E\left[\frac{n}{\theta^2}\right] = \frac{n}{\theta^2}.$$

(c) The likelihood is

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{y_i-1} = \theta^n(1-\theta)^{\sum_{i=1}^{n} y_i - n}.$$

The log-likelihood is

$$\ell(\theta) = \log L(\theta) = n \log\theta + \left(\sum_{i=1}^{n} y_i - n\right)\log(1-\theta).$$

The score is

$$u(\theta) = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} y_i - n}{1-\theta}$$

So a stationary point of the log-likelihood $\hat{\theta}$ solves

$$u(\hat{\theta}) = \frac{n}{\hat{\theta}} - \frac{\sum_{i=1}^{n} y_i - n}{1-\hat{\theta}} = 0,$$

which gives

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} y_i} = \frac{1}{\bar{y}}.$$

The Hessian is

$$H(\theta) = -\frac{n}{\theta^2} - \frac{\sum_{i=1}^{n} y_i - n}{(1-\theta)^2} < 0$$

for all θ, as each $y_i \geq 1$ so $\sum_{i=1}^{n} y_i - n \geq 0$. So $\hat{\theta}$ is the MLE.

The Fisher information is

$$\begin{aligned}
\mathcal{I}(\theta) &= E[-H(\theta)] \\
&= E\left[\frac{n}{\theta^2} + \frac{\sum_{i=1}^n Y_i - n}{(1-\theta)^2}\right] \\
&= \frac{n}{\theta^2} + \frac{nE(Y_i)}{(1-\theta)^2} - \frac{n}{(1-\theta)^2} \\
&= \frac{n}{\theta^2} + \frac{n}{\theta(1-\theta)^2} - \frac{n}{(1-\theta)^2} \\
&= \frac{n[(1-\theta)^2 + \theta - \theta^2]}{\theta^2(1-\theta^2)} \\
&= \frac{n(1-\theta)}{\theta^2(1-\theta)^2} \\
&= \frac{n}{\theta^2(1-\theta)}.
\end{aligned}$$

4. (a) The score is

$$U(\theta) = \frac{n}{\theta} - \sum_{i=1}^n Y_i.$$

Since $E(U(\theta)) = 0$,

$$E\left(\sum_{i=1}^n Y_i\right) = \frac{n}{\theta},$$

so

$$E(\bar{Y}) = \frac{1}{\theta},$$

and $\bar{Y}$ is an unbiased estimator of θ^{-1}.

(b) The score is

$$U(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \log Y_i.$$

Since $E(U(\theta)) = 0$,

$$E\left(-\sum_{i=1}^n \log Y_i\right) = \frac{n}{\theta},$$

so

$$E\left(-\frac{1}{n}\sum_{i=1}^n \log Y_i\right) = \frac{1}{\theta},$$

and $-\frac{1}{n}\sum_{i=1}^n \log Y_i$ is an unbiased estimator of θ^{-1}.

(c) The score is

$$u(\theta) = \frac{n}{\theta} - \frac{\sum_{i=1}^n Y_i - n}{1-\theta}.$$

Since $E(U(\theta)) = 0$,

$$\frac{E\left(\sum_{i=1}^n Y_i\right) - n}{1-\theta} = \frac{n}{\theta},$$

so we have

$$\frac{E(\bar{Y})-1}{1-\theta}=\frac{1}{\theta},$$

and

$$E(\bar{Y})=\frac{1}{\theta}.$$

So $\bar{Y}$ is an unbiased estimator of θ^{-1}.

5. The p.f. of each $\boldsymbol{Y}_i=(Y_{i1},Y_{i2},Y_{i3})^T\sim$ Multinomial$(1,\theta_1,\theta_2,1-\theta_1-\theta_2)$ is

$$f_Y(y;\theta_1,\theta_2)=\frac{1}{y_1!y_2!y_3!}\theta_1^{y_1}(\theta_2)^{y_2}(1-\theta_1-\theta_2)^{y_3},$$

so the likelihood is

$$L(\theta_1,\theta_2)=\prod_{i=1}^n f_Y(y_i;\theta_1,\theta_2)=\frac{\theta_1^{\sum_{i=1}^n y_{i1}}(\theta_2)^{\sum_{i=1}^n y_{i2}}(1-\theta_1-\theta_2)^{\sum_{i=1}^n y_{i1}}}{\prod_{i=1}^n y_{i3}!\prod_{i=1}^n y_{i2}!\prod_{i=1}^n y_{i3}!}.$$

The log-likelihood is

$$\begin{aligned}\ell(\theta_1,\theta_2)&=\log L(\theta_1,\theta_2)\\&=\left(\sum_{i=1}^n y_{i1}\right)\log\theta_1+\left(\sum_{i=1}^n y_{i2}\right)\log(\theta_2)+\left(\sum_{i=1}^n y_{i3}\right)\log(1-\theta_1-\theta_2)-\sum_{i=1}^n\sum_{j=1}^3\log(y_{ij}!).\end{aligned}$$

Differentiating the $\ell(\theta_1,\theta_2)$ with respect to θ_1 and θ_2, respectively, gives

$$u_1(\theta_1,\theta_2)=\frac{\partial\ell(\theta_1,\theta_2)}{\partial\theta_1}=\left(\sum_{i=1}^n y_{i1}\right)\frac{1}{\theta_1}-\left(\sum_{i=1}^n y_{i3}\right)\frac{1}{1-\theta_1-\theta_2}.$$

and

$$u_2(\theta_1,\theta_2)=\frac{\partial\ell(\theta_1,\theta_2)}{\partial\theta_2}=\left(\sum_{i=1}^n y_{i2}\right)\frac{1}{\theta_2}-\left(\sum_{i=1}^n y_{i3}\right)\frac{1}{1-\theta_1-\theta_2}.$$

Therefore, the negative Hessian is

$$-H(\theta_1,\theta_2)=-\begin{pmatrix}-\left(\sum_{i=1}^n y_{i1}\right)\frac{1}{\theta_1^2}-\left(\sum_{i=1}^n y_{i3}\right)\frac{1}{(1-\theta_1-\theta_2)^2}, & -\left(\sum_{i=1}^n y_{i3}\right)\frac{1}{(1-\theta_1-\theta_2)^2}\\ -\left(\sum_{i=1}^n y_{i3}\right)\frac{1}{(1-\theta_1-\theta_2)^2}, & -\left(\sum_{i=1}^n y_{i2}\right)\frac{1}{\theta_2^2}-\left(\sum_{i=1}^n y_{i3}\right)\frac{1}{(1-\theta_1-\theta_2)^2}\end{pmatrix}$$

It is straightforward that $E(Y_{i1})=\theta_1$, $E(Y_{i2})=\theta_2$, and $E(Y_{i3})=1-\theta_1-\theta_2$. Hence, the Fisher informaton matrix is

$$\mathcal{I}(\theta_1,\theta_2)=E(-H(\theta_1,\theta_2))=\begin{pmatrix}\frac{n}{\theta_1}+\frac{n}{1-\theta_1-\theta_2}, & \frac{n}{1-\theta_1-\theta_2}\\ \frac{n}{1-\theta_1-\theta_2}, & \frac{n}{\theta_2}+\frac{n}{1-\theta_1-\theta_2}\end{pmatrix}.$$

6. The p.d.f. of each $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{ip})^T \sim N(\boldsymbol{\mu}, \sigma^2 \boldsymbol{I}_p)$ is

$$f_{\boldsymbol{Y}}(\boldsymbol{y}; \boldsymbol{\mu}, \sigma^2) = \frac{1}{\sqrt{(2\pi)^p \sigma^{2p}}} \exp\left[-\frac{(\boldsymbol{y}-\boldsymbol{\mu})^T(\boldsymbol{y}-\boldsymbol{\mu})}{2\sigma^2} \right],$$

so the likelihood is

$$L(\boldsymbol{\mu}, \sigma^2) = \prod_{i=1}^n f_{\boldsymbol{Y}}(\boldsymbol{y}_i; \boldsymbol{\mu}, \sigma^2) = \frac{1}{\sqrt{(2\pi)^{np} \sigma^{2np}}} \exp\left[-\frac{\sum_{i=1}^n (\boldsymbol{y}_i-\boldsymbol{\mu})^T(\boldsymbol{y}_i-\boldsymbol{\mu})}{2\sigma^2} \right].$$

The log-likelihood is

$$\ell(\boldsymbol{\mu}, \sigma^2) = \log L(\boldsymbol{\mu}, \sigma^2) = -\frac{np}{2}\log(2\pi) - \frac{np}{2}\log(\sigma^2) - \frac{\sum_{i=1}^n (\boldsymbol{y}_i-\boldsymbol{\mu})^T(\boldsymbol{y}_i-\boldsymbol{\mu})}{2\sigma^2}.$$

Differentiating the ℓ with respect to $\boldsymbol{\mu}$ and σ^2 gives

$$\boldsymbol{u}_1(\boldsymbol{\mu}, \sigma^2) = \frac{\partial \ell(\boldsymbol{\mu}, \sigma^2)}{\partial \boldsymbol{\mu}} = \frac{1}{\sigma^2}\sum_{i=1}^n (\boldsymbol{y}_i - \boldsymbol{\mu})^T$$

and

$$u_2(\boldsymbol{\mu}, \sigma^2) = \frac{\partial \ell(\boldsymbol{\mu}, \sigma^2)}{\partial \sigma^2} = -\frac{np}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n (\boldsymbol{y}_i-\boldsymbol{\mu})^T(\boldsymbol{y}_i-\boldsymbol{\mu}).$$

Note here $\boldsymbol{u}_1(\boldsymbol{\mu}, \sigma^2) \in \mathbb{R}^p$ and $u_2(\boldsymbol{\mu}, \sigma^2) \in \mathbb{R}^1$.

So the MLE $\hat{\boldsymbol{\mu}}$ and $\hat{\sigma}^2$ solves $\boldsymbol{u}_1(\hat{\boldsymbol{\mu}}, \hat{\sigma}^2) = \boldsymbol{0}$ and $u_2(\hat{\boldsymbol{\mu}}, \hat{\sigma}^2) = 0$,which is easy to see the solutions are:

$$\hat{\boldsymbol{\mu}} = \frac{\sum_{i=1}^n \boldsymbol{y}_i}{n}, \qquad \hat{\sigma}^2 = \frac{\sum_{i=1}^n (\boldsymbol{y}_i-\boldsymbol{\mu})^T(\boldsymbol{y}_i-\boldsymbol{\mu})}{np},$$

which is quite similar to the univariate case.

Furthermore, we can derive the negative Hessian matrix is

$$-\boldsymbol{H}(\boldsymbol{\mu}, \sigma^2) = \begin{pmatrix} \frac{n}{\sigma^2}\boldsymbol{I}_p & \frac{\sum_{i=1}^n (\boldsymbol{y}_i-\boldsymbol{\mu})^T}{(\sigma^2)^2} \\ \frac{\sum_{i=1}^n (\boldsymbol{y}_i-\boldsymbol{\mu})}{(\sigma^2)^2} & \frac{1}{(\sigma^2)^3}\sum_{i=1}^n (\boldsymbol{y}_i-\boldsymbol{\mu})^T(\boldsymbol{y}_i-\boldsymbol{\mu}) - \frac{np}{2(\sigma^2)^2}, \end{pmatrix}$$

This is a $(p+1) \times (p+1)$ matrix.

Using $E(\boldsymbol{Y}_i) = \boldsymbol{\mu}$ and $E(\boldsymbol{Y}_i - \boldsymbol{\mu})^T(\boldsymbol{Y}_i - \boldsymbol{\mu}) = p\sigma^2$, we have the Fisher information matrix is

$$\mathcal{I}(\boldsymbol{\mu}, \sigma^2) = E\left[-\boldsymbol{H}(\boldsymbol{\mu}, \sigma^2)\right] = \begin{pmatrix} \frac{n}{\sigma^2}\boldsymbol{I}_p & \boldsymbol{0} \\ \boldsymbol{0} & \frac{np}{2(\sigma^2)^2} \end{pmatrix}$$

7. (a) The likelihood is

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{y_i-1} = \theta^n (1-\theta)^{\sum_{i=1}^n y_i - n}.$$

The log-likelihood is

$$\ell(\theta) = \log L(\theta) = n \log \theta + \left(\sum_{i=1}^{n} y_i - n \right) \log(1-\theta).$$

The score is

$$u(\theta) = \frac{n}{\theta} - \frac{\sum_{i=1}^n y_i - n}{1-\theta}$$

The Hessian is

$$H(\theta) = -\frac{n}{\theta^2} - \frac{\sum_{i=1}^n y_i - n}{(1-\theta)^2}.$$

The Fisher information is

$$\begin{aligned}
\mathcal{I}(\theta) &= E[-H(\theta)] \\
&= E\left[\frac{n}{\theta^2} + \frac{\sum_{i=1}^n Y_i - n}{(1-\theta)^2} \right] \\
&= \frac{n}{\theta^2} + \frac{nE(Y_i)}{(1-\theta)^2} - \frac{n}{(1-\theta)^2} \\
&= \frac{n}{\theta^2} + \frac{n}{\theta(1-\theta)^2} - \frac{n}{(1-\theta)^2} \\
&= \frac{n[(1-\theta)^2 + \theta - \theta^2]}{\theta^2(1-\theta^2)} \\
&= \frac{n(1-\theta)}{\theta^2(1-\theta)^2} \\
&= \frac{n}{\theta^2(1-\theta)}.
\end{aligned}$$

(b) A stationary point of the log-likelihood $\hat{\theta}$ solves

$$u(\hat{\theta}) = \frac{n}{\hat{\theta}} - \frac{\sum_{i=1}^n y_i - n}{1-\hat{\theta}} = 0,$$

which gives

$$\hat{\theta} = \frac{n}{\sum_{i=1}^n y_i} = \frac{1}{\bar{y}}.$$

The Hessian $H(\theta) < 0$ for all θ, as each $y_i \geq 1$ so $\sum_{i=1}^n y_i - n \geq 0$. So $\hat{\theta}$ is the MLE.

The asymptotic distribution of the $\hat{\theta}$ is

$$\hat{\theta} \sim N(\theta, [\mathcal{I}(\theta)]^{-1}) = N\left(\theta, \frac{\theta^2(1-\theta)}{n}\right).$$

A $100(1-\alpha)\%$ confidence interval for θ is

$$[\hat{\theta} - z_{1-\frac{\alpha}{2}}[\mathcal{I}(\hat{\theta})^{-1}]^{1/2}, \hat{\theta}_i + z_{1-\frac{\alpha}{2}}[\mathcal{I}(\hat{\theta})^{-1}]^{1/2}].$$

For $\alpha = 0.01$, we need to know $z_{1-\frac{\alpha}{2}} = z_{0.995}$, which we can find in R that $z_{0.995} = 2.575829$, or 2.58 to two decimal places. We have

$$\mathcal{I}(\hat{\theta})^{-1} = \frac{\hat{\theta}^2(1-\hat{\theta})}{n} = \frac{\bar{y}-1}{n(\bar{y})^3},$$

so a 99% confidence interval for θ is

$$\left[\frac{1}{\bar{y}} - 2.58\sqrt{\frac{\bar{y}-1}{n(\bar{y})^3}}, \frac{1}{\bar{y}} + 2.58\sqrt{\frac{\bar{y}-1}{n(\bar{y})^3}}\right].$$

8. (a) The log likelihood ratio statistic is

$$L_{01} = 2\log\left(\frac{L(\hat{\theta})}{L(0.5)}\right) = 2[\ell(\hat{\theta}) - \ell(0.5)].$$

From Question 7, the log-likelihood is

$$\ell(\theta) = n\left[\log\theta + (\bar{y}-1)\log(1-\theta)\right],$$

and $\hat{\theta} = \bar{y}^{-1}$, so so

$$\begin{aligned} L_{01} &= 2n\left[-\log\bar{y} + (\bar{y}-1)\log(1-\bar{y}^{-1}) - \log 0.5 - (\bar{y}-1)\log 0.5\right] \\ &= 2n\left[-\log\bar{y} + (\bar{y}-1)\log(1-\bar{y}^{-1}) - \bar{y}\log 0.5\right]. \end{aligned}$$

(b) Under H_0, $L_{01} \sim \chi^2_1$.

(c) We would reject H_0 is $L_{01} > k$, where k is the 99% point of the χ^2_1 distribution. We can find this value in R that $\chi^2_{1,0.99} = 6.634897$.

So we reject H_0 if $L_{01} > 6.63$.