MATH3091 problem sheet 2

Please attempt following questions, and questios 7&8 in problem sheet 1 before 18th Feb 2022.

1. For the simple linear regression model, $Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$, where $Y_1, \ldots, Y_n$ are independent:
   (a) find the maximum likelihood estimators of β_0 and β_1. You may assume that a stationary point of the log-likelihood is a maximum.
   (b) find the maximum likelihood estimator of σ^2. You may assume that a stationary point of the log-likelihood is a maximum.
   (c) write down the maximum likelihood estimator of σ.
   (d) in the multiple linear regression $Y_i \sim N(\boldsymbol{\beta}^T \boldsymbol{x}_i, \sigma^2)$ where $\boldsymbol{\beta}, \boldsymbol{x}_i \in \mathbb{R}^p$. Find the maximum likelihood estimators $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$, and $\hat{\sigma}^2$ for σ^2. You may assume that a stationary point of the log-likelihood is a maximum.
   (e) show that $\hat{\boldsymbol{\beta}}$ is an unbiased estimator. Which, if any, of the following common linear model assumptions are required for $\hat{\boldsymbol{\beta}}$ to be unbiased?
      i. $\{Y_1 \ldots, Y_n\}$ are all independent
      ii. $\{Y_1 \ldots, Y_n\}$ are all have the same variance
      iii. $\{Y_1 \ldots, Y_n\}$ are all normally distributed
   (f) Is $\hat{\sigma}^2$ an unbiased estimator for σ^2? If not please suggest an unbiased estimator.
2. Consider the linear model with a single explanatory variable x_i

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i,$$

   and suppose we would like to test $H_0 : \beta_1 = 0$ against the alternative H_1 : 'β_1 is unrestricted'. Suppose that $n = 50$, and that the deviances under H_0 and H_1 are $D_0 = 13$ and $D_1 = 12$ respectively.
   (a) Calculate the F statistic for this data. What is the distribution of F under H_0? Find the critical value for the F test of size 0.05 (you may need to use 'R' to do this). Would you reject H_0 using your F test?
   (b) Calculate the log likelihood ratio statistic L_{01} for this data. What is the distribution of L_{01} under H_0? Find the critical value for the log likelihood ratio test of approximate size 0.05 (you may need to use 'R' to do this). Would you reject H_0 using your log likelihood ratio test?

(c) Compare your answers from (a) and (b).

3. Consider the multiple linear regression

$$Y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + \epsilon_i, \quad i = 1, \ldots, n$$

Suppose we want to test $\boldsymbol{C}\boldsymbol{\beta} = \mathbf{0}$, where $\boldsymbol{C}$ is a $m \times p$ matrix of known coefficients.

(a) what test you think is suitable for this problem? Write down your test statistic.

(b) what is the distribution of your test statistic under H_0?