ON DOUBLE COSET SEPARABILITY AND THE WILSON-ZALESSKII PROPERTY

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Abstract. A residually finite group G has the Wilson-Zalesskii property if for all finitely generated subgroups $H, K \leqslant G$, one has $\overline{H} \cap \overline{K} = \overline{H \cap K}$, where the closures are taken in the profinite completion $\widehat{G}$ of G. This property played an important role in several papers, and is usually combined with separability of double cosets. In the present note we show that the Wilson-Zalesskii property is actually enjoyed by every double coset separable group. We also construct an example of a LERF group that is not double coset separable and does not have the Wilson-Zalesskii property.

1. Introduction

Every residually finite group G has a natural embedding into its profinite completion $\widehat{G}$, which is a compact topological group. The topology on $\widehat{G}$ induces the *profinite topology* on G. A subset $S \subseteq G$ is said to be *separable* if it is closed in this topology, i.e., $S = \overline{S} \cap G$, where $\overline{S}$ denotes the closure of S in $\widehat{G}$.

Many residual properties of G can be interpreted in terms of the profinite topology or the embedding of G into $\widehat{G}$. In establishing various such properties it is often useful to have control over the intersections of images of two subgroups $H, K \in \mathcal{S}$ in finite quotients of G, where $\mathcal{S}$ is a class of subgroups of G (e.g., $\mathcal{S}$ could consist of all cyclic subgroups, all abelian subgroups or all finitely generated subgroups). The best one can hope for is that for all $H, K \in \mathcal{S}$ we have

$$\overline{H} \cap \overline{K} = \overline{H \cap K} \text{ in } \widehat{G} \tag{1}$$

(see Remark 2.2 and Proposition 2.4 below, which explain how this is related to controlling the intersection of the images of H and K in finite quotients of G).

Condition (1) played an important role in the papers of Ribes and Zalesskii [10], Ribes, Segal and Zalesskii [9], Wilson and Zalesskii [13] and Antolín and Jaikin-Zapirain [2], to mention a few. In all of these papers, this condition was established along with (and after) the double coset separability condition, stating that for all $H, K \in \mathcal{S}$

$$HK \text{ is separable in } G. \tag{2}$$

The purpose of this note is to demonstrate that condition (2) implies (1), provided $\mathcal{S}$ is closed under taking finite index subgroups. More precisely, we prove the following.

Theorem 1.1. *Let H, K be subgroups of a residually finite group G. Then the following are equivalent:*

- *(a) the double coset HK is separable in G and $\overline{H} \cap \overline{K} = \overline{H \cap K}$ in $\widehat{G}$;*
- *(b) for every finite index subgroup $L \leqslant_f G$, with $H \cap K \subseteq L$, the double coset $(H \cap L)K$ is separable in G.*

The above theorem follows from Proposition 2.4 below, which restates condition (1) in terms of finite index subgroups of the group G, and Proposition 3.1, which characterises this restatement

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in terms of double cosets. Both of these propositions are stated in the general situation of a pro-$\mathcal{C}$ topology, where $\mathcal{C}$ is a formation of finite groups. In particular, analogues of Theorem 1.1 are also true for the pro-p topology, the pro-soluble topology, etc.

Following [2], we say that a group G has the *Wilson-Zalesskii property* if (1) holds for arbitrary finitely generated subgroups $H, K \leqslant G$. This property is named after Wilson and Zalesskii, who established it in the case of finitely generated virtually free groups in [13]. We will call a group G *double coset separable* if (2) holds for all finitely generated subgroups $H, K \leqslant G$.

Corollary 1.2. *Every double coset separable group satisfies the Wilson-Zalesskii property.*

Note that for (virtually) free groups the double coset separability was first proved by Gitik and Rips [3]. This was extended by Niblo [8] to finitely generated Fuchsian groups and fundamental groups of Seifert-fibred 3-manifolds. In [7] the author and Mineh showed that all finitely generated Kleinian groups and limit groups are double coset separable. Hence, by Corollary 1.2, such groups have the Wilson-Zalesskii property. For limit groups this answers a question of Antolín and Jaikin-Zapirain from [2, Subsection 2.2].

More generally, separability of double cosets of "convex" subgroups is known in many non-positively curved groups (see [6, 4, 12, 7]). By combining these results with Theorem 1.1 we gain control over the intersection of such subgroups in finite quotients. Our last corollary describes one such application.

Corollary 1.3. *Let G be a finitely generated group hyperbolic relative to a family of double coset separable subgroups. If every finitely generated relatively quasiconvex subgroup is separable in G then any two finitely generated relatively quasiconvex subgroups $H, K \leqslant G$ satisfy* (1).

Proof. Let G be a group from the statement. By [7, Corollary 1.4], the product of two finitely generated relatively quasiconvex subgroups is separable in G. Since a finite index subgroup of a relatively quasiconvex subgroup is again relatively quasiconvex [7, Lemma 5.22], the claim of the corollary follows from Theorem 1.1. □

We finish this note by constructing, in Section 4, an example of a finitely presented LERF group which is not double coset separable and does not have the Wilson-Zalesskii property.

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2. A restatement of condition (1)

Let us fix a formation $\mathcal{C}$ of finite groups; in other words, $\mathcal{C}$ is a non-empty class of finite groups which is closed under taking quotients and subdirect products (see [11, Section 2.1]).

2.1. Pro-$\mathcal{C}$ topology and completion.

In this subsection we summarise basic definitions and properties of pro-$\mathcal{C}$ topology and pro-$\mathcal{C}$ completions. We refer the reader to [11, Sections 3.1, 3.2] for a detailed exposition.

Given a group G, we define the *pro-$\mathcal{C}$ topology* on G by taking the family of normal subgroups $\mathcal{N}_{\mathcal{C}}(G) = \{N \lhd G \mid G/N \in \mathcal{C}\}$ as a basis of open neighborhoods of the identity element. A subset $A \subseteq G$ will be called *$\mathcal{C}$-open* if it is open in the pro-$\mathcal{C}$ topology on G. *$\mathcal{C}$-closed* and *$\mathcal{C}$-clopen* subsets are defined similarly. We will write $H \leqslant_o G$ and $N \lhd_o G$ to indicate that H is an open subgroup of G and N is an open normal subgroup of G in the pro-$\mathcal{C}$ topology. Note that a subgroup $H \leqslant G$ is $\mathcal{C}$-open if and only if it contains a $\mathcal{C}$-open normal subgroup; and $N \lhd G$ is $\mathcal{C}$-open if and only if $G/N \in \mathcal{C}$. If $H \leqslant_o G$ and $X \subseteq G$ then XH and $G \setminus XH$ are both open as unions of cosets modulo H, thus XH is a $\mathcal{C}$-clopen subset of G.

We will use $G_{\widehat{\mathcal{C}}}$ to denote the *pro-$\mathcal{C}$ completion* of a group G. Equipped with its pro-$\mathcal{C}$ topology, $G_{\widehat{\mathcal{C}}}$ is a profinite group; in particular, it is compact. The natural homomorphism $G \to G_{\widehat{\mathcal{C}}}$ has dense image. This homomorphism is injective if and only if G is *residually-$\mathcal{C}$*, i.e., $\bigcap_{N\in\mathcal{N}_{\mathcal{C}}(G)} N = \{1\}$.

2.2. **Tractable intersections.**

Definition 2.1. Let G be a group and let $H, K \leqslant G$ be two subgroups. We will say that the intersection $H \cap K$ is *pro-$\mathcal{C}$ tractable* in G if for every $M \lhd_o G$ there exists $N \lhd_o G$ such that $N \subseteq M$ and

$$HN \cap KN \subseteq (H \cap K)M \quad \text{in } G. \tag{3}$$

Remark 2.2. Note that condition (3) can be restated as $\phi(H)\cap\phi(K) \subseteq \phi(H\cap K)\phi(M)$ in the finite quotient $G/N \in \mathcal{C}$, where $\phi : G \to G/N$ denotes the natural homomorphism.

Remark 2.3. The following observation will be used throughout this note without further justification. If A, B are subsets of a group G and $H' \leqslant H \leqslant G$ are subgroups then

$$AH' \cap BH = (A \cap BH)H' \quad \text{and} \quad H'A \cap HB = H'(A \cap HB).$$

Proposition 2.4. *For subgroups H, K of a residually-$\mathcal{C}$ group G the following are equivalent:*
- *(i) the intersection $H \cap K$ is pro-$\mathcal{C}$ tractable in G;*
- *(ii) $\overline{H} \cap \overline{K} = \overline{H \cap K}$ in $G_{\widehat{\mathcal{C}}}$, where $\overline{H}$ denotes the closure of H in the pro-$\mathcal{C}$ completion $G_{\widehat{\mathcal{C}}}$.*

Proof. Since G is residually-$\mathcal{C}$, we will treat it as a subgroup of $G_{\widehat{\mathcal{C}}}$. Note that for an arbitrary $L \lhd_o G$ its closure $\overline{L}$ is a clopen subgroup of $G_{\widehat{\mathcal{C}}}$ and $\overline{L} \cap G = L$, so that $G_{\widehat{\mathcal{C}}}/\overline{L} = G/L$ (see [11, Proposition 3.2.2]). Given any $M \lhd_o G$, let $\mathcal{N}_{\mathcal{C}}(M, G) = \{N \lhd_o G \mid N \subseteq M\}$ and observe that $\{\overline{N} \mid N \in \mathcal{N}_{\mathcal{C}}(M, G)\}$ is a basis of open neighborhoods of the identity element in $G_{\widehat{\mathcal{C}}}$.

Let us start with showing that (i) implies (ii). Assuming (i), we know that for every $M \lhd_o G$ there exists $N \in \mathcal{N}_{\mathcal{C}}(M, G)$ such that (3) holds. After taking closures of both sides we obtain

$$\overline{HN \cap KN} \subseteq \overline{(H \cap K)M} \quad \text{in } G_{\widehat{\mathcal{C}}}. \tag{4}$$

Note that $HN, KN \leqslant_o G$, so, by [11, Proposition 3.2.2], $\overline{HN \cap KN} = \overline{HN} \cap \overline{KN}$. Clearly $\overline{H} \cap \overline{K} \subseteq \overline{HN} \cap \overline{KN}$ and $\overline{(H \cap K)M} = (H \cap K)\overline{M}$, because $\overline{M}$ is a clopen subgroup of $G_{\widehat{\mathcal{C}}}$. Hence, in view of (4), we obtain

$$\overline{H} \cap \overline{K} \subseteq (H \cap K)\overline{M} \quad \text{in } G_{\widehat{\mathcal{C}}}, \text{ for every } M \lhd_o G. \tag{5}$$

It is easy to see that $\overline{H \cap K} = \bigcap_{M \lhd_o G} (H \cap K)\overline{M}$, because $\mathcal{N}_{\mathcal{C}}(G_{\widehat{\mathcal{C}}}) = \{\overline{L} \mid L \in \mathcal{N}_{\mathcal{C}}(G)\}$. Therefore (5) implies that $\overline{H} \cap \overline{K} \subseteq \overline{H \cap K}$. The opposite inclusion is obvious, so (ii) has been established.

We will now prove that (ii) implies (i) (in the case of profinite topology this was done in [2, Corollary 10.4]). Suppose that (ii) holds and $M \lhd_o G$ is arbitrary. If (i) is false, then for every $N \in \mathcal{N}_{\mathcal{C}}(M, G)$, we have

$$(HN \cap KN) \setminus (H \cap K)M \neq \emptyset \text{ in } G,$$

hence

$$(H\overline{N} \cap K\overline{N}) \setminus (H \cap K)\overline{M} \neq \emptyset \text{ in } G_{\widehat{\mathcal{C}}}, \text{ for all } N \in \mathcal{N}_{\mathcal{C}}(M, G), \tag{6}$$

where we used the fact that $(H \cap K)\overline{M} \cap G = (H \cap K)(\overline{M} \cap G) = (H \cap K)M$.

The family $\{(H\overline{N} \cap K\overline{N}) \setminus (H \cap K)\overline{M} \mid N \in \mathcal{N}_{\mathcal{C}}(M, G)\}$ consists of clopen sets in $G_{\widehat{\mathcal{C}}}$ and has the finite intersection property by (6) (because the intersection of finitely subgroups from $\mathcal{N}_{\mathcal{C}}(M, G)$ is again in $\mathcal{N}_{\mathcal{C}}(M, G)$). Compactness of $G_{\widehat{\mathcal{C}}}$ now implies that

$$\bigcap_{N\in\mathcal{N}_{\mathcal{C}}(M,G)} (H\overline{N} \cap K\overline{N}) \setminus (H \cap K)\overline{M} \neq \emptyset. \tag{7}$$

Since $\bigcap_{N \in \mathcal{N}_\mathcal{C}(M,G)} H\overline{N} = \overline{H}$, $\bigcap_{N \in \mathcal{N}_\mathcal{C}(M,G)} K\overline{N} = \overline{K}$ and $\overline{H \cap K} \subseteq (H \cap K)\overline{M}$, (7) demonstrates that $(\overline{H} \cap \overline{K}) \setminus \overline{H \cap K} \neq \emptyset$, contradicting (ii). Thus we have proved that (ii) implies (i). □

3. Characterising tractableness of intersections using double cosets

As before we will work with a fixed formation of finite groups $\mathcal{C}$. For a subgroup H of a group G the pro-$\mathcal{C}$-topology on G induces a topology on H (which may, in general, be different from the pro-$\mathcal{C}$ topology of H). We will use $\mathcal{O}_\mathcal{C}(H,G)$ to denote the open subgroups of H in this induced topology. In other words,

$$\mathcal{O}_\mathcal{C}(H,G) = \{H \cap L \mid L \leqslant_o G\}.$$

Note that for every $H' \in \mathcal{O}_\mathcal{C}(H,G)$, the index $|H : H'|$ is finite because any $L \leqslant_o G$ has finite index in G.

Proposition 3.1. *Let G be a group with subgroups H, K. Then the following are equivalent:*

(i) the double coset HK is $\mathcal{C}$-closed and the intersection $H \cap K$ is pro-$\mathcal{C}$ tractable in G;

(ii) for every $H' \in \mathcal{O}_\mathcal{C}(H,G)$, with $H \cap K \subseteq H'$, the double coset $H'K$ is $\mathcal{C}$-closed in G.

Proof. Let us start with showing that (i) implies (ii). So, assume that (i) is true. Consider any $H' \in \mathcal{O}_\mathcal{C}(H,G)$, containing $H \cap K$. Then $H' = H \cap L$, for some $L \leqslant_o G$, with $H \cap K \subseteq L$. Let $M \lhd_o G$ denote the normal core of L (it is $\mathcal{C}$-open by [11, Lemma 3.1.2]). Since $H \cap K$ is pro-$\mathcal{C}$ tractable, there exists $N \lhd_o G$ such that $N \subseteq M$ and

$$HN \cap KN \subseteq (H \cap K)M \subseteq L.$$

Since $NK = KN$, as $N \lhd G$, we can conclude that

$$H \cap NK = H \cap KN \subseteq H \cap L = H'.$$

Therefore, we have

$$H'K \subseteq HK \cap H'KN = H'(HK \cap NK) = H'(H \cap NK)K \subseteq H'H'K = H'K,$$

whence $H'K = HK \cap H'KN$ in G. Note that the subset $H'KN$ is $\mathcal{C}$-clopen in G, as $N \lhd_o G$, and the double coset HK is $\mathcal{C}$-closed by the assumption (i). Thus $H'K$ is $\mathcal{C}$-closed as the intersection of closed subsets, so (ii) holds.

Now let us assume (ii) and deduce (i). Then the double coset HK is $\mathcal{C}$-closed in G because $H \in \mathcal{O}_\mathcal{C}(H,G)$ and $H \cap K \subseteq H$. Thus it remains to show that $H \cap K$ is pro-$\mathcal{C}$ tractable in G.

Take any $M \lhd_o G$ and set $L = (H \cap K)M \leqslant_o G$. Then $H' = H \cap L \in \mathcal{O}_\mathcal{C}(H,G)$ and we can write $H = \bigsqcup_{i=1}^n H'h_i$, where $h_1 = 1$ and $h_i \in H \setminus H'$, for $i = 2, \ldots, n$. Note that $H \cap K \subseteq H'$, by construction, which easily implies that $h_i \notin H'K$, for $i = 2, \ldots, n$ (indeed, if $h_i = xy$, where $x \in H'$ and $y \in K$, then $x^{-1}h_i = y \in H \cap K \subseteq H'$, so $h_i \in H'$, whence and $i = 1$). By the assumption (ii), the double coset $H'K$ is $\mathcal{C}$-closed in G, hence there exists $N \lhd_o G$ such that

$$h_i \notin H'KN, \text{ for } i = 2, \ldots, n. \tag{8}$$

After replacing N with $N \cap M$, we can suppose that $N \subseteq M$. Let us show that

$$HN \cap KN \subseteq L = (H \cap K)M.$$

Since $HN \cap KN = (H \cap KN)N$ and $N \subseteq L$, it is enough to check that $H \cap KN \subseteq L$. But, in view of (8), we know that $H'h_i \cap KN = \emptyset$, for $i = 2, \ldots, n$, hence $H \cap KN \subseteq H'h_1 = H' \subseteq L$, as required. Therefore $H \cap K$ is pro-$\mathcal{C}$ tractable in G and (i) holds. □

Corollary 3.2. *If H, K are subgroups of a group G then the following are equivalent:*

(i) the double coset HK is $\mathcal{C}$-closed and the intersection $H \cap K$ is pro-$\mathcal{C}$ tractable in G.

(ii) for every $H' \in \mathcal{O}_\mathcal{C}(H,G)$, with $H \cap K \subseteq H'$, the double coset $H'K$ is $\mathcal{C}$-closed in G;

(iii) for every $K' \in \mathcal{O}_\mathcal{C}(K,G)$, with $H \cap K \subseteq K'$, the double coset HK' is $\mathcal{C}$-closed in G;

(iv) for all $H' \in \mathcal{O}_\mathcal{C}(H,G)$ and $K' \in \mathcal{O}_\mathcal{C}(K,G)$, with $H \cap K = H' \cap K'$, the double coset $H'K'$ is $\mathcal{C}$-closed in G;

Proof. The equivalence between (i) and (ii) is the subject of Proposition 3.1, and the equivalence between (i) and (iii) follows by symmetry (or because $HK' = (K'H)^{-1}$). Evidently (iv) implies (ii). Conversely, (iv) follows from (ii) and (iii) because

$$H'K \cap HK' = H'(K \cap HK') = H'(K \cap H)K' = H'K',$$

where the last equality is valid since $K \cap H \subseteq H'$. □

4. A LERF group without the Wilson-Zalesskii property

Throughout this section we assume that $\mathcal{C}$ is the family of all finite groups. In this case the pro-$\mathcal{C}$ topology on a group G is the profinite topology, $\mathcal{C}$-open subgroups of G are precisely the finite index subgroups and the $\mathcal{C}$-closed subsets of G are called separable. Recall that G is said to be *ERF* if all subgroups are separable and *LERF* if all finitely generated subgroups are separable.

In this section we show that separability of a double coset HK does not necessarily yield that the intersection $H \cap K$ is profinitely tractable even for finitely generated subgroups H, K of a LERF group G. Our construction is based on examples of Grunewald and Segal from [5].

Let $A = M_2(\mathbb{Z})$ be the additive group of 2×2 matrices with integer entries, and let $H = SL_2(\mathbb{Z})$ act on A by left multiplication. We define the group G as the resulting semidirect product $A \rtimes H = M_2(\mathbb{Z}) \rtimes SL_2(\mathbb{Z})$. Recall that H is finitely generated and virtually free and A is the free abelian group of rank 4, hence A is ERF and H is LERF, so G is LERF (see [1, Theorem 4]).

Denote by $i \in A$ the identity matrix from $M_2(\mathbb{Z})$ and set $K = iHi^{-1} \leqslant G$. For any subgroup $F \leqslant H = SL_2(\mathbb{Z})$ the conjugacy class $i^F = \{fif^{-1} \mid f \in F\} \subseteq A$ is the orbit of the identity matrix under the left action of F, so it consists of matrices from F, but now considered as a subset of $M_2(\mathbb{Z}) = A$. Since the determinant map $\det : A = M_2(\mathbb{Z}) \to \mathbb{Z}$ is clearly continuous with respect to the profinite topologies on A and $\mathbb{Z}$, the conjugacy class $i^H = \det^{-1}(\{1\})$ is closed in the profinite topology on A.

Now let us show that the product $i^H H$ is closed in the profinite topology on G. Indeed, suppose that $xy \in G \setminus i^H H$, where $x \in A$ and $y \in H$. Then $x \notin i^H$, so there is $m \in \mathbb{N}$ such that for the finite index characteristic subgroup $A' = M_2(m\mathbb{Z}) \lhd A$ we have $x \notin i^H A'$. The latter implies that $xy \notin i^H A'H$. Since $A'H$ is a finite index subgroup of G, we see that $i^H A'H$ is a clopen subset in the profinite topology on G containing $i^H H$ but not containing xy. Thus $i^H H$ is indeed profinitely closed in G. Note that $i^H H = HiH$, thus the double coset $HK = (HiH)i^{-1}$ is separable in G.

As Grunewald and Segal observed in [5, Section 5], H contains a finite index free subgroup H' (in fact, $|H : H'| = 36$) such that the orbit of i under the action of H' is not separable in the profinite topology on A (equivalently, H' is not closed in the congruence topology on $H = SL_2(\mathbb{Z})$).

Observe that $H'iH = i^{H'}H$, so $H'iH \cap A = i^{H'}$. Since $i^{H'}$ is not separable in A, it follows that the double cosets $H'iH$ and $H'K = (H'iH)i^{-1}$ cannot be separable in G (this is true because the topology on the subgroup A, induced from the profinite topology on G, is always weaker than the profinite topology on A).

Finally, we note that $H \cap K = \{1\}$ because the H-stabiliser of i is trivial, and every finite index subgroup of H belongs to $\mathcal{O}_\mathcal{C}(H,G)$, as G is LERF and H is finitely generated (see, for example, [7, Lemma 4.17]). Thus we have constructed the following example.

Example 4.1. There is a LERF group G (isomorphic to a split extension of $\mathbb{Z}^4$ by $SL_2(\mathbb{Z})$) and finitely generated subgroups $H, K \leqslant G$ such that $H \cap K = \{1\}$ and the double coset HK is separable in G, but the double coset $H'K$ is not separable in G, for some finite index subgroup $H' \leqslant_f H$. We deduce, from Proposition 3.1, that the intersection $H \cap K$ is not profinitely tractable in G, so G does not have the Wilson-Zalesskii property by Proposition 2.4.

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