ON PRODUCTS OF QUASICONVEX SUBGROUPS IN HYPERBOLIC GROUPS

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Abstract
An interesting question about quasiconvexity in a hyperbolic group concerns finding classes of quasiconvex subsets that are closed under finite intersections. A known example is the class of all quasiconvex subgroups [1]. However, not much is yet learned about the structure of arbitrary quasiconvex subsets. In this work we study the properties of products of quasiconvex subgroups; we show that such sets are quasiconvex, their finite intersections have a similar algebraic representation and, thus, are quasiconvex too.

0. Introduction
Let \( G \) be a hyperbolic group, \( \Gamma(G, A) \) – its Cayley graph corresponding to a finite symmetrized generating set \( A \) (i.e., for each element \( a \in A \), \( a^{-1} \) also belongs to this set). A subset \( Q \subseteq G \) is said to be \( \varepsilon \)-quasiconvex, if any geodesic connecting two elements from \( Q \) belongs to a closed \( \varepsilon \)-neighborhood \( O_\varepsilon(Q) \) of \( Q \) in \( \Gamma(G, A) \) for some \( \varepsilon \geq 0 \). \( Q \) will be called quasiconvex if there exists \( \varepsilon > 0 \) for which it is \( \varepsilon \)-quasiconvex.

In [4] Gromov proves that the notion of quasiconvexity in a hyperbolic group does not depend on the choice of a finite generating set (it is easy to show that this is not true in an arbitrary group).

If \( A, B \subseteq G \) then their product is a subset of \( G \) defined by \( A \cdot B = \{ab \mid a \in A, b \in B\} \).

**Proposition 1.** If the sets \( A_1, \ldots, A_n \subseteq G \) are quasiconvex then their product set \( A_1 A_2 \cdot \cdot \cdot A_n = \{a_1 a_2 \cdot \cdot \cdot a_n \mid a_i \in G_i\} \subset G \) is also quasiconvex.

Proposition 1 was proved by Zeph Grunschlag in 1999 in [11; Prop. 3.14] and, independently, by the author in his diploma paper in 2000.

If \( H \) is a subgroup of \( G \) and \( x \in G \) then the subgroup conjugated to \( H \) by \( x \) will be denoted \( H^x = xHx^{-1} \). The main result of the paper is

**Theorem 1.** Suppose \( G_1, \ldots, G_n, H_1, \ldots, H_m \) are quasiconvex subgroups of the group \( G, n, m \in \mathbb{N} \); \( f, e \in G \). Then there exist numbers \( r, \ell \in \mathbb{N} \cup \{0\} \) and \( f_l, \alpha_{lk}, \beta_{lk} \in G, k = 1, 2, \ldots, t_l \) (for every fixed \( l \)), \( l = 1, 2, \ldots, r \), such that

\[
f G_1 G_2 \cdot \cdot \cdot G_n \cap eH_1 H_2 \cdot \cdot \cdot H_m = \bigcup_{l=1}^{r} f_l S_l
\]

where for each \( l, \ell = t_l \), there are indices \( 1 \leq i_1 \leq i_2 \leq \ldots \leq i_\ell \leq n, 1 \leq j_1 \leq \leq j_2 \leq \ldots \leq j_\ell \leq m \):

\[
S_l = (G_{i_1}^{\alpha_{1l}} \cap H_{j_1}^{\beta_{1l}}) \cdot \cdot \cdot (G_{i_\ell}^{\alpha_{\ell l}} \cap H_{j_\ell}^{\beta_{\ell l}}).
\]
This claim does not hold if the group \( G \) is not hyperbolic: set \( G_1 = \langle (1, 0) \rangle \), \( G_2 = \langle (0, 1) \rangle \), \( H = \langle (1, 1) \rangle \) – cyclic subgroups of \( \mathbb{Z}^2 \) (they are quasiconvex in \( \mathbb{Z}^2 \) with generators \( \{(1, 0), (0, 1), (1, 1)\} \). \( G_1 \cdot G_2 = \mathbb{Z}^2 \), thus, \( G_1 G_2 \cap H = H \) but \( G_1 \cap H = G_2 \cap H = \{(0, 0)\} \) and, if the statement of the theorem held for \( \mathbb{Z}^2 \) then \( H \) would be finite – a contradiction.

The above example can also be used as another argument to prove the well-known fact that \( \mathbb{Z}^2 \) can not be embedded into a hyperbolic group (because any cyclic subgroup is quasiconvex in a hyperbolic group).

The condition that the subgroups \( G_i \), \( H_j \) are quasiconvex is also necessary: using Rips’ Construction ([12]) one can achieve a group \( G \) satisfying the small cancellation condition \( C'(1/6) \) (and, therefore, hyperbolic) and its finitely generated normal subgroup \( K \) such that \( G/K \cong \mathbb{Z}^2 \). Let \( \phi \) be the natural epimorphism from \( G \) to \( \mathbb{Z}^2 \), \( G_1 = \phi^{-1}(\langle (1, 0) \rangle) \leq G \), \( G_2 = \phi^{-1}(\langle (0, 1) \rangle) \leq G \), \( H = \phi^{-1}(\langle (1, 1) \rangle) \leq G \). \( G_1, G_2 \) and \( H \) are finitely generated subgroups of \( G \), \( G_1 \cdot G_2 = G \) because \( \langle (1, 0) \rangle \cdot \langle (0, 1) \rangle = \mathbb{Z}^2 \) and \( K \leq G_2 \), thus \( G_1 \cdot G_2 \cap H = H \).

But for every \( \alpha, \beta \in G \) \( \phi(G_1^\alpha \cap H^\beta) \subseteq \phi(G_1^\alpha) \cap \phi(H)^{\phi(\beta)} = \{(0, 0)\}, i = 1, 2 \). Hence, it is impossible to obtain the infinite subgroup \( \phi(H) \) from products of cosets to such sets, and we constructed the counterexample needed.

**Definition:** let \( G_1, G_2, \ldots, G_n \) be quasiconvex subgroups of \( G \), \( f_1, f_2, \ldots, f_n \in G \), \( n \in \mathbb{N} \). Then the set

\[
f_1 G_1 f_2 G_2 \cdots f_n G_n = \{ f_1 g_1 f_2 g_2 \cdots f_n g_n \in G \mid g_i \in G_i, i = 1, \ldots, n \}
\]

will be called quasiconvex product.

**Corollary 2.** An intersection of finitely many quasiconvex products is a finite union of quasiconvex products.

Thus the class of finite unions of quasiconvex products is closed under taking finite intersections.

Recall that a group \( H \) is called elementary if it has a cyclic subgroup \( \langle h \rangle \) of finite index. An elementary subgroup of a hyperbolic group is quasiconvex (see remark 5, Section 4). It is well known that any element \( x \) of infinite order in \( G \) is contained in a unique maximal elementary subgroup \( E(x) \leq G \) [4, 5]. Every non-elementary hyperbolic group contains the free group of rank 2 [5, Cor. 6].

Suppose \( G_1, G_2, \ldots, G_n, H_1, H_2, \ldots, H_m \) are infinite maximal elementary subgroups of \( G \), \( f, e \in G \). And \( G_i \neq G_{i+1}, H_j \neq H_{j+1}, i = 1, \ldots, n - 1, j = 1, \ldots, m - 1 \). Then we present the following uniqueness result for the products of such subgroups:

**Theorem 2.** The sets \( f G_1 \cdots G_n \) and \( e H_1 \cdots H_m \) are equal if and only if \( n = m, G_n = H_n \), and there exist elements \( z_j \in H_j, j = 1, \ldots, n \), such that \( G_j = (z_n z_{n-1} \cdots z_{j+1}) \cdot H_j \cdot (z_n z_{n-1} \cdots z_{j+1})^{-1}, j = 1, 2, \ldots, n - 1, f = e z_1^{-1} z_2^{-1} \cdots z_n^{-1} \).
given in Section 4). The statement of Corollary 2 can be strengthened in this case:

**Theorem 3.** Intersection of any family (finite or infinite) of finite unions of ME-products is a finite union of ME-products.

An example which shows that an analogous property is not true for arbitrary quasiconvex products is constructed at the end of this paper.

Thus, all finite unions of ME-products constitute a topology $\mathcal{T}$ (of closed sets) on the set of elements of a hyperbolic group. Taking an inverse, left and right shifts in $G$ are continuous operations in $\mathcal{T}$. Also, by definition, any point is closed in $\mathcal{T}$, so $\mathcal{T}$ is weakly separated ($T_1$). However, if $G$ is infinite elementary, then $\mathcal{T}$ turns out to be the topology of finite complements which is not Hausdorff, also, in this case, the group multiplication is not continuous with respect to $\mathcal{T}$ (since any product of two non-empty open sets contains the identity of $G$).

1. Preliminary information

Let $d(\cdot, \cdot)$ be the usual left-invariant metric on the Cayley graph of the group $G$ with generating set $\mathcal{A}$. For any two points $x, y \in \Gamma(G, \mathcal{A})$ we fix a geodesic path between them and denote it by $[x, y]$.

If $Q \subset \Gamma(G, \mathcal{A})$, $N \geq 0$, the closed $N$-neighborhood of $Q$ will be denoted by

$$O_N(Q) \overset{df}{=} \{ x \in \Gamma(G, \mathcal{A}) \mid d(x, Q) \leq N \} .$$

If $x, y, w \in \Gamma(G, \mathcal{A})$, then the number

$$(x|y)_w \overset{df}{=} \frac{1}{2}(d(x, w) + d(y, w) - d(x, y))$$

is called the Gromov product of $x$ and $y$ with respect to $w$.

Let $abc$ be a geodesic triangle. There exist "special" points $O_a \in [b, c]$, $O_b \in [a, c]$, $O_c \in [a, b]$ with the properties: $d(a, O_b) = d(a, O_c) = \alpha$, $d(b, O_a) = = d(b, O_c) = \beta$, $d(c, O_a) = d(c, O_b) = \gamma$. From a corresponding system of linear equations one can find that $\alpha = (b|c)_a$, $\beta = (a|c)_b$, $\gamma = (a|b)_c$. Two points $O \in [a, b]$ and $O' \in [a, c]$ are called $a$-equidistant if $d(a, O) = d(a, O') \leq \alpha$.

The triangle $abc$ is said to be $\delta$-thin if for any two points $O, O'$ lying on its sides and equidistant from one of its vertices, $d(O, O') \leq \delta$ holds.

$abc$ is $\delta$-slim if each of its sides belongs to a closed $\delta$-neighborhood of the two others.

We assume the following equivalent definitions of hyperbolicity of $\Gamma(G, \mathcal{A})$ to be known to the reader (see [6], [2]):

1°. There exists $\delta \geq 0$ such that for any four points $x, y, z, w \in \Gamma(G, \mathcal{A})$ their Gromov products satisfy

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - \delta ;$$

2°. All triangles in $\Gamma(G, \mathcal{A})$ are $\delta$-thin for some $\delta \geq 0$;

3°. All triangles in $\Gamma(G, \mathcal{A})$ are $\delta$-slim for some $\delta \geq 0$.  

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Now and below we suppose that $G$ meets $1^\circ, 2^\circ$ and $3^\circ$ for a fixed (sufficiently large) $\delta \geq 0$. $3^\circ$ easily implies

**Remark 0.** Any side of a geodesic $n$-gon ($n \geq 3$) in $\Gamma(G, A)$ belongs to a closed $(n-2)\delta$-neighborhood of the union of the rest of its sides.

Let $p$ be a path in the Cayley graph of $G$. Further on by $p_-$, $p_+$ we will denote the startpoint and the endpoint of $p$, by $|p|$ its length; $\text{lab}(p)$, as usual, will mean the word in the alphabet $A$ written on $p$. $\text{elem}(p) \in G$ will denote the element of the group $G$ represented by the word $\text{lab}(p)$.

A path $q$ is called $(\lambda, c)$-quasigeodesic if there exist $0 < \lambda \leq 1$, $c \geq 0$, such that for any subpath $p$ of $q$ the inequality $\lambda|p| - c \leq d(p_-, p_+)$ holds. In a hyperbolic space quasigeodesics and geodesics with same ends are mutually close:

**Lemma 1.1.** ([6; 5.6,5.11], [2; 3.3]) There is a constant $N = N(\delta, \lambda, c)$ such that for any $(\lambda, c)$-quasigeodesic path $p$ in $\Gamma(G, A)$ and a geodesic $q$ with $p_- = q_-, p_+ = q_+$, one has $p \subset O_N(q)$ and $q \subset O_N(p)$.

An important property of cyclic subgroups in a hyperbolic group states

**Lemma 1.2.** ([6; 8.21], [2; 3.2]) For any word $w$ representing an element $g \in G$ of infinite order there exist constants $\lambda > 0$, $c \geq 0$, such that any path with a label $w^m$ in the Cayley graph of $G$ is $(\lambda, c)$-quasigeodesic for arbitrary integer $m$.

A broken line $p = [X_0, X_1, \ldots, X_n]$ is a path obtained as a consequent concatenation of geodesic segments $[X_{i-1}, X_i]$, $i = 1, 2, \ldots, n$. Later, in this paper, we will use the following fact concerning broken lines in a hyperbolic space:

**Lemma 1.3.** ([3, Lemma 21]) Let $p = [X_0, X_1, \ldots, X_n]$ be a broken line in $\Gamma(G, A)$ such that $|[X_{i-1}, X_i]| > C_1 \quad \forall \ i = 1, \ldots, n$, and $(X_{i-1}|X_{i+1})X_i \leq C_0 \quad \forall \ i = 1, \ldots, n-1$, where $C_0 \geq 14\delta$, $C_1 > 12(C_0 + \delta)$. Then $p$ is contained in the closed $2C_0$-neighborhood $O_{2C_0}([X_0, X_n])$ of the geodesic segment $[X_0, X_n]$.

Suppose $H = \langle X \rangle$ is a subgroup of $G$ with a finite symmetrized generating set $X$. If $h \in H$, then by $|h|_G$ and $|h|_H$ we will denote the lengths of the element $h$ in $A$ and $X$ respectively. The distortion function $D_H : \mathbb{N} \to \mathbb{N}$ of $H$ in $G$ is defined by $D_H(n) = \max\{|h|_H \mid h \in H, |h|_G \leq n\}$.

If $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ are two functions then we write $\alpha \preceq \beta$ if $\exists K_1, K_2 > 0: \alpha(n) \leq K_1\beta(K_2n)$. $\alpha(n)$ and $\beta(n)$ are said to be equivalent if $\alpha \preceq \beta$ and $\beta \preceq \alpha$.

Evidently, the function $D_H$ does not depend (up to this equivalence) on the choice of finite generating sets $A$ of $G$ and $X$ of $H$. One can also notice that $D_H(n)$ is always at least linear (provided that $H$ is infinite). If $D_H$ is equivalent to linear, $H$ is called *undistorted*.

**Lemma 1.4.** ([2; 3.8], [7; 10.4.2]) A quasiconvex subgroup $H$ of a hyperbolic group $G$ is finitely generated.

**Remark 1.** From the proof of this statement it also follows that $D_H$ is equivalent to linear for a quasiconvex subgroup $H$.

Indeed, it was observed in [2] that if $H$ is $\varepsilon$-quasiconvex, it is generated by
finitely many elements \( x_i, i = 1, \ldots, s \), such that \( |x_i|_G \leq 2\varepsilon + 1 \forall i \), and \( \forall h \in H \),
\[ h = a_1 \cdots a_r, \quad a_j \in A, \quad \text{hence } \exists i_1, \ldots, i_r \in \{1, 2, \ldots, s\}: h = x_{i_1}x_{i_2} \cdots x_{i_r}. \]

The proof of corollary 2 is based on

**Lemma 1.5.** ([1; Prop. 3]) Let \( G \) be a group generated by a finite set \( A \). Let \( A, B \) be subgroups of \( G \) quasiconvex with respect to \( A \). Then \( A \cap B \) is quasiconvex with respect to \( A \).

We will use the following notion in this paper :

**Definition :** let \( H = \langle X \rangle \leq G = \langle A \rangle, \quad card(X) < \infty, \quad card(A) < \infty \). A path \( P \) in \( \Gamma(G, A) \) will be called \( H \)-geodesic (or just \( H \)-path) if : a) \( P \) is labelled by the word \( a_{i_1} \cdots a_{i_k} \cdots a_{i_s} \) corresponding to an element \( \text{elem}(P) = x \in H \), where \( a_{i_j} \in A \); b) \( a_{i_1} \cdots a_{i_k} \) is a shortest word for generator \( x_j \in X \) (i.e. \( |x_j|_G = k_j \), \( j = 1, \ldots, s \); c) \( x = x_1 \cdots x_s \) in \( H \), \( |x|_H = s \).

I.e. \( P \) is a broken line in \( \Gamma(G, A) \) with segments corresponding to shortest representations of generators of \( H \) by means of \( A \).

**Lemma 1.6.** (see also [10; Lemma 2.4]) Let \( H \) be a (finitely generated) subgroup of a \( \delta \)-hyperbolic group \( G \). Then \( H \) is quasiconvex iff \( H \) is undistorted in \( G \).

**Proof.** The necessity is given by remark 1.

To prove the sufficiency, suppose \( H = \langle X \rangle, \quad card(X) < \infty, \quad D_H(n) \leq cn, \forall n \in \mathbb{N}, \quad c > 0 \). For arbitrary two vertices \( x, y \in H \) there is a \( H \)-path \( q \) connecting them in \( \Gamma(G, A) \). Let \( p \) be any its subpath. By definition, there exists a subpath \( p' \) of \( q \) such that \( p'_-, p'_+ \in H \), subpaths of \( q \) from \( p_- \) to \( p'_- \) and from \( p_+ \) to \( p'_+ \) are geodesic, and \( d(p_-, p'_-) \leq \varepsilon/2, d(p_+, p'_+) \leq \varepsilon/2, \) where \( \varepsilon = \max\{|h|_G : h \in X\} < \infty \). In particular, \( p' \) is also \( H \)-geodesic. Using the property c) from the definition of a \( H \)-path we obtain

\[ ||p'|| \leq \varepsilon \cdot |\text{elem}(p')|_H \leq \varepsilon \cdot c \cdot d(p'_-, p'_+). \]

Therefore, \( ||p'|| = ||p'|| + \varepsilon \leq \varepsilon \cdot c \cdot d(p'_-, p'_+) + \varepsilon \leq \varepsilon \cdot c \cdot d(p_-, p_+) + \varepsilon^2 c + \varepsilon, \) which shows that \( q \) is \((\frac{1}{2}, \varepsilon + \frac{1}{2})\)-quasigeodesic. By lemma 1.1 \( \exists N = N(\varepsilon, c) \) such that any geodesic path between \( x \) and \( y \) belongs to the closed \( N \)-neighborhood \( O_N(q) \) but \( q \subset O_{N/2}(H) \) in the Cayley graph of \( G \). Hence, \( H \) is quasiconvex with the constant \( (N + \varepsilon/2) \), and the lemma is proved. □

During this proof we showed

**Remark 2.** If \( H \) is a quasiconvex subgroup of a hyperbolic group \( G \) then any \( H \)-path is \((\lambda, c)\)-quasigeodesic for some \( \lambda, c \) depending only on the subgroup \( H \).

Let the words \( W_1, \ldots, W_l \) represent elements \( w_1, \ldots, w_l \) of infinite order in a hyperbolic group \( G \). For a fixed constant \( K \) consider the set \( S_M = S(W_1, \ldots, W_l; K, M) \) of words

\[ W = X_0 W_1^\alpha_1 X_1 W_2^\alpha_2 X_2 \cdots W_l^\alpha_l X_l \]
where \(|X_i| \leq K\) for \(i = 0, 1, \ldots, l\), \(|\alpha_2|, \ldots, |\alpha_{l-1}| \geq M\), and the element of \(G\) represented by \(X_i^{-1}W_iX_i\) does not belong to the maximal elementary subgroup \(E(w_{i+1}) \leq G\) containing \(w_{i+1}\) for \(i = 1, \ldots, l - 1\).

**Lemma 1.7.** ([5; Lemma 2.4]) There exist constants \(\lambda > 0\), \(c \geq 0\) and \(M > 0\) (depending on \(K, W_1, \ldots, W_l\)) such that any path in \(\Gamma(G, A)\) labelled by an arbitrary word \(W \in S_M\) is \((\lambda, c)\)-quasigeodesic.

**Lemma 1.8.** Suppose \(l \in \mathbb{N}, K > 0\), and \(w_1, \ldots, w_l \in G\) are elements of infinite order. Then there are \(\lambda > 0\), \(c \geq 0\) and \(M > 0\) (depending on \(K, w_1, \ldots, w_l\)) such that for arbitrary \(x_0, x_1, \ldots, x_l \in G\), \(|x_i|_G \leq K, i = 0, \ldots, l\), with conditions \(w_i \notin x_iE(w_{i+1})x_i^{-1} \forall i \in \{1, \ldots, l-1\}\), and any \(\alpha_i \in \mathbb{Z}, |\alpha_i| \geq M, i = 2, \ldots, l - 1\), the element

\[
w = x_0w_1^{\alpha_1}x_1w_2^{\alpha_2}x_2 \cdots w_l^{\alpha_l}x_l \in G
\]

satisfies \(|w|_G \geq \lambda|\alpha_1| - c\).

**Proof.** As follows from Lemma 1.7 and the definition of a \((\lambda, c)\)-quasigeodesic path, one has the following inequality:

\[
|w|_G \geq \lambda \cdot \left( |x_0|_G + \sum_{i=1}^{l} (|\alpha_i||w_i|_G + |x_i|_G) \right) - c \geq \lambda \cdot |\alpha_1||w_1|_G - c \geq \lambda|\alpha_1| - c.
\]

\[\square\]

2. Quasiconvex sets and their products

**Remark 3.** Any finite subset of \(G\) is \(d\)-quasiconvex (where \(d\) is the diameter of this set).

**Remark 4.** Let \(Q \subseteq G\) be \(\varepsilon\)-quasiconvex, \(g \in G\). Then (a) the left shift \(gQ = \{gx \mid x \in Q\}\) is quasiconvex with the same constant; (b) the right shift \(Qg = \{xg \mid x \in Q\}\) is quasiconvex (possibly, with a different quasiconvexity constant).

(a) holds because the metric \(d(\cdot, \cdot)\) is left-invariant.

\(x, y \in Q\) if and only if \(xg, yg \in Qg\). By remark 0

\[
[xg, yg] \subset O_{2\delta}([x, xg] \cup [x, y] \cup [y, yg]) \subset O_{2\delta+|g|_G}([x, y]) \subset O_{2\delta+|g|_G+\varepsilon}(Q) \subset O_{2\delta+2|g|_G+\varepsilon}(Qg)
\]

therefore (b) is true.

Therefore, a left coset of a quasiconvex subgroup and a conjugate subgroup to it are quasiconvex (in a hyperbolic group).

**Lemma 2.1.** (see also [11; Prop. 3.14]) A finite union of quasiconvex sets in a hyperbolic group \(G\) is quasiconvex.

**Proof.** It is enough to prove that if \(A, B \subset G\) are \(\varepsilon_i\)-quasiconvex, \(i = 1, 2\), respectively, then \(C = A \cup B\) is quasiconvex.
Hence \( d \) (see Figure 1).

By remark 0 we have \( [x, y] \subset O_{2\delta}(x, a \cup [a, b] \cup [b, y]). \) After denoting \( d(a, b) = 2\eta \) we obtain \( [x, a] \subset O_{\epsilon_1}(A), [b, y] \subset O_{\epsilon_2}(B), [a, b] \subset O_\eta(A \cup B). \) Hence \( [x, a] \cup [a, b] \cup [b, y] \subset O_{max(\epsilon_1, \epsilon_2, \eta)}(C), \) consequently, \( [x, y] \subset O_{max(\epsilon_1, \epsilon_2, \eta) + 2\delta}(C), \) and the lemma is proved. □

**Proof of Proposition 1.** Assume \( n = 2 \) (for \( n > 2 \) the statement will follow by induction).

So, let \( A, B \) be \( \epsilon_i \)-quasiconvex subsets of \( G \) respectively, \( i = 1, 2. \)

Consider arbitrary \( a_1b_1a_2b_2 \in AB, a_i \in A, b_i \in B_i, i = 1, 2, \) and fix an element \( b \in B, |b|_G = \eta. \) Then, since the triangles are \( \delta \)-slim,

\[
[b_1, 1_G] \subset O_\delta([b, 1_G] \cup [b, b_1]) \subset O_{3\delta + \eta}([b, b_1]) \subset O_{3\delta + \eta + \epsilon_2}(B).
\]

Denoting \( \epsilon_3 = \delta + \eta + \epsilon_2, \) one obtains \( [b_1, 1_G] \subset O_{\epsilon_3}(B) \) and, similarly, \( [b_2, 1_G] \subset O_{\epsilon_3}(B). \) Therefore, \( [a_1b_1, a_1] \subset O_{\epsilon_3}(a_1B), [a_2b_2, a_2] \subset O_{\epsilon_3}(a_2B). \)

Also, observe that \( \forall a \in A \quad d(a, ab) = |b|_G = \eta, \) i.e. \( A \subset O_\eta(AB) \subset O_\eta(AB), \) hence \( [a_1, a_2] \subset O_{\epsilon_1}(A) \subset O_{\epsilon_1 + \eta}(AB). \) And using remark 0 we achieve

\[
[a_1b_1, a_2b_2] \subset O_{3\delta}([a_1b_1, a_1] \cup [a_1, a_2] \cup [a_2b_2, a_2]) \subset O_{3\delta + max(\epsilon_1, \epsilon_2, \eta)}(AB),
\]

q.e.d. □

**Corollary 1.** In a hyperbolic group \( G \) every quasiconvex product is a quasiconvex set.

This follows directly from the proposition 1 and part (a) of remark 4.

### 3. Intersections of quasiconvex products

Set a partial order on \( \mathbb{Z}^2: (a_1, b_1) \leq (a_2, b_2) \) if \( a_1 \leq a_2 \) and \( b_1 \leq b_2. \) As usual, \((a_1, b_1) < (a_2, b_2)\) if \((a_1, b_1) \leq (a_2, b_2)\) and \((a_1, b_1) \neq (a_2, b_2).\)
Definition: a finite sequence \(((i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t))\) of pairs of positive integers will be called increasing if it is empty \((t = 0)\) or \((t > 0)\)
\((i_q, j_q) < (i_{q+1}, j_{q+1}) \forall q = 1, 2, \ldots, t - 1.\) This sequence will also be called \((n, m)\)-increasing \((n, m \in \mathbb{N})\) if \(1 \leq i_q \leq n, 1 \leq j_q \leq m\) for all \(q \in \{1, 2, \ldots, t\} \).

Note that the length \(t\) of an \((n, m)\)-increasing sequence never exceeds \((n + m - 1)\).

Instead of proving theorem 1 we will prove

**Theorem 1’**: Suppose \(G_1, \ldots, G_n, H_1, \ldots, H_m\) are quasiconvex subgroups of the group \(G, n, m \in \mathbb{N}; f, e \in G.\) Then there exist numbers \(r, t \in \mathbb{N} \cup \{0\}\) and \(f_t, \alpha_{t_k}, \beta_{t_k} \in G, k = 1, 2, \ldots, t\) (for every fixed \(l),
\(l = 1, 2, \ldots, r,\) such that

\[
\begin{align*}
(1) & \quad fG_1G_2 \cdots G_n \cap eH_1H_2 \cdots H_m = \bigcup_{l=1}^r f_l S_l
\end{align*}
\]

where for each \(l, t = t_l,\) there are indices \(1 \leq i_1 \leq i_2 \leq \ldots \leq i_t \leq n, 1 \leq j_1 \leq \ldots \leq j_t \leq m ;
\]

\[
\begin{align*}
(2) & \quad S_l = (G_{i_1}^{\alpha_{i_1}} \cap H_{j_1}^{\beta_{i_1}}) \cdots (G_{i_t}^{\alpha_{i_t}} \cap H_{j_t}^{\beta_{i_t}})
\end{align*}
\]

and the sequence \(((i_1, j_1), \ldots, (i_t, j_t))\) is \((n, m)\)-increasing.

For our convenience, let us also introduce the following

**Definition**: the unions as in the right-hand side of (1) will be called special \((n, m)\)-products.

**Lemma 3.1**: Consider a geodesic polygon \(X_0X_1 \ldots X_n\) in the Cayley graph \(\Gamma(G, A), n \geq 2.\) Then there are points \(X_i \in [X_i; X_{i+1}], i = 1, 2, \ldots, n - 1,\) such that setting \(X_0 = X_0, X_n = X_n,\) we have \((X_i-1; X_{i+1}) \leq \delta\) and \(d(X_i, [X_{i-1}; X_i]) \leq \delta,\) for \(1 \leq i \leq n - 1.\)

**Proof** of the lemma. First, we recursively construct the vertices \(\bar{X}_i.\) Let \(X_1 \in [X_1; X_2], \bar{U}_1 \in [X_0; X_1]\) be the "special" points of the geodesic triangle \(X_0X_1X_2,\) i.e. \(|X_1 - X_1| = |X_1 - \bar{U}_1| = (X_0|X_2)X_1.\) Now, if \(X_{i-1}\) is constructed, denote by \(\bar{X}_i \in [X_i; X_{i+1}], \bar{U}_i \in [X_{i-1}; X_i]\) the special points of triangle \(X_{i-1}X_iX_{i+1}((X_i - \bar{X}_i) = |X_i - \bar{U}_i| = (X_{i-1}|X_{i+1})X_i.\) (Figure 2)

Then \(d(\bar{X}_i, [X_{i-1}; X_i]) \leq |X_i - \bar{U}_i| \leq \delta, \forall i = 1, 2, \ldots, n - 1.\)

For the other part of the claim we will use induction on \(n.\)

\(\text{If } n = 2,\) then
\[
(X_0|X_2)_{X_1} \overset{\text{def}}{=} \frac{1}{2}(|X_0 - \bar{X}_1| + |X_2 - \bar{X}_1| - |X_0 - X_2|) \leq \frac{1}{2}(|X_0 - \bar{U}_1| + |\bar{U}_1 - X_1| + |X_2 - X_1| - |X_0 - X_2|) = \frac{1}{2} |\bar{U}_1 - X_1| \leq \frac{\delta}{2} \leq \delta
\]

Suppose, now, that \(n \geq 3.\) Let us evaluate the Gromov product \((X_0|X_2)_{X_1}.\)
\[
(X_0|X_2)_{X_1} = \frac{1}{2}(|X_0 - \bar{X}_1| + |X_2 - \bar{X}_1| - |X_0 - X_2|)
\]
The lemma is proved.

Now we notice that $\bar{\triangle} X_0 | X \in i = 1$ and ending at $0$ and a path corresponding to $X_0 - X_2$.

Consider a pair of (non-geodesic) polygons associated with $X_0$. Choose an arbitrary $x = g_1 g_2 \ldots g_n = e h_1 h_2 \ldots h_m$ where $g_i \in G_i$, $h_j \in H_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Choose an arbitrary $x \in T$, $x = f g_1 g_2 \ldots g_n = e h_1 h_2 \ldots h_m$ where $g_i \in G_i$, $h_j \in H_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Figure 2

$$|\bar{X}_2 - \bar{X}_1| = |\bar{X}_1 - \bar{U}_2| + |\bar{X}_2 - \bar{U}_2| \leq |\bar{X}_1 - \bar{U}_2| + \delta, |X_0 - \bar{X}_1| \leq |X_0 - \bar{U}_1| + \delta.$$

$|X_0 - \bar{U}_1| + |X_2 - \bar{X}_1| = |X_0 - X_2|$ by the definition of special points of the triangle $X_0 X_2$. Therefore

$$|X_0 - \bar{X}_1| + |\bar{X}_2 - \bar{X}_1| \leq |X_0 - \bar{U}_1| + |\bar{X}_1 - \bar{U}_2| + 2\delta =$$

$$= |X_0 - \bar{U}_1| + (|X_2 - \bar{X}_1| - |X_2 - \bar{U}_2|) + 2\delta = |X_0 - X_2| - |X_2 - \bar{X}_2| + 2\delta.$$

Now we notice that $|X_0 - X_2| - |X_2 - \bar{X}_2| \leq |X_0 - X_2|$ and obtain:

$$(\bar{X}_0|X_2)_X \leq \frac{1}{2}(|X_0 - X_2| + 2\delta - |X_0 - \bar{X}_2|) = \delta.$$

To the $n$-gon $\bar{X}_1 X_2 \ldots X_n$ we can apply the induction hypothesis.

The lemma is proved. $\square$

Proof of theorem 1'. Define $T = f G_1 G_2 \ldots G_n \cap e H_1 H_2 \ldots H_m$.

Fix some finite generating sets in every $G_i, H_j$ and denote

$$K_1 = \max \{1 : |f| G : |\text{generators of } G_i| G : i = 1, 2, \ldots, n\} < \infty,$$

$$K_2 = \max \{1 : |e| G : |\text{generators of } H_j| G : j = 1, 2, \ldots, m\} < \infty.$$

Induction on $(n + m)$.

If $n = 0$ or $m = 0$, then $\text{card}(T) \leq 1$ and the statement is true.

Let $n \geq 1$ and $m \geq 1$, $n + m \geq 2$.

Consider a pair of (non-geodesic) polygons associated with $x$ in $\Gamma(G, A)$:

$P = X_0 p_1 X_1 p_2 \ldots p_n X_0 p_0$ and $Q = Y_0 q_1 Y_1 q_2 \ldots q_m Y_0 q_0$ with vertices $X_0 = Y_0 = 1_G$, $X_i = f g_1 \ldots g_i \in G$, $Y_j = e h_1 \ldots h_j \in G$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and edges $p_0, p_1, \ldots, p_n$, $q_0, q_1, \ldots, q_m$. Such that $p_1$, starting at $X_0$ and ending at $X_1$, is a union of a geodesic path corresponding to $f$ and a $G_i$-path corresponding to $g_i$, $p_i$ is a $G_i$-path labelled by a word representing the element $g_i$ in $G$ from $X_{i-1}$ to $X_i$, $i = 2, \ldots, n$; $p_0$ is the geodesic path $[X_n, X_0]$ (Figure 3).
By construction, there are constants \( \lambda_i, c_i \) (not depending on \( x \in T \)) such that the segments \( p_i, i = 1, \ldots, n \) are \( (\lambda_i, c_i) \)-quasigeodesics respectively.

Similarly one constructs the paths \( q_j, j = 0, \ldots, m \).

Therefore the geodesic path \( p_0 = [X_0; X_n] = [Y_0; Y_m] = q_0 \) will be labelled by a word representing \( x \) in our Cayley graph.

We will also consider the geodesic polygons \( X_0X_1 \ldots X_n \) and \( Y_0Y_1 \ldots Y_m \) with same vertices as \( P \) and \( Q \) respectively.

Recalling the property of quasigeodesic paths, for each \( i = 1, \ldots, n \)

\[
\text{[} j = 1, \ldots, m \text{]} \quad \text{we obtain a constant } N_i > 0 \quad [M_j > 0] \quad \text{(not depending on the element } x \in T \text{)} \quad \text{such that}
\]

\[
(i) \quad [X_{i-1}, X_i] \subset O_{N_i}(p_i) \quad [\quad [Y_{j-1}, Y_j] \subset O_{M_j}(q_j) \quad].
\]

Define \( L = \max\{N_1, \ldots, N_n, M_1, \ldots, M_m\} \).

a) Suppose \( n, m \geq 2 \) (after considering this case, we will see the other cases, when \( n = 1 \) or \( m = 1 \) are easier).

Let’s focus our attention on the polygons \( X_0 \ldots X_n \) and \( P \) since everything for the two others can be done analogously.

One can apply lemma 3.1 and obtain \( \bar{X}_1 \in [X_i; X_{i+1}], i = 1, \ldots, n-1 \), such that \( (\bar{X}_{i-1}\bar{X}_{i+1})_{\bar{X}_i} \leq \delta, i = 1, \ldots, n-1, (\bar{X}_0 = X_0 = 1_G, \bar{X}_n = X_n = x) \), along with \( \bar{U}_i \in [\bar{X}_{i-1}; \bar{X}_i], |\bar{X}_i - \bar{U}_i| \leq \delta, i = 1, \ldots, n-1 \).

Now, using (i), we obtain points \( \bar{X}_i \in p_{i+1}, i = 1, \ldots, n-1 \), satisfying \( d(\bar{X}_i; X_i) \leq L \quad (X_0 = X_0 = 1_G, X_n = X_n = x) \) and \( \bar{U}_1 \in p_1 \) satisfying \( d(\bar{U}_1, \bar{U}_1) \leq L \). For each \( i \in \{1, 2, \ldots, n-2\} \) the triangle \( \bar{X}_i \bar{X}_{i+1} \bar{X}_{i+1} \) is \( \delta \)-slim, hence \( \exists U_{i+1} \in [X_i; X_{i+1}] : d(U_{i+1} \bar{U}_{i+1}) \leq L + \delta \). The segment of \( p_{i+1} \) between \( \bar{X}_i \) and \( X_{i+1} \) is quasigeodesic with the same constants as \( p_{i+1} \), therefore there is a point \( \bar{U}_{i+1} \in p_{i+1} \) between \( \bar{X}_i \) and \( X_{i+1} \) such that \( d(U_{i+1} \bar{U}_{i+1}) \leq L \), and, consequently, \( d(U_{i+1} \bar{U}_{i+1}) \leq 2L + \delta \) (see Figure 4).
Let $\alpha_t$ denote the segment of $p_t$ from $\bar{X}_{t-1}$ to $X_t$, $t = 2, \ldots, n$, and $\beta_s$ – the subpath of $p_s$ from $X_{s-1}$ to $U_s$, $s = 1, \ldots, n - 1$. Shifting the points $\bar{X}_i, \bar{U}_i, i = 1, \ldots, n - 1$, along their sides of $P$ (so that $\bar{U}_i$ still stays between $\bar{X}_{i-1}$ and $X_i$ on $p_i$) by distances at most $K_1$, we can achieve $\text{elem}(\beta_1) \in fG_1$ (i.e., $\text{lab}(\beta_1)$ represents an element of $fG_1$), $\text{elem}(\alpha_t) \in G_{t+1}$, $\text{elem}(\beta_s) \in G_s$, $t = 2, \ldots, n$, $s = 1, 2, \ldots, n - 1$. And after this, setting, for brevity, $K = \max\{K_1 + \frac{3}{2}L, K_2 + \frac{3}{2}L\}$, one obtains

$$(X_{i-1}, X_{i+1}) \leq \delta + 3K_1 + 3L \leq \delta + 3K \leq 14\delta + 3K \overset{\text{def}}{=} C_0,$$

$$|\bar{X}_i - \bar{U}_i| \leq \delta + 2K_1 + 3L + \delta \leq 2\delta + 2K, \quad i = 1, \ldots, n - 1.$$ 

Let $\text{elem}(\beta_1) = f\bar{g}_1$, $\text{elem}(\beta_i) = \bar{g}_i$, $i = 1, \ldots, n - 1$, $\text{elem}(\alpha_n) = \bar{g}_n$, where $\bar{g}_k \in G_k$, $k = 1, 2, \ldots, n$. $\text{elem}([U_i; X_i]) = u_i \in G, G_{i+1}$, $i = 1, \ldots, n - 1$. 

Then $|u_i| \leq 2\delta + 2K$, and there are only finitely many of possible $u_i$'s for every $i \in \{1, 2, \ldots, n - 1\}$. Hence, we achieved the following representation for $x$:

$$x \overset{\alpha}{=} f\bar{g}_1u_1\bar{g}_2u_2 \cdot \cdot \cdot \bar{g}_{n-1}u_{n-1}\bar{g}_n.$$ 

Similarly, one can obtain

$$x \overset{\alpha}{=} e\bar{h}_1v_1\bar{h}_2v_2 \cdot \cdot \cdot \bar{h}_{m-1}v_{m-1}\bar{h}_m.$$ 

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where $\bar{h}_j \in H_j$, $j = 1, \ldots, m$; $v_j \in H_jH_{j+1}$ and $|v_j|_G \leq 2\delta + 2K$ for every $j = 1, 2, \ldots, m - 1$ (see Figure 5).

$\mathcal{U}_i \overset{\text{def}}{=} \{ u \in G_iG_{i+1} : |u|_G \leq 2\delta + 2K \} \subset G$, $i = 1, \ldots, n - 1$, card($\mathcal{U}_i$) $< \infty$, $\forall i = 1, \ldots, n - 1$. For convenience, $\mathcal{U}_0 = \mathcal{U}_n = G_0 = G_{n+1} \overset{\text{def}}{=} \{1_G\}$.

Analogously, define $\mathcal{V}_j \subset H_jH_{j+1}$, $j = 1, \ldots, m - 1$, and again,

$\mathcal{V}_0 = V_m = H_0 = H_{m+1} \overset{\text{def}}{=} \{1_G\}$.

Set $D = 14(\delta + C_0) + 3K = \text{const}$, and $\mathcal{L} = \{ g \in G : |g|_G \leq D \}$. At last, we denote

$$\Delta_i = \mathcal{U}_{i-1} \cdot (\mathcal{L} \cap G_i) \cdot \mathcal{U}_i \subset G_{i-1}G_iG_{i+1} \subset G, \ i = 1, 2, \ldots, n,$$

$$\Theta_i = \mathcal{V}_{j-1} \cdot (\mathcal{L} \cap H_j) \cdot \mathcal{V}_j \subset H_{j-1}H_jH_{j+1} \subset G, \ j = 1, 2, \ldots, m.$$

By construction, card($\Delta_i$) $< \infty$, card($\Theta_i$) $< \infty$, $\forall i, j$.

Take any $i \in \{1, 2, \ldots, n\}$ and consider the intersection

$$T \supseteq fG_1G_2 \cdot \ldots \cdot G_{i-1} \Delta_i G_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m =$$

$$= \bigcup_{g \in \Delta_i} [fG_1G_2 \cdot \ldots \cdot G_{i-1}gG_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m] =$$

$$= \bigcup_{g \in \Delta_i} [fg(g^{-1}G_1g)(g^{-1}G_2g) \cdot \ldots \cdot (g^{-1}G_{i-1}g)G_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m].$$

Because of remark 4, one can apply the induction hypothesis to the last expression and conclude that it is a (finite) "special" union. Hence,

$$(3) \quad T_1 \overset{\text{def}}{=} \bigcup_{i=1}^n [fG_1G_2 \cdot \ldots \cdot G_{i-1} \Delta_i G_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m]$$

is also a finite special union.

Because of the symmetry, we parallely showed that

$$(4) \quad T_2 \overset{\text{def}}{=} \bigcup_{j=1}^m [fG_1 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{j-1} \Theta_j H_{j+1} \cdot \ldots \cdot H_m]$$

is a finite "special" union.

We have just proved that there exist $r_1 \in \mathbb{N} \cup \{0\}$, $f_1 \in G$ and increasing $(n, m)$-products $S_l$, $l = 1, 2, \ldots, r_1$, such that

$$T_1 \cup T_2 = \bigcup_{l=1}^{r_1} f_1 S_l \subset T.$$

$$T = T_1 \cup T_2 \cup T_3, \text{ where } T_3 \overset{\text{def}}{=} T \setminus (T_1 \cup T_2).$$

Now, let’s consider the case $x \in T_3$. It means that in representations ($\ast$) and ($\ast\ast$) for $x$, $|\bar{y}_i|_G > D$,

$|h_j|_G > D$, for $D = 14(\delta + C_0) + 3K$ and $\forall i = 1, \ldots, n, \forall j = 1, \ldots, m$.

Therefore, returning to the pair of polygons we constructed, one will have:
We also possess the following inequalities: \( |\bar{X}_n - \bar{X}_1| \geq |\bar{y}_1| \sigma - |f| \sigma - |u_1| \sigma > 14(\delta + C_0) + 3K - K - 2\delta - 2K = 12(\delta + C_0) + 2C_0 \).\(^{\text{def}}\) \( C_1, |||X_{n-1}; X_1||| \geq |\bar{y}_1| \sigma - |u_1| \sigma > 14(\delta + C_0) + 3K - 2\delta - 2K > C_1, i = 2, \ldots, n - 1, |||X_{n-1}; X_1||| = |\bar{y}_1| \sigma > C_1.\)

We also possess the following inequalities: \( (\bar{X}_{n-1}; \bar{X}_1)_{X_i} \subset C_0, i = 1, \ldots, n - 1, \)

\( C_0 \geq 14C, C_1 > 12(\delta + C_0).\)

By lemma 1.3, the broken line \([\bar{X}_0; \bar{X}_1; \ldots; \bar{X}_n]\) is contained in the closed \( C = 2C_0\)-neighborhood of the geodesic segment \([\bar{X}_0; \bar{X}_n]\). In particular,

\[
(5) \quad d(X_{n-1}, [X_0; X_n]) \leq C.
\]

A similar argument shows that \( d(Y_{m-1}, [Y_0; Y_m]) = C, \) and, since \([\bar{X}_0; \bar{X}_n] = [X_0; X_n] = [Y_0; Y_m] = [Y_0; Y_m],\) one has

\[
(6) \quad d(Y_{m-1}, [X_0; X_n]) \leq C.
\]

**b)** In the previous case we assumed that \( n, m \geq 2 \) and we needed quite a long argument to prove (5) and (6). On the other hand, if, for example, \( n = 1, \) then \( X_0 = X_{n-1} \) and (5) is trivial.

Because of (5) and (6) one can choose \( W, Z \in [X_0; X_n] \) with the properties \( |W - X_{n-1}| \leq C, |Z - Y_{m-1}| \leq C.\)

The first possibility is, when the point \( W \) on \([X_0; X_n]\) lies between \( Z \) and \( X_n, \) i.e. \( W \in [Z; X_n].\)

Then, since triangles are \( \delta \)-thin in the hyperbolic space \( \Gamma(G, A),\) \( d(W, [Y_{m-1}; X_1]) \leq C + \delta.\) Hence \( d(X_{n-1}, [Y_{m-1}; X_1]) \leq 2C + \delta.\) Consequently, because \( q_m \) is quasigeodesic, there exists a point \( R \) on the subpath \( \gamma \) of \( q_m \) from \( Y_{m-1} \) to \( Y_m \) such that \( d(X_{n-1}, R) \leq 2C + \delta + K + M_m\) ( \( M_m \) is the same as in \( \gamma \) ) and \( \text{elem}([R; Y_m]) = \text{elem}(\gamma) = h_m \in H_m.\)

Define \( \Omega = \{ g \in G_n H_m : |g| \leq 2C + \delta + K + M_m \}.\) Therefore \( \text{card}(\Omega) < \infty \) and \( \text{elem}(\{X_{n-1}; R\}) \in \Omega.\)

For each element \( g \in \Omega \) take a pair \( g' \in G_n, h' \in H_m \) such that \( g = g'h'.\) By \( G' \subset G_n \) denote the set of all elements \( g' \) which we have chosen, by \( H' \subset H_m - \text{set of all } h'\)’s.

\[
x = f g_1 u_1 \ldots u_{n-1} g_n = e h_1 v_1 \ldots v_{m-1} h_m.
\]
From the triangle $X_nXnR$ we obtain $\bar{g}_n\hat{h}_m^{-1} = g'h' \in \Omega$, $g' \in G'$, $h' \in H'$. Thus $(g')^{-1}\bar{g}_n = h'\hat{h}_m \in G_n \cap H_m$.

$$x \in fG_1G_2 \cdot \ldots \cdot G_{n-1} \cdot u_{n-1}g' \cdot ((g')^{-1}\bar{g}_n) \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}H_m \subset$$

$$\subset fG_1G_2 \cdot \ldots \cdot G_n \cdot U_{n-1}G' \cdot (G_n \cap H_m) \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}H_m \subset T \cdot$$

Denote $I = U_{n-1} \cdot G' \subset G_n \cap G_n$ - a finite subset of $G$. Then

$$x \in fG_1G_2 \cdot \ldots \cdot G_{n-1}I \cdot (G_n \cap H_m) \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}H_m =$$

$$= [fG_1G_2 \cdot \ldots \cdot G_{n-1}I \cap eH_1H_2 \cdot \ldots \cdot H_m] \cdot (G_n \cap H_m) \subset T \cdot$$

The second possibility, when $Z \in [W; X_n]$ is considered analogously, and, in this case, one obtains a finite subset $J \subset H_{m-1}H_m$ such that

$$x \in [fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}J] \cdot (G_n \cap H_m) \subset T \cdot$$

Therefore, we showed that $T_3 \subseteq [T_3^1 \cup T_3^2] \cdot (G_n \cap H_m) \subset T$ where

(7) $$T_3^1 \overset{def}{=} fG_1G_2 \cdot \ldots \cdot G_{n-1}I \cap eH_1H_2 \cdot \ldots \cdot H_m \cdot$$

(8) $$T_3^2 \overset{def}{=} fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}J \cdot$$

Combining the formulas (3),(4),(7),(8) and the property that if $H \leq G$ and $a \in H$ then $aH = Ha = H$, we obtain the following

**Lemma 3.2.** In notations of the theorem 1

$$fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_m = T_1 \cup T_2 \cup [T_3^1 \cup T_3^2] \cdot (G_n \cap H_m) \cdot$$

where

$$T_1 = \bigcup_{i=1}^{n} \left(fG_1G_2 \cdot \ldots \cdot G_{i-1} \Delta_i \Delta_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m \right) \cdot$$

$$T_2 = \bigcup_{j=1}^{m} \left(fG_1 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{j-1} \Theta_i \Theta_{j+1} \cdot \ldots \cdot H_m \right) \cdot$$

$$T_3^1 = fG_1G_2 \cdot \ldots \cdot G_{n-1}I \cap eH_1H_2 \cdot \ldots \cdot H_m \cdot$$

$$T_3^2 = fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}J \cdot$$

for some finite subsets $\Delta_i \subset G_i, \Theta_j \subset H_j$, $I \subset G_n, J \subset H_m$, $1 \leq i \leq n$, $1 \leq j \leq m$.

Now, to finish the proof of the theorem, we apply the inductive hypothesis:

$$T_3 \subseteq \bigcup_{g \in I} \left[fG_1G_2 \cdot \ldots \cdot G_{n-1}g \cap eH_1H_2 \cdot \ldots \cdot H_m \right] \cdot (G_n \cap H_m) \cup$$

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∪ \bigcup_{h \in J} [fG_1 G_2 \ldots \cdot G_n \cap eH_1 H_2 \ldots \cdot H_{m-1} h] \cdot (G_n \cap H_m) = \\
= \bigcup_{g \in I} \left[ fG_1 G_2 \ldots \cdot G_{n-1} \cap eH_1 H_2 \ldots \cdot H_m \right] \cdot (G_n \cap H_m) \cup \\
\bigcup_{h \in J} [fG_1 G_2 \ldots \cdot G_n \cap eH_1 H_2 \ldots \cdot H_{m-1} \cdot eH_1 H_2 \ldots \cdot H_{m-1} h] \cdot (G_n \cap H_m) = \\
= \left( \bigcup_{g \in I} \left[ \bigcup_{k=1}^{\tilde{r}} \tilde{f}_k \tilde{S}_k \right] \cup \bigcup_{h \in J} \left[ \bigcup_{q=1}^{\hat{r}} \hat{f}_q \hat{S}_q \right] \right) \cdot (G_n \cap H_m) = \\
= \bigcup_{l=r_1+1}^{r} f_l S_l \subseteq T .

Here \( \tilde{r}, \hat{r}, r \in \mathbb{N} \cup \{0\} \), \( r \geq r_1, \tilde{f}_k, \hat{f}_q, f_l \in G \); \( \tilde{S}_k \) is an \((n-1,m)\)-increasing product,
\( \hat{S}_q \) is an \((n,m-1)\)-increasing product and \( S_l \) is an \((n,m)\)-increasing product;
\( k = 1, \ldots, \tilde{r} \); \( q = 1, \ldots, \hat{r} \); \( l = r_1 + 1, \ldots, r \).

Hence,

\[ T = T_1 \cup T_2 \cup T_3 \subseteq \bigcup_{l=1}^{r} f_l S_l \subseteq T , \]

and, thus

\[ T = \bigcup_{l=1}^{r} f_l S_l . \]

So, the theorem is proved. \( \square \)

Proof of Corollary 2. Observe that arbitrary quasiconvex product \( f_1 G_1 f_2 G_2 \cdot \ldots \cdot f_n G_n \) is equal to a "transformed" product \( fG_1' G_2' \cdot \ldots \cdot G_n' \) where
\( G_i' = (f_{i+1} \cdot \ldots \cdot f_n)^{-1} G_i (f_{i+1} \cdot \ldots \cdot f_n), i = 1, \ldots, n-1, G_n' = G_n, \) are quasiconvex subgroups of \( G \) by remark 4 and \( f = f_1 f_2 \cdot \ldots \cdot f_n \in G \). It remains to apply theorem 1 to the intersection of "transformed products" several times because a \((n,m)\)-increasing product is also a quasiconvex product. \( \square \)

4. Products of elementary subgroups

Recall that a group \( H \) is called elementary if it has a cyclic subgroup \( \langle h \rangle \) of
finite index.

Remark 5. An elementary subgroup \( H \) of a hyperbolic group \( G \) is quasiconvex.

Indeed, we have : \( |H : \langle h \rangle| < \infty \). If the element \( h \) has a finite order, then
\( H \) is finite and, thus, quasiconvex. In the case, when the order of \( h \) is infinite,
by lemmas 1.2,1.1 \( \langle h \rangle \) is a quasiconvex subgroup of \( G \). By remark 4 and lemma
2.1 \( H \) is quasiconvex.

It is well known that any element \( x \) of infinite order in \( G \) is contained in a
unique maximal elementary subgroup \( E(x) \leq G \) (see [4]). And the intersection
Let \( \beta \) be a property:

\[
|\langle 1 \rangle| \leq 1
\]

Then, \( \exists \beta \) elements of infinite order. Also, assume \( G_i \neq G_{i+1} \), \( H_j \neq H_{j+1} \), \( i = 1, \ldots, n - 1 \), \( j = 1, \ldots, m - 1 \). If there is a sequence of positive integers \( (t_k)_{k=1}^{\infty} \) with the properties:

\[
\lim_{k\to\infty} t_k = \infty \quad \text{and} \quad f g_1^{h_k} g_2^{h_k} \cdots g_n^{h_k} \in eH_1 H_2 \cdots H_m \quad \text{for all} \quad k \in \mathbb{N},
\]

then \( n = m, G_n = H_n \), and there exist elements \( z_i \in H_i, i = 1, \ldots, n \), such that \( G_i = \langle z_n z_{n-1} \cdots z_{i+1} \rangle \cdot H_i \cdot \langle z_n z_{n-1} \cdots z_{i+1} \rangle^{-1}, i = 1, 2, \ldots, n - 1 \), \( f = e z_1^{-1} z_2^{-1} \cdots z_n^{-1} \). Consequently, \( f G_1 \cdot \cdots \cdot G_n = eH_1 \cdot \cdots \cdot H_m \).

In the conditions of theorem 2', let \( h_j \in H_j \) be fixed elements of infinite order, \( j = 1, 2, \ldots, m \). Then \( G_i = E(y_i), H_j = E(h_j) \) and \( |G_i : \langle y_i \rangle| < \infty \), \( |H_j : \langle h_j \rangle| < \infty \). Hence, there exists \( T \in \mathbb{N} \) such that for all \( j \) and \( \forall v \in H_j \) there exists \( \beta \in \mathbb{Z}, y \in H_j : v = y \cdot h_j^\beta \) and \( |y|_G \leq T \). Thus, every element \( h \in eH_1 \cdot \cdots \cdot H_m \) can be presented in the form

\[
h = cy_1 h_1^{\beta_1} y_2 h_2^{\beta_2} \cdots y_m h_m^{\beta_m}
\]

where \( \beta_j \in \mathbb{Z}, y_j \in H_j, |y_j|_G \leq T, j = 1, 2, \ldots, m \).

**Definition:** the representation (9) for \( h \) will be called reduced if for any \( i, j \), \( 1 \leq i \neq j \leq m \), such that \( \beta_i, \beta_j \neq 0 \), one has

\[
(y_{i+1} h_{i+1}^{\beta_{i+1}} \cdots h_{j-1}^{\beta_{j-1}} y_j)^{-1} \cdot h_i \cdot (y_{i+1} h_{i+1}^{\beta_{i+1}} \cdots h_{j-1}^{\beta_{j-1}} y_j) \notin H_j = E(h_j).
\]

Observe that each element \( h \in eH_1 \cdot \cdots \cdot H_m \) has a reduced representation. Indeed, if \( (y_{i+1} h_{i+1}^{\beta_{i+1}} \cdots h_{j-1}^{\beta_{j-1}} y_j)^{-1} \cdot h_i \cdot (y_{i+1} h_{i+1}^{\beta_{i+1}} \cdots h_{j-1}^{\beta_{j-1}} y_j) \in H_j \) for some \( 1 \leq i < j \leq m \) then there are \( \beta'_j \in \mathbb{Z}, y'_j \in H_j, |y'_j|_G \leq T \):

\[
y_j \cdot (y_{i+1} h_{i+1}^{\beta_{i+1}} \cdots h_{j-1}^{\beta_{j-1}} y_j)^{-1} \cdot h_i \cdot (y_{i+1} h_{i+1}^{\beta_{i+1}} \cdots h_{j-1}^{\beta_{j-1}} y_j) \cdot h_j^{\beta_j} = y'_j h_j^{\beta_j'}.
\]

Therefore,

\[
h = ey_1 h_1^{\beta_1} \cdots y_{i-1} h_{i-1}^{\beta_{i-1}} y_i y_{i+1} h_{i+1}^{\beta_{i+1}} \cdots y_{j-1} h_{j-1}^{\beta_{j-1}} y_j y_{j+1} h_{j+1}^{\beta_{j+1}} \cdots y_m h_m^{\beta_m}
\]

and the number of non-zero \( \beta_j \)'s is decreased. Continuing this process, we will obtain a reduced representation for \( h \) after a finite number of steps.
Proof of Theorem 2'. Let $h_j \in H_j$, $1 \leq j \leq m$, $T$, be as above. Induction on $n$.

If $n=1$, then, evidently, $m = 1$, and $\forall k \in \mathbb{N}$ there is $y_k \in H_1$, $|y_k| \leq T$, and $\beta_k \in \mathbb{Z}$ such that $f g_1^{t_k} = e y_k h_1^{\beta_k}$. Because of having $\lim_{k \to \infty} t_k = \infty$, one can choose $p, q \in \mathbb{N}$ so that $t_p < t_q$ and $y_p = y_q$. Therefore,

$$f g_1^{t_p} h_1^{-\beta_p} = e y_p = f g_1^{t_q} h_1^{-\beta_q},$$

and, thus, $g_1^{t_q-t_p} = h_1^{\beta_q-\beta_p}$ — an element of infinite order in the intersection of $G_1$ and $H_1$. Consequently, $G_1 = H_1$, because these subgroups are maximal elementary.

Assume, now, that $n > 1$. For every $k \in \mathbb{N}$ one has

$$f g_1^{t_k} g_2^{t_k} \cdots g_n^{t_k} = e y_{k+1} h_1^{\beta_1} \cdots y_{k+n} h_1^{\beta_n} m$$

where the product in the right-hand side is reduced. Obviously, there exists a subsequence $(t_k)_{k=1}^\infty$ of $(t_k)$ and $C \in \mathbb{N}$ such that for each $j \in \{1, 2, \ldots, m\}$ either $|\beta_{kj}| \leq C$ for all $k$ or $\lim_{k \to \infty} |\beta_{kj}| = \infty$.

Therefore, since $|y_{kj}| \leq T \forall k \in \mathbb{N}, \forall j$, there is a subsequence $(s_k)_{k=1}^\infty$ of $(t_k)$ such that $y_{kj} = y_j \in H_j \forall j$, and if for $j \in \{1, \ldots, m\}$ we had $|\beta_{kj}| \leq C$ and $k \in \mathbb{N}$ then $\lim_{k \to \infty} |\beta_{kj}| = \infty$ for all other $j$'s.

Thus, $\{1, 2, \ldots, m\} = J_1 \cup J_2$ where if $j \in J_1$ then $|\beta_{kj}| = \beta_j$ for every $k$, and if $j \in J_2$ then $\lim_{k \to \infty} |\beta_{kj}| = \infty$. Let $J_2 = \{j_1, j_2, \ldots, j_{\lambda}\} \subset \{1, 2, \ldots, m\}$, $j_1 < j_2 < \ldots < j_{\lambda}$, and denote

$$w_1 = y_1^{-1} \in H_1$$

if $j_1 = 1$, otherwise, if $j_1 > 1$,

$$w_1 = y_j^{-1} h_{j_1-1}^{\beta_j-1} \cdots h_1^{\beta_1-1} y_1^{-1} \in H_j H_{j_1-1} \cdots H_1,$$

$$\ldots$$

$$w_{\lambda} = y_{j_{\lambda}}^{-1} \in H_{j_{\lambda}}$$

if $j_{\lambda} = j_1 + 1$, otherwise, if $j_{\lambda} > j_1 + 1$,

$$w_{\lambda} = y_j^{-1} h_{j_{\lambda}-1}^{\beta_j-1} \cdots h_{j_{\lambda}+1-1} y_{j_{\lambda}+1-1}^{-1} \in H_{j_{\lambda}} H_{j_{\lambda}+1} \cdots H_{j_{\lambda}+1-1},$$

$$w_{\lambda+1} = 1_G$$

if $j_{\lambda} = m$, otherwise, if $j_{\lambda} < m$,

$$w_{\lambda+1} = y_1^{-1} h_{j_{\lambda}+1}^{\beta_1-\beta_{j_{\lambda}+1}} y_{j_{\lambda}+1}^{-1} \in H_1 H_{j_{\lambda}+1} \cdots H_{j_{\lambda}+1}.$$

To simplify the formulas, denote $\delta_{k\nu} = -\beta_{j_{k\nu}}$, $1 \leq \nu \leq \lambda$.

Then $\lim_{k \to \infty} |\delta_{k\nu}| = \infty$ for every $\nu = 1, 2, \ldots, \lambda$. (10) is equivalent to

$$w_k \overset{\text{def}}{=} f g_1^{s_k} g_2^{s_k} \cdots g_{n-1}^{s_k} w_{\lambda+1} h_{j_{\lambda}+1}^{\delta_{j_{\lambda}+1}} w_{j_{\lambda}+1}^{-1} \cdots w_2 h_{j_1}^{\delta_{j_1}} w_1 e^{-1} = 1_G$$

So, $|u_k| \leq 0$ for all $k \in \mathbb{N}$. Denote $K = \max\{|f_G|, |w_1 e^{-1}|_G, |w_2|_G, \ldots, |w_{\lambda+1}|_G\}$, and assume that $g_n \notin w_{\nu+1} E(h_{j_{\nu+1}}) w_{\nu+1}^{-1}$. The product in the right-hand side of (10) was reduced, therefore $h_{j_{\nu}} \notin w_{\nu} E(h_{j_{\nu-1}}) w_{\nu-1}^{-1}$, $\nu = 2, 3, \ldots, \lambda$. Thus, we can apply Lemma 1.6 to (11) and obtain $\lambda > 0$, $c \geq 0$. 

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and $M > 0$ (depending on $K$, $g_1, \ldots, g_n, h_{j_1}, \ldots, h_{j_n}$) such that if $s_k \geq M$ and $|\delta_{jk}| \geq M$, $\nu = 2, 3, \ldots, \infty$, then $|u_k| \geq \lambda \cdot s_k - c$. Now, by the choice of the sequence $(s_k)$, there exists $N \in \mathbb{N}$ such that $s_k \geq M$ and $|\delta_{jk}| \geq M \forall k \geq N$, $\nu = 2, 3, \ldots, \infty$. Thus, taking $k \geq \max\{N, c/\lambda\} + 1$, we achieve a contradiction: $0 = |u_k| \geq \lambda \cdot s_k - c$.

Hence, $g_n \in w_{x'}E(h_{j_n})w_{x'}^{-1}$ which implies

$$
G_n = E(g_n) = w_{x'}E(h_{j_n})w_{x'}^{-1} = E(w_{x'}h_{j_n}w_{x'}^{-1}).
$$

Consequently, for every $k \in \mathbb{N}$, $w_{x'}g_k w_{x'}h_{j_n}^{x'} = y_k h_{j_n}^{y_k} \in H_{j_n}$ where $|y_k| \leq T$. By passing to a subsequence of $(s_k)$ we can assume that $y_{j_n} = y_{j_n}' \in H_{j_n}$ for every $k$. Therefore

$$
u_k = f g_k g_2 \cdots g_{n-1} w_{x'}h_{j_n}^{y_k} \cdots w_{x'}h_{j_n}^{y_k} \cdots w_{x'}h_{j_n}^{y_k}w_{x'}^{-1} = 1_G.
$$

Suppose $\limsup_{k \to \infty} |\gamma_k| = \infty$. Since $E(g_{n-1}) = G_{n-1} \neq G_n = E(g_n)$, we have $g_{n-1} \notin w_{x'}E(h_{j_n})w_{x'}^{-1} = w_{x'}y_{j_n}'E(h_{j_n})(y_{j_n}')^{-1}w_{x'}^{-1}$ (because $y_{j_n}' \in H_{j_n}$).

Then for $K' = \max\{K, w_{x'}y_{j_n}'|G\}$ by Lemma 1.8 there exist $\lambda > 0$, $c > 0$ and $M > 0$ (depending on $K'$, $g_1, \ldots, g_{n-1}, h_{j_1}, \ldots, h_{j_n}$) such that if $s_k \geq M$, $|\delta_{jk}| \geq M$, $\nu = 2, 3, \ldots, \infty$, and $|\gamma_k| \geq M$ then $|\nu_k| \geq \lambda \cdot s_k - c$. Now, by the assumption on $(\gamma_k)$ and $(\gamma_n)$, there exists $N \in \mathbb{N}$, $N > c/\lambda$, such that $s_N \geq M$, $|\delta_{Nj}| \geq M$, $\nu = 2, 3, \ldots, \infty$, and $|\gamma_N| > M$. Which leads us to a contradiction: $0 = |\nu_k| \geq \lambda \cdot s_k - c$.

Thus, $|\gamma_k| \leq C_1$ for some constant $C_1$, so, by passing to a subsequence as above, we can assume that $\gamma_k = \gamma \forall k \in \mathbb{N}$. Hence, after setting $z_\nu = w_{x'}y_{j_n}'h_{j_n}^{y_k}w_{x'}$, for every natural index $k$ we will have

$$
u_k = f g_k g_2 \cdots g_{n-1} z_\nu h_{j_n}^{y_k} \cdots w_{x'}h_{j_n}^{y_k} \cdots w_{x'}h_{j_n}^{y_k}w_{x'}^{-1} = 1_G.
$$

Which implies $f g_k g_2 \cdots g_{n-1} \in \nu w_{x'}^{-1} h_{j_n} w_{x'}^{-1} h_{j_n} z_\nu - 1 = u H_{j_n}^2 \cdots H_{j_n}^{v_\nu} \cdots H_{j_n}^{w_\nu} \nu = 2, 3, \ldots, \infty - 1, v_\nu = z_\nu w_{x'}^{-1} \cdots w_{x'}^{-1} w_{x'}^{-1} w_{x'}^{-1} \cdots w_{x'}^{-1}$.

$n - 1 \geq m - 1 > \infty - 1$ and the other conditions of the theorem 2' are satisfied, therefore one can apply the induction hypothesis and obtain that $n - 1 = \infty - 1$, hence, $n = m = n$, $j_\nu = \nu$, $1 \leq \nu \leq \infty$ and, by definition, $w_\nu = y_{j_n} \in H_{\nu}$, $\nu = 1, 2, \ldots, n$, $w_{x'} = 1_G$, $z_\nu = z_\nu H_{n}$, and also $G_{n-1} = H_{j_n}^{v_\nu} = H_{n-1}^{v_\nu}$, and there exist $z_\nu \in H_{n}, 1 \leq i < n - 1$, such that

$$
G_i = (z_{n-1} z_{n-2} \cdots z_{i+1}) \cdot H_{i}^{v_\nu} \cdot (z_{n-1} z_{n-2} \cdots z_{i+1})^{-1} = (z_{n-1} \cdots z_{i+1}) H_{i} \cdot (z_{n-1} \cdots z_{i+1})^{-1}, i = 1, 2, \ldots, n - 2,
$$

where $z_\nu \in H_{n}$, $1 \leq \nu \leq n - 1$, $f = u (z_{n-1} \cdots z_{i+2})^{-1} = e_{n-1} \cdots e_{i+2}$.

By (12) $G_n = E(h_n) = H_n$. The proof of the theorem 2' is finished. $\square$
Suppose \( G_1, G_2, \ldots, G_n \) are infinite maximal elementary subgroups of \( G \), \( f_1, \ldots, f_n \in G \), \( n \in \mathbb{N} \cup \{0\} \).

**Definition:** the set \( P = f_1 G_1 f_2 G_2 \cdots f_n G_n \) will be called ME-product. Thus, if \( n = 0 \), we have the empty set. For convenience, we will also consider every element \( g \in G \) to be a ME-product. As in the proof of corollary 2, every such ME-product can be brought to a form (however, not unique)

\[ P' = fG'_1 G'_2 \cdots G'_k \]

where \( 0 \leq k \leq n \), \( f \in G \), \( G'_i \) are infinite maximal elementary subgroups, \( i = 1, 2, \ldots, k \), and \( G'_i \neq G'_{i+1} \), \( 1 \leq i \leq k - 1 \). The number \( k \) in this case will be called rank of the ME-product \( P \) (thus, \( \text{rank}(P) = \text{rank}(P') = k \leq n \)).

A set \( U \) which can be presented as a finite union of ME-products has rank \( k \), by definition, if \( U = \bigcup_{i=1}^{t} P_i \), where \( P_i \), \( i = 1, \ldots, t \), are ME-products, and \( k = \max\{\text{rank}(P_i) | 1 \leq i \leq t\} \).

Note: an empty set is defined to have rank \( -1 \); any element of the group \( G \) is a ME-product of rank 0; thus any finite non-empty subset of \( G \) is a finite union of ME-products of rank 0.

**Remark 6.** the rank of a ME-product is defined correctly by theorem 2. By theorem 2' the definition of the rank of a finite union of ME-products is correct.

**Lemma 4.1.** Suppose \( P,R \) are ME-products in a hyperbolic group \( G \). Then the intersection \( T \overset{\text{def}}{=} P \cap R \) is a finite union of ME-products and its rank is at most \( \text{rank}(P) \). If \( \text{rank}(T) = \text{rank}(P) \) then \( T = P \).

**Proof.** Since a conjugate to an infinite maximal elementary subgroup is also infinite maximal elementary, it follows from theorem 1 that \( T \) is a finite union of ME-products \( P_i \), \( 1 \leq i \leq t \) (for some \( t \in \mathbb{N} \cup \{0\} \)):

\[ T = P \cap R = \bigcup_{i=1}^{t} P_i \]

For each \( i = 1, \ldots, t \), \( P_i \subseteq P \), therefore by theorem 2', \( \text{rank}(P_i) \leq \text{rank}(P) \) (otherwise we would get a contradiction), and \( \text{rank}(P_i) = \text{rank}(P) \) if and only if \( P_i = P \). Thus \( \text{rank}(T) = \max\{\text{rank}(P_i) | 1 \leq i \leq t\} \leq \text{rank}(P) \). If \( \text{rank}(T) = \text{rank}(P) \) then \( \text{rank}(P_i) = \text{rank}(P) \) for some \( i \), and so, \( P_i = P = T \). Q.e.d. □

As an immediate consequence of lemma 4.1 one obtains

**Corollary 3.** let \( P \) be a ME-product of rank \( n \) and \( U \) be a finite union of ME-products. Then the set \( P \cap U \) is a finite union of ME-products, \( \text{rank}(P \cap U) \leq n \), and if \( \text{rank}(P \cap U) = n \) then \( P \cap U = P \).

**Corollary 4.** A non-elementary hyperbolic group \( G \) can not be equal to a finite union of its ME-products.

**Proof.** Suppose, by the contrary, that \( G \) is a finite union of ME-products: \( G = P_1 \cup \ldots \cup P_1 \) and \( \text{rank}(G) = m \). Since \( G \) is not elementary, there exist
two elements $x, y \in G$ of infinite order such that $E(x) \neq E(y)$. Hence, one can construct a ME-product $P = G_1G_2 \cdots G_{m+1}$ in $G$ where $G_i = E(x)$ if $i$ is even, and $G_i = E(y)$ if $i$ is odd. Consequently, $\text{rank}(P) = m + 1$, but $P \subset G$, thus

$$P \cap G = P = \bigcup_{j=1}^{l} (P_j \cap P).$$

By lemma 4.1, $\text{rank}(P_j \cap P) \leq \text{rank}(P_j) \leq m$ for every $j = 1, 2, \ldots, l$. Therefore, we achieve a contradiction with the definition of rank: $m + 1 = \text{rank}(P) = \text{rank}(P \cap G) \leq m$. □

A group $H$ is called **bounded-generated** if it is a product of finitely many cyclic subgroups, i.e. there are elements $x_1, x_2, \ldots, x_k \in H$ such that every $h \in H$ is equal to $x_1^{s_1} x_2^{s_2} \cdots x_k^{s_k}$ for some $s_1, \ldots, s_k \in \mathbb{Z}$.

**Corollary 5.** Any bounded-generated hyperbolic group is elementary.

**Proof.** Indeed, any cyclic subgroup of a hyperbolic group either is finite or is contained in some infinite maximal elementary subgroup. Hence, their product is contained in a finite union of ME-products and we can apply corollary 4. □

**Proof of Theorem 3.** Since there exist at most countably many different ME-products in $G$, it is enough to consider only their countable intersections. Let $P_{ji}, 1 \leq i \leq k_j, k_j, j \in \mathbb{N}$, be ME-products, and $U_j = \bigcup_{i=1}^{k_j} P_{ji}$ - their finite unions. Let

$$T = \bigcap_{j=1}^{\infty} U_j.$$

One has to show that there exist ME-products $R_1, \ldots, R_s$, $s \in \mathbb{N} \cup \{0\}$, such that $T = R_1 \cup \ldots \cup R_s$.

Induct on $n = \text{rank}(U_1)$.

$$T = \left( \bigcup_{i=1}^{k_1} P_{i1} \right) \cap \bigcap_{j=2}^{\infty} U_j = \bigcup_{i=1}^{k_1} \left( P_{i1} \cap \bigcap_{j=2}^{\infty} U_j \right)$$

So, it is enough to consider the case when $k_1 = 1$, $U_1 = P_{11} = P$.

If $n = 0$ then $P$ is finite and there is nothing to prove.

Assume that $n > 0$ and let $J \in \mathbb{N}$ be the smallest index such that $P \cap U_J \neq P$ (if there is no such $J$ then $T = P$ and the theorem is true).

Therefore

$$T = P \cap \bigcap_{j=J}^{\infty} U_j = (P \cap U_J) \cap \bigcap_{j=J+1}^{\infty} U_j.$$

By corollary 3, $P \cap U_J$ is a finite union of ME-products:

$$P \cap U_J = \bigcup_{l=1}^{t} R_{lJ}, \ t \in \mathbb{N} \cup \{0\}.$$
and \( \text{rank}(P \cap U_j) < n \) because of the choice of \( J \), therefore \( \text{rank}(R'_l) < n \), \( \forall l = 1, 2, \ldots, t \).

Hence, by the induction hypothesis,

\[
T = \bigcup_{l=1}^{t} \left[ R'_l \cap \bigcap_{j=J+1}^{\infty} U_j \right] = \bigcup_{l=1}^{t} [R_{l1} \cup \ldots \cup R_{ln}]
\]

for some ME-products \( R_{l1}, \ldots, R_{ln} \), \( n_i \in \mathbb{N} \cup \{0\} \), \( 1 \leq l \leq t \). \( \square \)

The statement of the theorem 3 fails to be true if maximal elementary subgroups in the definition of ME-products one substitutes by arbitrary elementary subgroups. Below we construct an example to demonstrate that.

Let \( G = F(x, y) \) be the free group with two generators, \( q_1 < q_2 < q_3 < \ldots \) be an infinite sequence of prime numbers. Define \( d_i = q_i q_2 \ldots q_i, c_i = q_1 q_2 \ldots q_i \), \( i \in \mathbb{N} \), and the sets \( P_i, i \in \mathbb{N} \), as follows:

\[
P_1 = \langle x^{d_i} \rangle - \text{cyclic subgroup of } G \text{ generated by } x^{d_i} = x^{q_1},
\]

\[
P_2 = \langle y \rangle \cdot \langle y x^{c_i} y^{-1} \rangle \cdot \langle y^2 x^{q_2} y^{-2} \rangle \cdot \langle y \rangle,
\]

\[
P_3 = \langle y \rangle \cdot \langle y x^{c_i} y^{-1} \rangle \cdot \langle y^2 x^{q_2} y^{-2} \rangle \cdot \langle y^3 x^{q_3} y^{-3} \rangle \cdot \langle y \rangle,
\]

\[\ldots.
\]

\[
P_t = \langle y \rangle \cdot \langle y x^{c_i} y^{-1} \rangle \cdot \langle y^2 x^{q_2} y^{-2} \rangle \cdot \ldots \cdot \langle y^{t-1} x^{c_i-1} y^{-t(i-1)} \rangle \cdot \langle y^t x^{q_i} y^{-i} \rangle \cdot \langle y \rangle,
\]

\[\ldots.
\]

Now consider the intersection \( T = \bigcap_{i=1}^{\infty} P_i \). Let us observe that

\[
P_1 \cap P_2 = \langle x^{c_1} \rangle \cup \langle x^{d_1} \rangle, \ldots, \bigcap_{i=1}^{k} P_i = \langle x^{c_1} \rangle \cup \ldots \cup \langle x^{c_{k-1}} \rangle \cup \langle x^{d_k} \rangle, \ldots.
\]

Indeed, \( P_1 \cap P_2 = \langle x^{d_1} \rangle \cap (\langle x^{c_1} \rangle \cup \langle x^{d_2} \rangle) = \langle x^{c_1} \rangle \cup \langle x^{d_2} \rangle \). Inducting on \( k \), we get

\[
\bigcap_{i=1}^{k} P_i = \left( \bigcap_{i=1}^{k-1} P_i \right) \cap P_k = \langle x^{c_1} \rangle \cup \ldots \cup \langle x^{c_{k-1}} \rangle \cup \langle x^{d_{k-1}} \rangle \cap (\langle x^{c_1} \rangle \cup \ldots \cup \langle x^{c_{k-1}} \rangle \cup \langle x^{d_k} \rangle)
\]

\[
= \langle x^{c_1} \rangle \cup \ldots \cup \langle x^{c_{k-1}} \rangle \cup \langle x^{d_k} \rangle.
\]

Since \( \bigcap_{i=1}^{\infty} \langle x^{d_i} \rangle = \{1\} \), therefore \( T = \bigcup_{i=1}^{\infty} \langle x^{c_i} \rangle \).

If \( q_1 = 2, q_2 = 3, q_3 = 5, \ldots \), is chosen to be the enumeration of all primes, one can show directly that the set \( T \) can not be presented as a finite union of products \( f_1 G_1 f_2 G_2 \ldots f_n G_n \), where \( f_1, \ldots, f_n \in G \) and \( G_1, \ldots, G_n \) are elementary (in this case cyclic) subgroups of \( G \). We are not going to do that, instead we will use a set-theoretical argument: there are only countably many such finite unions, hence there is an infinite sequence of primes \( q_1 < q_2 < q_3 < \ldots \) such that the corresponding set \( \bigcap_{i=1}^{\infty} P_i \) is the example sought (because the sets \( \bigcap_{i=1}^{\infty} P_i \) and \( \bigcap_{i=1}^{\infty} P'_i \) corresponding to different increasing sequences of prime numbers \( \alpha = \{q_1, q_2, q_3, \ldots\} \) and \( \alpha' = \{q_1', q_2', q_3', \ldots\} \) are distinct: if \( q_i \in \alpha \setminus \alpha' \) then \( x^{c_i} \in \bigcup_{i=1}^{\infty} \langle x^{c_i} \rangle \setminus \bigcup_{i=1}^{\infty} \langle x^{c_i} \rangle \).
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References