MATH3091 problem sheet 1

Please attempt following questions before 11th Feb 2022.

1. Write down the likelihood and log-likelihood functions for observations $y_1, \ldots, y_n$ from i.i.d. random variables/vectors $Y_1, \ldots, Y_n$ with following distributions:
   (a) $Y_i \sim \text{Poisson}(\lambda)$
   (b) $Y_i \sim \text{Unif}(a, b)$
   (c) $\boldsymbol{Y}_i = (Y_{i1}, Y_{i2}, Y_{i3})^T \sim \text{Multinomial}(1, \theta, 2\theta, 1-3\theta)$
   (d) $Y_i \sim \text{Weibull}(\beta)$, with p.d.f $f_{Y_i}(y) = \alpha y^{\alpha-1} e^{-y^\alpha}$ for $y > 0$ and 0 otherwise.
   (e) $Y_i \sim \text{Pareto}(\alpha)$, with p.d.f $f_{Y_i}(y) = \alpha(1+y)^{-\alpha-1}$ for $y > 0$ and 0 otherwise.
2. In each of the cases from question 1, find the maximum likelihood estimate of
   (a) parameter λ in Poisson distribution
   (b) *parameters a and b in uniform distribution
   (c) parameter θ in multinomial distribution (you do not need to verify your MLE is the maximiser for this question)
   (d) parameter α in Pareto distribution
3. Suppose $y_1, y_2, \cdots, y_n$ are observations from random variables $Y_1, \ldots, Y_n$, which are i.i.d. with p.d.f. (or p.f.) $f_Y(y; \theta)$ for a scalar parameter θ. In each case below, derive the maximum likelihood estimate of θ, and find the score and Fisher information.
   (a) $f_Y(y; \theta) = \theta \exp(-\theta y)$, $y > 0$, $\theta > 0$ (exponential distribution);
   (b) $f_Y(y; \theta) = \theta y^{\theta-1}$, $y \in (0, 1)$, $\theta > 0$; (beta distribution)
   (c) $f_Y(y; \theta) = \theta(1-\theta)^{y-1}$, $y \in \{1, 2, 3, \ldots\}$, $\theta \in (0, 1)$ (geometric distribution).
4. In each of the cases from question 3, use the fact that the expected score is zero to find an unbiased estimator of θ^{-1}.
5. *Suppose $y_1, y_2, \cdots, y_n$ are observations from random variables $Y_1, \ldots, Y_n$, which are i.i.d. Multinomial$(1, \theta_1, \theta_2, 1-\theta_1-\theta_2)$ distributed. Derive the Fisher information matrix for parameters (θ_1, θ_2).
6. *Suppose $\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n$ are observations from p-dimensional random vectors $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$, each follows $N(\boldsymbol{\mu}, \sigma^2 \boldsymbol{I}_p)$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)$ is the p-dimensional mean vector and

$\boldsymbol{I}_p$ is the $p \times p$ identity matrix. Determine the MLE for $(\boldsymbol{\mu}, \sigma^2)$, the score function and the Fisher information matrix.

7. Suppose $y_1, \ldots, y_n$ are observations of $Y_1, \ldots, Y_n$, i.i.d. Geometric(θ) random variables, each with p.d.f. $f_Y(y; \theta) = \theta(1-\theta)^{y-1}$, $y \in \{1, 2, 3, \ldots\}$, where $0 < \theta < 1$ is an unknown parameter.

   (a) Calculate the log-likelihood function, the score function and the Fisher information.

   (b) Calculate the maximum likelihood estimator $\hat{\theta}$ of θ and construct its approximate (large sample) confidence interval at 99% confidence level.

8. Continue Question 7. Suppose that we would like to test $H_0 : \theta = 0.5$ against the general alternative H_1 : 'θ is unrestricted'.

   (a) Find the log likelihood ratio statistic L_{01} for testing H_0 against H_1.

   (b) What is the asymptotic distribution of L_{01} under H_0?

   (c) Suppose we want a test of approximate size $\alpha = 0.01$. For which values of L_{01} would you reject H_0? You may need to use the 'qchisq' function in 'R' to find the critical value.