MATH3091 problem sheet 3

Please attempt following questions before 25th Feb 2022.

1. Try to prove Lemma 4.1 by yourself.

2. For the linear mixed model (expressed in matrix form),

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$$

   (a) Assume $\boldsymbol{\theta}$ is known, write down the joint log-likelihood function of $(\boldsymbol{Y}, \boldsymbol{\gamma}; \boldsymbol{\beta})$ (your can discard the some constant terms). Note that $\ell(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \log L(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \ln f(\boldsymbol{y}|\gamma; \boldsymbol{\beta}) f(\gamma; \boldsymbol{\beta})$.

   (b) find a system of equations to solve the estimator of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ from maximising $\ell(\boldsymbol{\beta}, \boldsymbol{\gamma})$. You may assume that a stationary point of the log-likelihood is a maximum.

   hint: (i) for a function of vector $f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{\alpha}$, *we have* $\dfrac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}$; *(ii) for a function of vector* $g(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x}$, *we have* $\dfrac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}} = 2\boldsymbol{A}\boldsymbol{x}$.

   (c) Show that the following estimators we mentioned in Lecture 7-8:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T \boldsymbol{V}^{-1} \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{V}^{-1} \boldsymbol{y}, \qquad \hat{\boldsymbol{\gamma}} = \mathcal{G} \boldsymbol{U}^T \boldsymbol{V}^{-1} (\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})$$

   solve the system of equations you just derived in (b).

3. Consider the linear mixed model with a single explanatory variable in fixed effect and an intercept in random effect:

$$Y_{ij} = \beta_0 + \beta_1 x_{ij} + \gamma_{i0} + \epsilon_{ij},$$

   and suppose we would like to test $H_0 : \beta_1 = 0$ against the alternative H_1 : 'β_1 is unrestricted'. Suppose that $\hat{\beta}_1 = 2.0$ for each group i, and $\text{Var}(\hat{\beta}_1) = 1.69$.

   Calculate the Wald test statistic W for this data. What is the distribution of W under H_0? Find the critical value for W of size 0.05 (you may need to use `R` to do this). Would you reject H_0 using your Wald test?