ON RESIDUALIZING HOMOMORPHISMS PRESERVING QUASICONVEXITY

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Abstract

H is called a G-subgroup of a hyperbolic group G if for any finite subset $M \subset G$ there exists a homomorphism from G onto a non-elementary hyperbolic group G_1 that is surjective on H and injective on M. In his paper in 1993 A. Ol'shanskii gave a description of all G-subgroups in any given non-elementary hyperbolic group G. Here we show that for the same class of G-subgroups the finiteness assumption on M (under certain natural conditions) can be replaced by an assumption of quasiconvexity.

1 Introduction

Suppose G is a group with some finite generating set $\mathcal{A}$ (assume that $\mathcal{A}$ is *symmetrized*, i.e. $\mathcal{A}^{-1} = \mathcal{A}$). Then one can construct the Cayley graph $\Gamma(G, \mathcal{A})$ which is a geodesic metric space. Assume $\eta \geq 0$ is some number. A subset Q of G (or of $\Gamma(G, \mathcal{A})$) is said to be η-quasiconvex (or just quasiconvex) if for any pair of elements $u, v \in Q$ and any geodesic segment p connecting u and v, p belongs to a closed η-neighborhood of the subset Q in $\Gamma(G, \mathcal{A})$.

If, in addition, the group G is (word) hyperbolic, the fact that a subset of this group is quasiconvex does not depend on the choice of a generating set $\mathcal{A}$ (see [7]).

If $S \subseteq G$, $C_H(S)$ will denote the centralizer subgroup of S in H, i.e.

$$C_H(S) = \{h \in H \mid hg = gh \ \ \forall\, g \in S\} \ .$$

If A, B are subsets of the group G, their product is the subset of G defined by

$$A \cdot B = AB \stackrel{def}{=} \{ab \mid a \in A, b \in B\} \ .$$

$A^{-1} \subset G$ will denote the subset $\{a^{-1} \mid a \in A\}$.

Now let G be a δ-hyperbolic group, $\delta \geq 0$. Every element $g \in G$ of infinite order belongs to a unique *maximal elementary subgroup* $E(g)$ (a group is called *elementary* if it has a cyclic subgroup of finite index). Then by [16, Lemmas 1.16 and 1.17]

$$E(g) = \left\{ \ x \in G \mid xg^n x^{-1} = g^{\pm n} \text{ for some } n \in \mathbb{N} \ \right\} \ \text{ and}$$

$$E(g) = \left\{ \ x \in G \mid xg^k x^{-1} = g^l \text{ for some } k, l \in \mathbb{Z}\backslash\{0\} \ \right\} \qquad (1)$$

It is easy to see that the subgroup

$$E^+(g) = \{ \ x \in G \mid xg^n x^{-1} = g^n \text{ for some } n \in \mathbb{N} \ \}$$

is of index at most 2 in $E(g)$.

For any subgroup H of G denote by H^0 the set of elements of infinite order in H. Using the terminology from [16], a non-elementary subgroup H of G will be called a *G-subgroup* if for any finite subset $M \subset G$ there is a non-elementary hyperbolic quotient G_1 of the group G, such that the natural homomorphism $G \to G_1$ is surjective on H and injective on M.

Denote $E(H) = \bigcap_{x \in H^0} E(x)$. If H is a non-elementary subgroup of G, then $E(H)$ is the unique maximal finite subgroup of G normalized by H ([16, Prop. 1]). Hence H acts on $E(H)$ by conjugation and we have a homomorphism of H into the permutation group on the set of elements of $E(H)$. The kernel of that homomorphism is $C_H\big(E(H)\big)$ which sometimes will be denoted by $K(H)$. The index $|H : K(H)|$ is finite because of the finiteness of $E(H)$.

The following characterization of all G-subgroups was given in [16, Thm. 1]:

Lemma 1.1. *A non-elementary subgroup H of a hyperbolic group G is a G-subgroup if and only if $E(H) = E(G)$ and $|H : K(H)| = |G : K(G)|$ (i.e. the actions by conjugation of H and G on $E(H) = E(G)$ are similar: for every $g \in G$ there exists an element $h \in H$ with $gag^{-1} = hah^{-1}$ for all $a \in E(G)$).*

Definition. Let H be a subgroup of the group G and $Q \subseteq G$ be a quasiconvex subset. The subset Q will be called *small relatively to H* if for any two finite subsets P_1, P_2 of the group G one has

$$H \not\subseteq \left(P_1 \cdot Q^{-1} \cdot Q \cdot P_2\right) \ . \qquad (*)$$

As we know, any generating set induces a left-invariant metric on the set of elements of a group. So, if G_1 is a quotient of G, the group G_1 will be generated by the image of $\mathcal{A}$ under the natural homomorphism $\phi : G \to G_1$. Therefore, later G_1 will be assigned the metric corresponding to the generating set $\phi(\mathcal{A})$.

The main result of this paper is

Theorem 1. *Let $H_1, H_2, \dots, H_k$ be G-subgroups of a non-elementary hyperbolic group G and $H'_1, \dots, H'_{k'}$ be some non-elementary subgroups of G. Assume $Q \subseteq G$ is an η-quasiconvex subset (for some $\eta \geq 0$) that is small relatively to H_i for every $i = 1, 2 \dots, k$. Then there exist a group G_1 and an epimorphism $\phi : G \to G_1$ such that*

1) *G_1 is a non-elementary hyperbolic group;*

2) *The homomorphism ϕ is an isometry between Q and $\phi(Q)$ (if the metrics on G and G_1 are chosen as explained above) and for any quasiconvex subset $S \subseteq Q$, its image $\phi(S)$ is quasiconvex in G_1. In particular, ϕ is injective on Q;*

3) *ϕ is surjective on each of the subgroups $H_1, \dots, H_k$, i.e. $\phi(H_i) = G_1$ for each $i = 1, 2, \dots, k$;*

4) *ϕ-images of two elements from Q are conjugate in G_1 if and only if these elements are conjugate in G;*

5) *The centralizer* $C_{G_1}(\phi(a))$ *for every* $a \in Q$ *is the* ϕ*-image of the centralizer* $C_G(a)$*;*

6) $ker\phi$ *is a torsion-free subgroup;*

7) ϕ *induces a bijective map on sets of conjugacy classes of elements having finite orders in* G *and* G_1 *respectively;*

8) $\phi(H'_1), \dots, \phi(H'_{k'})$ *are non-elementary subgroups of* G_1*;*

9) $E(G_1) = \phi(E(G))$.

Parts 1)-8) of Theorem 1 were proved by A. Ol'shanskii in [16, Thm. 2] for the case when the subset Q is finite; part 9) was proved in [16, Thm. 4] with additional conditions imposed on G. If $card(Q) < \infty$, since each H_i is infinite, the condition $(*)$ becomes trivial and can be omitted (Q will be small relatively to any infinite subgroup). In our proof the tools and techniques developed in the paper [16] will be crucial, so an interested reader is referred to that article in advance.

Definition. A subgroup H of a group G will be called a *quasiretract* of G if there exists a normal subgroup $N \lhd G$ such that $|G : HN| < \infty$ and the intersection $H \cap N$ is finite.

In particular, any retract is a quasiretract. The proof of the following lemma is not difficult and will be given in the beginning of section 3.

Lemma 1.2. *Assume that* G *is a hyperbolic group and* H *is a quasiretract of* G*. Then the subgroup* H *is quasiconvex in* G*.*

One can observe that if the group G is torsion-free then any its non-elementary subgroup will be a G-subgroup. However, by far, not every subgroup in G will be quasiconvex (or a quasiretract). In the next theorem we show that demanding Q to be small relatively to H_i is necessary if one doesn't impose additional limitations on the subgroups H_i, $i = 1, 2, \dots, k$.

Theorem 2. *Let* H *be an infinite subgroup of a hyperbolic group* G *and* $Q \subset G$ *be a subset (not necessarily quasiconvex). Suppose that*

$$H \subseteq P_1 Q^{-1} Q P_2$$

for some finite subsets P_1*,* P_2 *of* G *and there is a group* G_1 *and an epimorphism* $\phi : G \to G_1$ *such that* ϕ *is surjective on* H *(i.e.* $\phi(H) = G_1$*) and* $\phi|_Q$ *is a quasiisometry between* Q *and* $\phi(Q)$*. Then the subgroup* H *is a quasiretract of* G*.*

In section 5 we investigate some properties of condition $(*)$; we show that if two quasiconvex subsets satisfy this condition then so do their union and product (lemma 5.4). If, in addition, one demands that Q^{-1} is quasiconvex then we are able to simplify $(*)$: the set $Q^{-1}Q$ in it can be substituted by just Q (corollary 4).

As examples of quasiconvex sets Q satisfying $(*)$ we can consider special subsets that are finite unions of products of quasiconvex subgroups. More precisely, suppose $F_1, \dots, F_n$ are quasiconvex subgroups of a hyperbolic group G

and $g_0, g_1, \ldots, g_n \in G$. Following [13], the subset

$$P = g_0 F_1 \cdot \ldots \cdot g_{n-1} F_n g_n = \{g_0 f_1 \cdot \ldots \cdot g_{n-1} f_n g_n \mid f_i \in F_i, i = 1, \ldots, n\} \subseteq G$$

will be called a *quasiconvex product*. The quasiconvex subgroups F_i, $i = 1, \ldots, n$, are said to be *members* of the product P. A finite union of quasiconvex products is always a quasiconvex subset in a hyperbolic group ([13, Prop. 1,Lemma 2.1]).

Let $U = \bigcup_{k=1}^{N} P_k$ be a finite union of quasiconvex products P_k, $k = 1, \ldots, N$. A subgroup $F \le G$ will be called a *member* of U, by definition, if F is a member of P_k for some $1 \le k \le N$. For any such set U we fix its representation as a finite union of quasiconvex products and fix its members.

Theorem 3. *Suppose G is a hyperbolic group, H is its subgroup and Q is a finite union of quasiconvex products in G with members $F_1, \ldots, F_l$. Assume also that*

$$|H : (H \cap gF_j g^{-1})| = \infty \ , \quad \textit{for every } g \in G \textit{ and } j = 1, 2, \ldots, l \ .$$

Then for arbitrary two finite subsets $P_1, P_2 \subset G$, the subgroup H is not contained inside of $P_1 \cdot Q^{-1} \cdot Q \cdot P_2$.

Let $\mathcal{A}$ be an alphabet and $\mathcal{R}_i$ – subsets of words in $\mathcal{A}^{\pm 1}$, $i \in \mathbb{N}$, satisfying $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ for all i. Let the groups G_i have presentations

$$G_i = \langle \mathcal{A} \parallel \mathcal{R}_i \rangle, \ i \in \mathbb{N}.$$

Then $G_{i+1} \cong G_i / N_i$ where $N_i \lhd G_i$ is the normal closure of $\mathcal{R}_{i+1} \backslash \mathcal{R}_i$ in G_i, i.e. there is an epimorphism $\phi_i : G_i \to G_{i+1}$ with $ker(\phi_i) = N_i$.

Set $\mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{R}_i$. The group M defined by the presentation

$$M = \langle \mathcal{A} \parallel \mathcal{R} \rangle$$

is said to be an *inductive limit* of the groups G_i, $i \in \mathbb{N}$.

Thus we obtain an infinite sequence of epimorphisms

$$G_1 \overset{\phi_1}{\to} G_2 \overset{\phi_2}{\to} G_3 \overset{\phi_3}{\to} \ldots$$

and $M = \lim\limits_{\to} (G_i, \phi_i)$.

The statement below follows from theorem 1 and can be compared with [18, Theorems 1,3].

Corollary 1. *Suppose G, H are hyperbolic groups and G is non-elementary. Then H can be isomorphically embedded into some simple quotient M of the group G. Moreover, the group M is an inductive limit of hyperbolic groups.*

D. Osin noticed that using a similar argument one can obtain

Corollary 2. *There exists a simple group M that is a quotient of every non-elementary hyperbolic group and contains every hyperbolic group (isomorphically embedded).*

Recall that a non-trivial proper subgroup H of a group G is called *malnormal* if for any $g \in G \backslash H$ the intersection $H \cap gHg^{-1}$ is trivial.

In the torsion-free case we can strengthen the claim of corollary 1 to obtain a "thrifty" embedding (cf. [15]):

Corollary 3. *Suppose G, H are torsion-free hyperbolic groups, G is non-elementary and H is non-trivial. Then there exists a simple torsion-free quotient M of G and an injective homomorphism $\pi : H \to M$ such that $\pi(H)$ is malnormal in M and any proper subgroup of M is conjugate (in M) to a subgroup of $\pi(H)$.*

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2 Preliminary Information

Assume $\big(\mathcal{X}, d(\cdot,\cdot)\big)$ is a proper geodesic metric space. If $Q \subset \mathcal{X}$, $N \geq 0$, the closed N-neighborhood of Q will be denoted by

$$\mathcal{O}_N(Q) \stackrel{def}{=} \{x \in \mathcal{X} \mid d(x,Q) \leq N\} \ .$$

If $x, y, w \in \mathcal{X}$, then the number

$$(x|y)_w \stackrel{def}{=} \frac{1}{2}\Big(d(x,w) + d(y,w) - d(x,y)\Big)$$

is called the *Gromov product* of x and y with respect to w.

Let abc be a geodesic triangle in the space $\mathcal{X}$ and $[a,b]$, $[b,c]$, $[a,c]$ be its sides between the corresponding vertices. There exist "special" points $O_a \in [b,c]$, $O_b \in [a,c]$, $O_c \in [a,b]$ with the properties: $d(a,O_b) = d(a,O_c) = \alpha$, $d(b,O_a) = = d(b,O_c) = \beta$, $d(c,O_a) = d(c,O_b) = \gamma$. From a corresponding system of linear equations one can find that $\alpha = (b|c)_a$, $\beta = (a|c)_b$, $\gamma = (a|b)_c$. Two points $O \in [a,b]$ and $O' \in [a,c]$ are called *a-equidistant* if $d(a,O) = d(a,O') \leq \alpha$. The triangle *abc* is said to be δ-*thin* if for any two points O, O' lying on its sides and equidistant from one of its vertices, $d(O,O') \leq \delta$ holds.

A geodesic n-gon in the space $\mathcal{X}$ is said to be δ-*slim* if each of its sides belongs to a closed δ-neighborhood of the union of the others.

Later we will use three equivalent definitions of *hyperbolicity* of the space $\mathcal{X}$ (see [5],[1]):

1°. (M. Gromov) There exists $\delta \geq 0$ such that for any four points $x, y, z, w \in \mathcal{X}$ their Gromov products satisfy

$$(x|y)_w \geq min\{(x|z)_w, (y|z)_w\} - \delta \ ;$$

2°. All triangles in $\mathcal{X}$ are δ-thin for some $\delta \geq 0$;

3°. (E. Rips) All triangles in $\mathcal{X}$ are δ-slim for some $\delta \geq 0$.

Remark 1. It is easy to see that the definition 3° implies that any geodesic n-gon in the space $\mathcal{X}$ is $(n-2)\delta$-slim if $n \geq 3$.

Now, suppose G is a finitely generated group with a fixed finite generating set $\mathcal{A}$. For every element $g \in G$ its length $|g|_G$ is the length of a shortest word in the alphabet $\mathcal{A}$ representing g in G. Then, if $g, h \in G$ we define the distance $d(g,h) = |g^{-1}h|_G$. This distance function can be extended to a metric on the Cayley graph invariant under the action of G by left multiplication:

$$\forall\ x, y \in \Gamma(G, \mathcal{A}) \text{ and } g \in G \ \ d(g \circ x, g \circ y) = d(x, y)\ .$$

This implies that for any elements $x, y, z, w \in G$

$$(x|y)_w = (zx|zy)_{zw}\ .$$

Thus the Cayley graph $\Gamma(G, \mathcal{A})$ becomes a proper geodesic metric space. There is a natural (metric space) embedding of the group G into its Cayley graph. Later we will identify G with its image under it.

The group G is called *hyperbolic* if $\Gamma(G, \mathcal{A})$ is a hyperbolic metric space. This definition does not depend on the choice of the finite generating set $\mathcal{A}$ in G ([7],[5, 5.2.14]), thus hyperbolicity is a group-theoretical property. Among well-known examples of hyperbolic groups are free groups of finite rank, finite groups, fundamental groups of negatively curved compact manifolds, etc.

Further on we will assume that $\Gamma(G, \mathcal{A})$ meets $1°, 2°$ and $3°$ for a fixed (sufficiently large) $\delta \geq 0$.

For any two points $x, y \in \Gamma(G, \mathcal{A})$ we fix a geodesic path between them and denote it by $[x, y]$. Let p be a path in the Cayley graph of G. Then p_-, p_+ will denote the startpoint and the endpoint of p, $||p||$ – its length; $lab(p)$, as usual, will mean the word in the alphabet $\mathcal{A}$ written on p. $elem(p) \in G$ will denote the element of the group G represented by the word $lab(p)$. p^{-1} will be the inverse path to p, i.e. the path with the same set of points but traced in the opposite direction.

A path q is called (λ, c)- *quasigeodesic* if there exist $0 < \lambda \leq 1$, $c \geq 0$, such that for any subpath p of q the inequality $\lambda||p|| - c \leq d(p_-, p_+)$ holds.
In a hyperbolic space quasigeodesics and geodesics with same ends are mutually close :

Lemma 2.1. ([5, 5.6,5.11],[1, 3.3]) *There is a constant $\nu = \nu(\delta, \lambda, c)$ such that for any (λ, c)-quasigeodesic path p in $\Gamma(G, \mathcal{A})$ and a geodesic q with $p_- = q_-$, $p_+ = q_+$, one has $p \subset \mathcal{O}_\nu(q)$ and $q \subset \mathcal{O}_\nu(p)$.*

An important property of cyclic subgroups in a hyperbolic group states

Lemma 2.2. ([5, 8.21],[1, 3.2]) *For any word w representing an element $g \in G$ of infinite order there exist constants $\lambda > 0$, $c \geq 0$, such that any path with a label w^m in the Cayley graph of G is (λ, c)-quasigeodesic for arbitrary integer m.*

It is easy to see that any finite subset of a group is $D/2$-quasiconvex where D is the diameter of that subset.

It follows from lemmas 2.1 and 2.2 that any cyclic subgroup of a hyperbolic group is quasiconvex.

Lemma 2.3. ([8, Prop. 3.14],[13, Lemma 2.1,Prop. 1]) *Let G be a hyperbolic group and let A, B be its quasiconvex subsets. Then the subsets $A \cup B$ and $A \cdot B$ are also quasiconvex.*

If $X_1, X_2, \ldots, X_n$ are points in $\Gamma(G, \mathcal{A})$, the notation $X_1 X_2 \ldots X_n$ will be used for the geodesic n-gon with vertices X_i, $i = 1, \ldots, n$, and sides $[X_i, X_{i+1}]$, $i = 1, 2, \ldots, n-1$, $[X_n, X_0]$. $[X_1, X_2, \ldots, X_n]$ will denote the broken line with these vertices in the corresponding order (i.e. the path $[X_1, X_2, \ldots, X_n]$ will consist of consecutively concatenated geodesic segments $[X_i, X_{i+1}]$, $i = 1, 2, \ldots, n$).

Lemma 2.4. ([17, Lemma 21]) *Let $p = [X_0, X_1, \ldots, X_n]$ be a broken line in $\Gamma(G, \mathcal{A})$ such that $||[X_{i-1}, X_i]|| \geq C_1$ $\forall$ $i = 1, \ldots, n$, and $(X_{i-1}|X_{i+1})_{X_i} \leq C_0$ $\forall$ $i = 1, \ldots, n-1$, where $C_0 \geq 14\delta$, $C_1 > 12(C_0 + \delta)$. Then p is contained in the closed $2C_0$-neighborhood $\mathcal{O}_{2C_0}([X_0, X_n])$ of the geodesic segment $[X_0, X_n]$.*

Lemma 2.5. ([14, Lemma 3.5]) *In the conditions of lemma 2.4 the inequality $\|[X_0, X_n]\| \geq \|p\|/2$ holds.*

Lemma 2.6. ([14, Lemma 4.1]) *Consider a geodesic quadrangle $X_1 X_2 X_3 X_4$ in $\Gamma(G, \mathcal{A})$ with $d(X_2, X_3) > d(X_1, X_2) + d(X_3, X_4)$. Then there are points $U, V \in [X_2, X_3]$ such that $d(X_2, U) \leq d(X_1, X_2)$, $d(V, X_3) \leq d(X_3, X_4)$ and the geodesic subsegment $[U, V]$ of $[X_2, X_3]$ lies 2δ-close to the side $[X_1, X_4]$.*

Lemma 2.7. ([14, Lemma 4.5]) *Let G be a δ-hyperbolic and let H_i be ε_i-quasiconvex subgroups of G, $i = 1, 2$. If one has*

$$\sup\{(h_1|h_2)_{1_G} \ : \ h_1 \in H_1, h_2 \in H_2\} = \infty$$

then $card(H_1 \cap H_2) = \infty$.

Lemma 2.8. ([14, Lemma 4.2]) *Let A be an infinite ε-quasiconvex set in G and $g \in G$. Then if the intersection $A \cap gAg^{-1}$ is infinite, there exists an element $r \in G$ with $|r|_G \leq 4\delta + 2\varepsilon + 2\varkappa$ such that $g \in ArA^{-1}$, where $\varkappa$ is the length of a shortest element from A.*

Lemma 2.9. ([14, Lemma 4.3]) *Let G be a δ-hyperbolic group, H and K – its subgroups with K quasiconvex. If $\quad H \subset \bigcup_{j=1}^{N} K s_j K \quad$ for some $s_1, \ldots, s_N \in G$ then $|H : (K \cap H)| < \infty$.*

Lemma 2.10. ([5, 8.3.36]) *Any infinite subgroup of a hyperbolic group contains an element of infinite order.*

If $g \in G^0$, $T(g)$ will be used to denote the set of elements of finite order in the subgroup $E(g)$.

Definition. Let G be a hyperbolic group and H be its non-elementary subgroup. An element $g \in H^0$ will be called *H-suitable* if $E(H) = T(g)$ and

$$E(g) = E^+(g) = C_G(g) = T(g) \times \langle g \rangle \ .$$

In particular, if the element g is H-suitable then $g \in C_H\big(E(H)\big)$.

Two elements $g, h \in G$ of infinite order are called *commensurable* if $g^k = ah^l a^{-1}$ for some non-zero integers k, l and some $a \in G$.

Now we recall the statement of [16, Lemma 3.8]:

Lemma 2.11. *Every non-elementary subgroup H of a hyperbolic group G contains an infinite set of pairwise non-commensurable H-suitable elements.*

We will need the following modification of [16, Lemma 3.7]:

Lemma 2.12. *Let g be an H-suitable element in a non-elementary subgroup H of a hyperbolic group G. Suppose $l \in \mathbb{N}$ and K is a non-elementary subgroup of H. Then for any number $C_1 \geq 0$ there exist elements $x_i \in K$, $i = 1, \dots, l$, satisfying the following properties:*

0) *$|x_i|_G > C_1$ for every $i = 1, 2 \dots, l$;*
1) *$x_i \notin E(g)$ for every $i = 1, 2 \dots, l$;*
2) *$x_i \in C_G(E(H))$ for every $i = 1, 2 \dots, l$;*
3) *$ax_i = x_i b$ for $a, b \in E(g)$ implies that $a = b \in E(H)$, $i = 1, 2 \dots, l$;*
4) *if $a, b \in E(g)$ and $ax_i = x_j b$ for some $i, j \in \{1, \dots, l\}$ then $i = j$.*

Proof. Indeed, it is shown in the proof of [16, Lemma 3.7] that if the elements $g, h_1, \dots, h_l \in H$ are pairwise non-commensurable in G then for any sufficiently large $t \in \mathbb{N}$, the elements $x_i = h_i^t$ satisfy the conditions $1) - 4)$. By lemma 2.11 we can choose such $h_1, \dots, h_l$ inside of K, thus $x_i = h_i^t \in K$. Obviously, if $t \in \mathbb{N}$ is sufficiently large the property 0) will also be satisfied. $\square$

In this paper we will also use the concept of *Gromov boundary* of a hyperbolic group G (for a detailed theory the reader is referred to the corresponding chapters in [5],[3] or [1]).) In order to define it, call a sequence $(x_i)_{i \in \mathbb{N}} \subset \Gamma(G, \mathcal{A})$ *converging to infinity* if

$$\lim_{i,j \to \infty} (x_i | x_j)_{1_G} = \infty \ \ (1_G \text{ is the identity element of the group } G).$$

Two sequences $(x_i)_{i \in \mathbb{N}}, (y_j)_{i \in \mathbb{N}}$ converging to infinity are said to be equivalent if

$$\lim_{i \to \infty} (x_i | y_i)_{1_G} = \infty \ .$$

The points of the boundary ∂G are identified with the equivalence classes of sequences converging to infinity. (It is easy to see that this definition does not depend on the choice of a basepoint: instead of 1_G one could use any fixed point p of $\Gamma(G,\mathcal{A})$ [1].) If α is the equivalence class of $(x_i)_{i\in\mathbb{N}}$ we will write $\lim\limits_{i\to\infty} x_i = \alpha$.

The space ∂G can be topologized so that it becomes compact, Hausdorff and metrizable (see [1],[5]).

Every isometry ψ of the space $\Gamma(G,\mathcal{A})$ induces a homeomorphism of ∂G in a natural way: for every equivalence class of sequences converging to infinity $[(x_i)_{i\in\mathbb{N}}] \in \partial G$ choose a representative $(x_i)_{i\in\mathbb{N}} \subset \Gamma(G,\mathcal{A})$ and set

$$\psi\left([(x_i)_{i\in\mathbb{N}}]\right) \stackrel{def}{=} \left[(\psi(x_i))_{i\in\mathbb{N}}\right] \in \partial G \ .$$

Left multiplication by elements of the group induces an isometric action of G on $\Gamma(G,\mathcal{A})$. Hence, G acts homeomorphically on the boundary ∂G as described above.

If $g \in G$ is an element of infinite order then the sequences $(g^i)_{i\in\mathbb{N}}$ and $(g^{-i})_{i\in\mathbb{N}}$ converge to infinity and we will use the notation

$$\lim_{i\to\infty} g^i = g^{\infty}, \ \lim_{i\to\infty} g^{-i} = g^{-\infty} \ .$$

For a subset $A \subseteq \Gamma(G,\mathcal{A})$ the *limit set* $\Lambda(A)$ of A is the collection of the points $\alpha \in \partial G$ that are limits of sequences (converging to infinity) from A.

The following properties of limit sets are well-known:

Lemma 2.13. ([9],[19]) *Suppose A, B are arbitrary subsets of* $\Gamma(G,\mathcal{A})$, $g \in G$. *Then*

(a) $\Lambda(A) = \emptyset$ if and only if A is finite;

(b) $\Lambda(A)$ is a closed subset of the boundary ∂G;

(c) $\Lambda(A \cup B) = \Lambda(A) \cup \Lambda(B)$;

(d) $\Lambda(Ag) = \Lambda(A)$, $g \circ \Lambda(A) = \Lambda(gA)$;

Lemma 2.14. *Suppose A and B are subsets of the hyperbolic group G and $\Lambda(A) \cap \Lambda(B) = \emptyset$. Then* $\sup\limits_{a\in A,\ b\in B} \{(a|b)_{1_G}\} < \infty$.

Proof. This statement is an easy consequence of the definition of a limit set. Indeed, assume, by the contrary, that there are sequences of elements $(a_i)_{i\in\mathbb{N}} \subset A$ and $(b_i)_{i\in\mathbb{N}} \subset B$ such that $\lim_{i\to\infty}(a_i|b_i)_{1_G} = \infty$. Then the subsets $\{a_i \mid i \in \mathbb{N}\} \subset G$ and $\{b_i \mid i \in \mathbb{N}\} \subset G$ are infinite, hence each of them has at least one limit point (by lemma 2.13.(a)). Thus, there are subsequences $(a_{i_j})_{j\in\mathbb{N}}$ of (a_i) and $(b_{i_j})_{j\in\mathbb{N}}$ of (b_i) satisfying

$$\lim_{j\to\infty} a_{i_j} = \alpha \in \Lambda(A), \ \lim_{j\to\infty} b_{i_j} = \beta \in \Lambda(B) \ .$$

But $\lim_{j\to\infty}(a_{i_j}|b_{i_j})_{1_G} = \infty$ by our assumption, hence $\alpha = \beta$. A contradiction with the condition $\Lambda(A) \cap \Lambda(B) = \emptyset$. $\square$

If H is a subgroup of G, it is known that ΛH is either empty (if H is finite) or consists of two distinct points (if H is infinite elementary), or is uncountable (if H is non-elementary) ([9],[5]). In the second case, when there exists $g \in H^0$ such that $|H : \langle g\rangle| < \infty$, $\Lambda H = \{g^{\infty}, g^{-\infty}\}$.

As the hyperbolic group G acts on its boundary, for every subset $\Omega \subset \partial G$ one can define the stabilizer subgroup by $St_G(\Omega) = \{g \in G \mid g \circ \Omega = \Omega\}$. For our convenience, we set $St_G(\emptyset) = G$.

It is proved in [5, thm. 8.3.30] that for any point $\alpha \in \partial G \quad St_G(\{\alpha\})$ is an elementary subgroup of the group G (in fact, if $\alpha = g^\infty$ for some element of infinite order $g \in G$ then

$$St_G(\{g^\infty\}) = E^+(g) \ ,$$

otherwise the subgroup $St_G(\{\alpha\})$ is finite). $St_G(\{g^\infty, g^{-\infty}\}) = E(g)$.

If $A \subseteq G$ and $\Omega \subseteq \partial G$ then the orbit of Ω under the action of A will be denoted

$$A \circ \Omega \stackrel{def}{=} \{g \circ \alpha \mid g \in A, \alpha \in \Omega\} \subseteq \partial G \ .$$

Lemma 2.15. ([9, Lemma 3.3]) *If H is an infinite subgroup of G then $\Lambda(H)$ contains at least two distinct points and the set $\Lambda(H)$ is H-invariant, i.e. for every $h \in H \quad h \circ \Lambda(H) = \Lambda(H)$.*

Thus every infinite subgroup H of G acts on its limit set $\Lambda(H)$ and this action is induced by the action of G on the boundary ∂G described above.

Lemma 2.16. ([9, Lemma 3.8]) *Let A be an infinite normal subgroup of a subgroup H in G. Then $\Lambda(A) = \Lambda(H)$.*

For an arbitrary subset $\Omega \subseteq \partial G$ denote by $cl(\Omega)$ its closure in the topology of the boundary ∂G.

Lemma 2.17. ([14, Lemma 6.4]) *Suppose $\Omega \subset \partial G$ is a subset having at least two distinct points. Then $\Lambda\big(St_G(\Omega)\big) \subseteq cl(\Omega)$.*

Combining lemmas 8.1.(5) and 8.2 from [14] together one obtains

Lemma 2.18. *Assume H is a subgroup of G and $A \subseteq G$ is a non-empty quasiconvex subset. If $\Lambda(H) \subseteq \Lambda(A)$ then there is a finite subset $P \subset G$ such that $H \subseteq A \cdot P = \bigcup_{x \in P} Ax$.*

3 Proofs of Theorems 2,3

Suppose $H = \langle \mathcal{B} \rangle$ is a subgroup of G with a finite generating set $\mathcal{B}$. If $h \in H$, then by $|h|_G$ and $|h|_H$ we will denote the lengths of the element h in the alphabets $\mathcal{A}$ and $\mathcal{B}$ respectively. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$. The *distortion function* (see [6]) $D_H : \mathbb{N}_0 \to \mathbb{N}_0$ of H in G is defined by

$$D_H(n) = max\{|h|_H \mid h \in H, \ |h|_G \leq n\} \ .$$

If $\alpha, \beta \ : \mathbb{N}_0 \to \mathbb{N}_0$ are two functions then we write $\alpha \preceq \beta$ if there are constants $K_1, K_2 > 0 : \ \alpha(n) \leq K_1 \beta(K_2 n)$ for every $n \in \mathbb{N}$. α and β are said to be equivalent if $\alpha \preceq \beta$ and $\beta \preceq \alpha$.

Evidently, the function D_H does not depend (up to this equivalence) on the choice of finite generating sets $\mathcal{A}$ of G and $\mathcal{B}$ of H. One can also notice that $D_H(n)$ is always at least linear (provided that H is infinite). Fix a linear function $L : \mathbb{N}_0 \to \mathbb{N}_0$ (for example, $L(n) = n$). If $D_H \preceq L$, H is called *undistorted.*

Lemma 3.1. [13, Lemma 1.6] *Let H be a finitely generated subgroup of a hyperbolic group G. Then H is quasiconvex if and only if H is undistorted in G.*

<u>Proof of lemma 1.2.</u> Let H be a quasiretract of the hyperbolic group G and let $N \triangleleft G$ satisfy $|G : HN| < \infty$, $card(H \cap N) < \infty$. Denote $\hat{G} = HN \leq G$. Then for the quotient group $K = \hat{G}/N$ there is a natural epimorphism $\varphi : \hat{G} \to K$ such that $K = \varphi(H)$ and $M \overset{def}{=} ker(\varphi) \cap H$ is finite.

The group $\hat{G}$ is generated by a finite set $\hat{\mathcal{A}}$ (because it is of finite index in a finitely generated group G). Hence K is generated by the finite set $\mathcal{C} = \varphi(\hat{\mathcal{A}})$. For every element $x \in \mathcal{C}$ choose one element $y \in H$ from its preimage under φ and denote by $\bar{\mathcal{C}}$ the subset of H consisting of them. Since

$$H/M \cong K \ ,$$

H is generated by the finite set $\mathcal{B} = \bar{\mathcal{C}} \cup M$.

Now one can define the corresponding length functions $|\cdot|_H$ and $|\cdot|_{\hat{G}}$ which satisfy the following properties:

$$\forall\ g \in \hat{G} \quad |\varphi(g)|_K \leq |g|_{\hat{G}} \ ,$$

$$\forall\ h \in H \quad |h|_H \leq |\varphi(h)|_K + 1 \ .$$

Combining these inequalities we get

$$\forall\ h \in H \quad |h|_H \leq |h|_{\hat{G}} + 1 \ .$$

Therefore, $D_H(n) \leq n+1$, i.e. H is undistorted in $\hat{G}$. The group G is hyperbolic and any its subgroup of finite index is quasiconvex, thus, by lemma 3.1, $\hat{G}$ is undistorted in G. Evidently the property that a subgroup is undistorted in a group is transitive, hence H is undistorted in G. It remains to apply lemma 3.1 to complete the proof. $\square$

<u>Proof of theorem 2.</u> It is enough to show that $M = ker\phi \cap H$ is finite. By the conditions,

$$M \subset P_1 Q^{-1} Q P_2 = \bigcup_{x \in P_1, y \in P_2} x Q^{-1} Q y \ .$$

Proving by contradiction, assume that M is infinite. Then, since the subsets P_1, P_2 are finite, there are elements $g \in P_1$ and $h \in P_2$ such that the intersection

$$A \overset{def}{=} M \cap g Q^{-1} Q h$$

is infinite. Therefore, $B = g^{-1} A h^{-1}$ is an infinite subset of $Q^{-1}Q$ satisfying

$$\phi(B) = \{\phi(g^{-1}h^{-1})\} \text{ – a one-element subset .}$$

It is easy to see that the latter is impossible if ϕ is a quasiisometry between Q and $\phi(Q)$ (since for any $u, v \in Q \;\; d(u,v) = |u^{-1}v|_G$, $d_1(\phi(u), \phi(v)) = |\phi(u^{-1}v)|_{G_1}$, $u^{-1}v \in Q^{-1}Q$). □

To prove theorem 3 we will need the following statement:

Lemma 3.2. [14, Thm. 2] *Assume that U is a finite union of quasiconvex products in a hyperbolic group G and the subgroups $F_1, F_2, \dots, F_l$ are all the members of U. If H is a subgroup of G and $H \subseteq U$ then for some $g \in G$ and $j \in \{1, 2, \dots, l\}$ one has $|H : (H \cap gF_jg^{-1})| < \infty$.*

Proof of theorem 3. Fix arbitrary finite subsets P_1, P_2 of G. Then the set $U = P_1Q^{-1}QP_2$ is also a finite union of quasiconvex products in the group G with the same members $F_1, \dots, F_l$. Now the claim of the theorem follows directly from lemma 3.2. □

4 Auxiliary Lemmas

Assume $\mathcal{X}$ is a δ-hyperbolic metric space with metric $d(\cdot,\cdot)$.

Lemma 4.1. *Suppose $\varkappa \ge 0$, $X, Y, Z, X', Y' \in \mathcal{X}$ and $X' \in \mathcal{O}_\varkappa([X,Z])$, $Y' \in \mathcal{O}_\varkappa([Y,Z])$. Then $(X'|Y')_Z \le (X|Y)_Z + 2\varkappa$.*

Proof. Let $X'' \in [X,Z]$, $Y'' \in [Y,Z]$ satisfy $d(X', X'') \le \varkappa$, $d(Y', Y'') \le \varkappa$. By the triangle inequality

$$(X'|Y')_Z = \frac{1}{2}\big(d(X',Z) + d(Y',Z) - d(X',Y')\big) \le$$

$$\le \frac{1}{2}\big(d(X'',Z) + d(Y'',Z) - d(X'',Y'') + 2d(X',X'') + 2d(Y',Y'')\big)\ .$$

Now, since $d(X,Z) = d(X'',Z) + d(X'',X)$ and $d(Y,Z) = d(Y'',Z) + d(Y'',Y)$, we achieve

$$(X'|Y')_Z \le \frac{1}{2}\big(d(X,Z) + d(Y,Z) - [d(X'',Y'') + d(X'',X) + d(Y'',Y)]\big) + 2\varkappa \le$$

$$\le \frac{1}{2}\big(d(X,Z) + d(Y,Z) - d(X,Y)\big) + 2\varkappa = (X|Y)_Z + 2\varkappa\ .$$

Q.e.d □

Lemma 4.2. *Let $\bar\lambda > 0$, $\bar c \ge 0$, $C_0 \ge 14\delta$, $C_1 = 12(C_0 + \delta) + \bar c + 1$ be given. Then for $\lambda = \bar\lambda/4 > 0$ there exist $c = c(\bar\lambda, \bar c, C_0) \ge 0$ satisfying the statement below.*
Assume $N \in \mathbb{N}$, $X_i \in \mathcal{X}$, $i = 0, \dots, N$, and q_i are $(\bar\lambda, \bar c)$-quasigeodesic paths between X_{i-1} and X_i in $\mathcal{X}$, $i = 1, \dots, N$. If $\|q_i\| \ge (C_1 + \bar c)/\bar\lambda$, $i = 1, \dots, N$, and $(X_{i-1}|X_{i+1})_{X_i} \le C_0$ for all $i = 1, \dots, N-1$, then the path q obtained as a consequent concatenation of $q_1, q_2, \dots, q_N$, is (λ, c)-quasigeodesic.

Proof. Let the number $\nu = \nu(\bar\lambda, \bar c) \geq 0$ be chosen according to the claim of lemma 2.1. Set $c = \frac{5}{2}(\nu + C_1) \geq 0$.

Suppose p is an arbitrary subpath of q. Then $p_- \in q_j$, $p_+ \in q_k$ for some $1 \leq j \leq k \leq N$. If $j = k$, p is a subpath of q_j and therefore, it is $(\bar\lambda, \bar c)$-quasigeodesic, hence it is (λ, c)-quasigeodesic.

Now let's assume that $j < k$. By our conditions and the choice of ν, there are points $U \in [X_{j-1}, X_j]$, $V \in [X_{k-1}, X_k]$ such that $d(p_-, U) \leq \nu$ and $d(p_+, V) \leq \nu$. $\|[X_{j-1}, X_j]\| \geq \bar\lambda \|q_j\| - \bar c \geq C_1$, similarly, $[X_{k-1}, X_k] \geq C_1$, hence after shifting the points U and V along the segments $[X_{j-1}, X_j]$ and $[X_{k-1}, X_k]$ (correspondingly) by distances at most C_1 we will obtain $\|[U, X_j]\| \geq C_1$, $\|[X_{k-1}, V]\| \geq C_1$, $d(U, p_-) \leq \nu + C_1$, $d(V, p_+) \leq \nu + C_1$ and, therefore, $d(p_-, p_+) \geq d(U, V) - 2\nu - 2C_1$.

According to the lemma 4.1, all the conditions of lemma 2.5 applied to the broken line $[U, X_j, \dots, X_{k-1}, V]$ are satisfied, hence

$$d(U, V) \geq \frac{1}{2} \|[U, X_j, \dots, X_{k-1}, V]\| \ .$$

Consequently,

$$d(p_-, p_+) \geq \frac{1}{2} \left(d(U, X_j) + \sum_{i=j}^{k-2} d(X_i, X_{i+1}) + d(X_{k-1}, V) \right) - 2\nu - 2C_1 \ .$$

Finally, we observe that $d(U, X_j) \geq \frac{1}{2} d(U, X_j) + \frac{C_1}{2} > \frac{1}{2}(d(U, X_j) + \bar c)$ and analogously for the other summands. Denote by q'_j, q'_k the segments of q_j and q_k from p_- to X_j and from X_{k-1} to p_+ correspondingly. We obtain

$$d(p_-, p_+) \geq \frac{1}{4} \left(d(U, X_j) + \bar c + \sum_{i=j}^{k-2} \Big(d(X_i, X_{i+1}) + \bar c \Big) + d(X_{k-1}, V) + \bar c \right) -$$
$$-2\nu - 2C_1 \geq$$

$$\geq \frac{1}{4} \left(d(p_-, X_j) + \bar c + \sum_{i=j}^{k-2} \Big(d(X_i, X_{i+1}) + \bar c \Big) + d(X_{k-1}, p_+) + \bar c \right) - \frac{5}{2}\nu - \frac{5}{2} C_1 \geq$$

$$\geq \frac{1}{4} \left(\bar\lambda \|q'_j\| + \sum_{i=j}^{k-2} \bar\lambda \|q_{i+1}\| + \bar\lambda \|q'_k\| \right) - \frac{5}{2}\nu - \frac{5}{2} C_1 \geq \frac{\bar\lambda}{4} \|p\| - \frac{5}{2}(\nu + C_1) \ .$$

The statement is proved. $\square$

Now let G be a δ-hyperbolic group, $\delta \geq 0$.

Lemma 4.3. *Let H be a non-elementary subgroup of a hyperbolic group G, and g be an H-suitable element. If $y \in C_H\big(E(H)\big)\backslash E(g)$ then there exists $N \in \mathbb{N}$ such that the element yg^n has infinite order in H and is H-suitable for every $n \geq N$.*

Proof. Observe that $E(g) \neq E(ygy^{-1})$ because, otherwise, we would have $yg^ky^{-1} = g^l$ for some non-zero integers k, l, and (1) would imply $y \in E(g)$ which is not true by the conditions of the lemma. Therefore $E(g) \cap E(ygy^{-1})$ is finite, hence, by lemma 2.7 there is $C_0' \geq 0$ such that $(g^l|yg^ky^{-1})_{1_G} \leq C_0'$ for any $k, l \in \mathbb{Z}$. Set $C_0 = C_0' + 2|y|_G + 14\delta$, then $C_0 \geq 14\delta$ and

$$(g^{-n}y^{-1}|yg^n)_{1_G} \leq (g^{-n}|yg^ny^{-1})_{1_G} + 2|y|_G \leq C_0 \quad \forall\, n \in \mathbb{N}. \tag{2}$$

First, let us show that $yg^n \in H^0$ if $n \in \mathbb{N}$ is sufficiently big.

Choose $C_1 = 12(C_0 + \delta) + 1$ and $N_1 \in \mathbb{N}$ so that $|yg^n|_G \geq C_1$ for all $n \geq N_1$. Suppose $(yg^n)^t = 1_G$ for some $t \in \mathbb{N}$ and $n \geq N_1$. Consider the broken line $[X_0, X_1, \dots, X_t]$ in $\Gamma(G, \mathcal{A})$ with $X_i = (yg^n)^i$, $i = 0, 1 \dots, t$. We can estimate

$$\big(X_{i-1}|X_{i+1}\big)_{X_i} = \big((yg^n)^{i-1}|(yg^n)^{i+1}\big)_{(yg^n)^i} = \big(g^{-n}y^{-1}|yg^n\big)_{1_G} \leq C_0\ .$$

It satisfies all the conditions of lemma 2.5, therefore

$$\|[X_0, X_t]\| \geq \frac{1}{2}\|[X_0, X_1, \dots, X_t]\| \geq C_1/2 > 0\ ,$$

but we assumed that $X_0 = X_t$. A contradiction. Hence the element yg^n has infinite order for each $n \geq N_1$.

Now, if $n \geq N_1$, $E(yg^n) \neq E(g)$ because, otherwise, we would obtain $yg^n \in E(g)$ which implies $y \in E(g)$ contradicting to the conditions of the lemma. Hence,

$$E(g) \cap E(yg^n) \subset T(g) = E(H) \subset E(yg^n), \ \text{ thus } \ E(g) \cap E(yg^n) = E(H).$$

Let w_1, w_2 be shortest words in the alphabet $\mathcal{A}$ representing y and g correspondingly. By lemma 2.2 there exist $\bar{\lambda} > 0$ and $\bar{c}' \geq 0$ such that any path in $\Gamma(G, \mathcal{A})$ labelled by the word w_2^n is $(\bar{\lambda}, \bar{c}')$-quasigeodesic for any $n \in \mathbb{N}$. Consequently, any path labelled by $w_1w_2^n$ is $(\bar{\lambda}, \bar{c})$-quasigeodesic where $\bar{c} = \bar{c}' + 2\|w_1\|$. Set $C_1 = 12(C_0 + \delta) + \bar{c} + 1$. Suppose that $n \geq (C_1 + \bar{c})/\bar{\lambda}$. Then $\|w_1w_2^n\| \geq (C_1 + \bar{c})/\bar{\lambda}$ and by (2) we can apply lemma 4.2 to find $\lambda > 0$ and $c \geq 0$ (not depending on n) such that any path labelled by $(w_1w_2^n)^t$ is (λ, c)-quasigeodesic for any $t \in \mathbb{Z}$.

Suppose $x \in E(yg^n)$. There is $k \in \mathbb{N}$ such that $x(yg^n)^kx^{-1} = (yg^n)^{\epsilon k}$ where $\epsilon \in \{1, -1\}$, hence $x(yg^n)^{lk}x^{-1} = (yg^n)^{\epsilon lk}$ for any $l \in \mathbb{N}$. Consider the geodesic quadrangle $Y_1Y_2Y_3Y_4$ in $\Gamma(G, \mathcal{A})$ with $Y_1 = 1_G$, $Y_2 = x$, $Y_3 = x(yg^n)^{lk}$, $Y_4 = x(yg^n)^{lk}x^{-1}$ and (λ, c)-quasigeodesic paths p between Y_2 and Y_3 and q between Y_1 and Y_4 labelled by the words $(w_1w_2^n)^{lk}$ and $(w_1w_2^n)^{\epsilon lk}$ correspondingly. Choose $\nu = \nu(\lambda, c)$ to be the constant given by lemma 2.1. Thus

$$p \subset \mathcal{O}_\nu([Y_2, Y_3]),\ [Y_2, Y_3] \subset \mathcal{O}_\nu(p),\ q \subset \mathcal{O}_\nu([Y_1, Y_4]),\ [Y_1, Y_4] \subset \mathcal{O}_\nu(q)\ .$$

Obviously, by taking the number l sufficiently large, one can find a subpath r of p labelled by w_2^n with its endpoints r_- and r_+ having distances at least $(|x|_G + \nu)$ from both of the vertices Y_2 and Y_3. Then an application of lemma 2.6 will give us

$$r_-, r_+ \in \mathcal{O}_{\nu+2\delta}([Y_1, Y_4]) \subset \mathcal{O}_{2\nu+2\delta}(q) .$$

Let denote u, v denote the points on the path q with $d(r_-, u) \le 2\nu + 2\delta$ and $d(r_+, v) \le 2\nu + 2\delta$, and let r' be the subpath of q (or q^{-1}) starting at u, ending at v. The length of r (and, hence, the length of $[r_-, r_+]$) depends on n, thus it can be made as large as we please, therefore r' will also be long (compared to $\|w_1\|$), consequently r' will have a subpath q' labelled by w_2^t, $t \in \mathbb{Z}$, with $|t| \ge \bar{\lambda} n/3$. Since the quadrangles in $\Gamma(G, \mathcal{A})$ are 2δ-slim, we achieve

$$[u, v] \subset \mathcal{O}_{2\delta}([r_-, r_+] \cup [r_-, u] \cup [r_+, v]) \subset \mathcal{O}_{2\nu+4\delta}([r_-, r_+]) \ , \quad \text{hence}$$

$$q' \subset \mathcal{O}_{\nu}([u, v]) \subset \mathcal{O}_{3\nu+4\delta}([r_-, r_+]) \subset \mathcal{O}_{4\nu+4\delta}(r) .$$

Consider the vertices $a_0 = q'_-, a_2, \dots, a_{|t|} = q'_+$ of the path q' such that the subpaths between a_{i-1} and a_i are labelled by w_2 (respectively, w_2^{-1} if $t < 0$) for every $1 \le i \le |t|$ (we will call them *phase* vertices). Then each of them is at distance at most $(4\nu + 4\delta + \|w_2\|)$ from some phase vertex of r. There are only finitely many words over the alphabet $\mathcal{A}$ of length at most $(4\nu + 4\delta + \|w_2\|)$, therefore, if n is sufficiently large, there will be two paths α and β connecting two different phase vertices of q' with some vertices of r having the same word w_3 written on them. Thus we achieve an equality in the group G:

$$w_2^s = w_3 w_2^{s'} w_3^{-1} \ \text{ for some } s, s' \in \mathbb{Z}\backslash\{0\} .$$

So, if z denotes the element of G represented by the word w_3, we have

$$z g^{s'} z^{-1} = g^s \tag{3}$$

Then $z \in E(g)$ by (1). By the construction, $x = (yg^n)^{\zeta} g^{\xi} z g^{\xi'} (yg^n)^{\zeta'}$ for some $\zeta, \xi, \zeta', \xi' \in \mathbb{Z}$. Note that $(yg^n)^{-\zeta} x (yg^n)^{-\zeta'} \in E(yg^n)$ and $g^{\xi} z g^{\xi'} \in E(g)$. Hence,

$$g^{\xi} z g^{\xi'} = (yg^n)^{-\zeta} x (yg^n)^{-\zeta'} \in E(yg^n) \cap E(g) = E(H) .$$

Now, since $yg^n \in C_H\big(E(H)\big)$ we obtain $x \in \langle yg^n \rangle \cdot E(H) = \langle yg^n \rangle \times E(H)$ for arbitrary x from $E(yg^n)$. This implies that yg^n is H-suitable. $\square$

Lemma 4.4. *Let G be a hyperbolic group, $k \in \mathbb{N}$ and let $g_1, g_2, \dots, g_k$ be pairwise non-commensurable elements of G. Consider $y_i \in G\backslash E(g_i)$ for each $i = 1, 2, \dots, k$. Then there exists $N \in \mathbb{N}$ such that the elements $y_1 g_1^n, \dots, y_k g_k^n$ have infinite order and are pairwise non-commensurable if $n \ge N$.*

Proof. By the same argument as above, $y_i g_i^n \in G^0$ for each $i \in \{1, 2, \dots, k\}$. Suppose that $y_i g_i^n$ is commensurable with $y_j g_j^n$, $1 \le i < j \le k$. Then there is $x \in G$ satisfying $x (y_i g_i^n)^t x^{-1} = (y_j g_j^n)^{t'}$ for some $t, t' \in \mathbb{Z}\backslash\{0\}$. Therefore, if n is sufficiently large, we can prove that there exist $z \in G$ and $s, s' \in \mathbb{Z}\backslash\{0\}$ such that $z g_i^{s'} z^{-1} = g_j^s$ in exactly the same way as we proved (3). Which leads to a contradiction with the assumptions of the lemma. $\square$

Lemma 4.5. *Suppose G is a hyperbolic group, H is its non-elementary subgroup and $\alpha_1, \ldots, \alpha_n$ are points on the boundary ∂G. Then H has a non-elementary subgroup K that is quasiconvex in G and $\alpha_i \notin \Lambda(K) \subset \partial G$ for every $i = 1, 2, \ldots, n$.*

Proof. Induction on n.

Let $n = 1$. Choose elements of infinite order $g_1, g_2 \in H$ with $E(g_1) \neq E(g_2)$ and let the words w_1, w_2 over the alphabet $\mathcal{A}$ represent them. Following the proof of [16, Cor. 6], we get $M \in \mathbb{N}$, $\lambda > 0$ and $c \geq 0$ such that any word of the form

$$w_{i_1}^{mk_1} \cdot \ldots \cdot w_{i_s}^{mk_s}$$

where $i_j \in \{1, 2\}$, $i_j \neq i_{j+1}$, $k_j \in \mathbb{Z}\backslash\{0\}$ and $m \geq M$, is (λ, c)-quasigeodesic. Take $\nu = \nu(\lambda, c)$ from lemma 2.1. Then the subgroup $K_1 = \langle g_1^m, g_2^m \rangle \leq H$ is free of rank 2 and ε-quasiconvex in G ($\varepsilon = \nu + m \cdot \max\{\|w_1\|, \|w_2\|\}$). By taking m large enough, we can obtain $|H : K_1| = \infty$.

Now, if $\alpha_1 \notin \Lambda(K_1)$, there is nothing to prove. So, assume that $\alpha_1 \in \Lambda(K_1)$. By lemmas 2.8 and 2.9 there exists $h \in H$ such that $card(K_1 \cap hK_1h^{-1}) < \infty$. The subgroup $hK_1h^{-1} \leq H$ is non-elementary and quasiconvex in G, hence, using lemma 2.7 and the definition of the boundary ∂G we obtain

$$\Lambda(K_1) \cap \Lambda(hK_1h^{-1}) = \emptyset \text{ in } \partial G \ .$$

Consequently, $\alpha_1 \notin \Lambda(hK_1h^{-1})$.

Assume, now, that $n > 1$. And the induction hypothesis is verified for $\alpha_1, \ldots, \alpha_{n-1} \in \partial G$. I.e. there is a non-elementary subgroup $K' \leq H$ with $\alpha_i \notin \Lambda(K')$, $1 \leq i \leq n-1$. Using the base of our induction, we obtain a non-elementary subgroup $K \leq K' \leq H$ that is quasiconvex in G and $\alpha_n \notin \Lambda(K)$. Since $\Lambda(K) \subseteq \Lambda(K')$, K satisfies all the conditions needed.

The proof of the lemma is complete. □

It is a well-known fact that the set of all rational points $\{g^\infty \mid g \in G^0\}$ is dense in the group boundary ∂G (see, for example, [2, Theorem], [7, 4,8.2D]). We will need a bit stronger statement:

Lemma 4.6. *Assume H is a non-elementary subgroup of a hyperbolic group G and $\alpha \in \partial G$. Then $\Lambda(H) \subseteq cl(H \circ \alpha)$ where $H \circ \alpha$ is the orbit of α under the action of H and $cl(H \circ \alpha)$ is its closure inside of ∂G.*

Proof. Since H is non-elementary, the set $H \circ \alpha$ consists of more than one point. By definition, $H \subset St_G(H \circ \alpha)$, hence, applying lemma 2.17, we achieve

$$\Lambda(H) \subseteq cl(H \circ \alpha) \ .$$

Q.e.d. □

5 On condition $(*)$

Let G be a δ-hyperbolic group, $Q \subseteq G$ – η-quasiconvex subset.

Lemma 5.1. *The subset $Q^{-1}Q \subseteq G$ is $(\eta+\delta)$-quasiconvex.*

Proof. Consider arbitrary $x \in Q^{-1}Q$, $x = u^{-1}v$ where $u, v \in Q$. Then $[u,v] \subset \mathcal{O}_\eta(Q)$. Since the metric on $\Gamma(G,\mathcal{A})$ is invariant under the action of G by left translations, we have

$$[1_G, x] = u^{-1} \circ [u,v] \subset \mathcal{O}_\eta(u^{-1}Q) \subset \mathcal{O}_\eta(Q^{-1}Q) \ . \tag{4}$$

Since the geodesic triangles in $\Gamma(G,\mathcal{A})$ are δ-slim, for any two $x_1, x_2 \in Q^{-1}Q$ using (4) one obtains

$$[x_1, x_2] \subset \mathcal{O}_\delta([1_G, x_1] \cup [1_G, x_2]) \subseteq \mathcal{O}_{\delta+\eta}(Q^{-1}Q) \ .$$

The lemma is proved. □

Lemma 5.2. *Suppose $S, Q \subseteq G$ and the subset Q is η-quasiconvex. Then on the boundary ∂G of the group G one has*

$$\Lambda(S \cdot Q) \subseteq \Lambda(S) \cup (S \cdot Q) \circ \Lambda(Q^{-1} \cdot Q) \ ,$$

$$\Lambda(S \cdot Q^{-1}) \subseteq \Lambda(S) \cup (S \cdot Q^{-1} \cdot Q) \circ \Lambda(Q^{-1}) \ .$$

Proof. Let $P \subseteq G$, consider an arbitrary limit point $\alpha \in \Lambda(SP)$. There is a sequence $(z_i)_{i\in\mathbb{N}}$ converging to infinity in G with $z_i = x_i y_i$, $x_i \in S$, $y_i \in P$ for all $i \in \mathbb{N}$.

I. Suppose, first, that $\sup_{i\in\mathbb{N}} (z_i|x_i)_{1_G} = \infty$. Then one can find a sequence $(i_j)_{j\in\mathbb{N}}$ of natural numbers such that

$$\lim_{j\to\infty} (z_{i_j}|x_{i_j})_{1_G} = \infty \ .$$

But $\lim\limits_{j\to\infty} z_{i_j} = \lim\limits_{i\to\infty} z_i = \alpha$, which implies that $(x_{i_j})_{j\in\mathbb{N}}$ also converges to infinity and

$$\lim_{j\to\infty} x_{i_j} = \lim_{j\to\infty} z_{i_j} = \alpha \ .$$

Thus, $\alpha \in \Lambda(S)$.

II. Therefore, we can now assume that there is a number $M \geq 0$ such that $(z_i|x_i)_{1_G} \leq M$ for every $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ consider a geodesic triangle in $\Gamma(G,\mathcal{A})$ with vertices 1_G, x_i and z_i. It is δ-thin, hence $d(1_G, [x_i, z_i]) \leq M + \delta$.

a). Suppose $P = Q$. Fix an arbitrary element $q \in Q$ and let $\varkappa = |q|_G$. Then

$$[1_G, y_i] \in \mathcal{O}_\delta([1_G, q] \cup [q, y_i]) \subset \mathcal{O}_{\delta+\varkappa}([q, y_i]) \subset \mathcal{O}_{\delta+\varkappa+\eta}(Q) \ .$$

Using the left translation-invariance of the word metric, we get

$$[x_i, z_i] = [x_i, x_i y_i] \subset \mathcal{O}_{\delta+\varkappa+\eta}(x_i Q) \ .$$

Consequently, there exists $q_i \in Q$ satisfying

$$d(1_G, x_i q_i) = |x_i q_i|_G \leq M + 2\delta + \varkappa + \eta \ \text{ for every } i \in \mathbb{N}.$$

The group G has only finitely many elements in a ball of finite radius, hence, by passing to a subsequence, we can assume that $x_i q_i = p \in SQ$ for all $i \in \mathbb{N}$. Thus, $z_i = x_i q_i q_i^{-1} y_i = p q_i^{-1} y_i \in pQ^{-1}Q$ for every i, which implies

$$\alpha \in \Lambda(pQ^{-1}Q) = p \circ \Lambda(Q^{-1}Q) \subset (SQ) \circ \Lambda(Q^{-1}Q) \ .$$

b). Assume, $P = Q^{-1}$. Then $y_i^{-1} \in Q$ hence

$$[x_i, z_i] = [z_i y_i^{-1}, z_i] \subset \mathcal{O}_{\delta+\varkappa+\eta}(z_i Q) \ .$$

So, there are elements $q_i \in Q$ such that

$$d(1_G, z_i q_i) = |z_i q_i|_G \leq M + 2\delta + \varkappa + \eta \ \text{ for every } i \in \mathbb{N}.$$

As before, we can suppose that $z_i q_i = p \in SQ^{-1}Q$ for all $i \in \mathbb{N}$. Thus $z_i = p q_i^{-1} \in pQ^{-1}$ for every i, implying

$$\alpha \in \Lambda(pQ^{-1}) = p \circ \Lambda(Q^{-1}) \subset (SQ^{-1}Q) \circ \Lambda(Q^{-1}) \ .$$

Q.e.d. □

Lemma 5.3. *Assume that H is a subgroup and A is a non-empty quasiconvex subset of a hyperbolic group G. The following conditions are equivalent:*
1. *There are finite subsets $P_1, P_2 \subset G$ such that $H \subseteq P_1 \cdot A \cdot P_2$;*
2. $\Lambda(H) \subseteq G \circ \Lambda(A)$.

Proof. The implication $1 \Rightarrow 2$ is an immediate consequence of lemma 2.13 since

$$\Lambda(P_1 \cdot A \cdot P_2) = \Lambda(P_1 A) = P_1 \circ \Lambda(A) \subset G \circ \Lambda(A) \ .$$

Now let's show that 2 implies 1. Denote $\Omega = \Lambda(A)$.

If the subgroup H is finite then the claim is trivial.

If H is infinite elementary then $card\big(\Lambda(H)\big) = 2$, hence, according to the condition 2, there are elements $g_1, g_2 \in G$ such that

$$\Lambda(H) \subset g_1 \circ \Lambda(A) \cup g_2 \circ \Lambda(A) = \Lambda(g_1 A \cup g_2 A) \ .$$

The subset $g_1 A \cup g_2 A \subset G$ is quasiconvex by lemma 2.3, hence we can apply lemma 2.18 to get a finite subset P_2 of G satisfying $H \subseteq (g_1 A \cup g_2 A) P_2 = P_1 A P_2$ where $P_1 = \{g_1, g_2\}$.

Thus we can assume that the subgroup H is non-elementary.

Case 1. Suppose that for some $g \in G$ $g \circ \Omega$ contains a non-empty open set U of the subspace $\Lambda(H)$, i.e. $U = U' \cap \Lambda(H)$ for some open set $U' \subseteq \partial G$. Then, by lemma 4.6, for any $\beta \in \Lambda(H)$ there exists $h \in H$ such that $h \circ \beta \in U'$. On the other hand, $h \circ \beta \in \Lambda(H)$ by lemma 2.15, thus, $h \circ \beta \in U$, i.e. $\beta \in h^{-1} \circ U$. Consequently,

$$\Lambda(H) \subseteq \bigcup_{h \in H} h \circ U \ . \qquad (5)$$

The space $\Lambda(H)$ is a closed subspace of the compact metric space ∂G, hence it is compact itself and one can choose a finite subcover of the open cover from (5). Thus

$$\Lambda(H) \subseteq \bigcup_{i=1}^{N} h_i \circ U \subseteq \bigcup_{i=1}^{N} h_i \circ (g \circ \Omega) = \bigcup_{i=1}^{N} h_i \circ \Lambda\big(gA\big) = \Lambda\left(\bigcup_{i=1}^{N} h_i g A\right) = \Lambda(P_1 A)$$

according to lemma 2.13, where $P_1 = \bigcup_{i=1}^{N} h_i g \subset G$, $card(P_1) < \infty$.

The set $P_1 A = \bigcup_{y \in P_1} yA$ is quasiconvex as a finite union of quasiconvex sets, therefore, we can apply lemma 2.18 to find a finite subset P_2 of the group G such that $H \subseteq P_1 \cdot A \cdot P_2$ as we needed.

Hence, we can proceed to

Case 2. For every $g \in G$ $\ g \circ \Omega$ contains no non-empty open subsets of $\Lambda(H)$. Ω is a closed subset of the boundary ∂G by lemma 2.13.(b), thus $g \circ \Omega$ is also closed and, hence, $\big(g\circ\Omega\big)\cap\Lambda(H)$ is a closed nowhere dense subset of the compact metric space $\Lambda(H)$. Evidently, $\Lambda(H)$ is a Baire space (it is locally compact and Hausdorff). Since the group G is countable, the set

$$(G \circ \Omega) \cap \Lambda(H) = \bigcup_{g \in G} (g \circ \Omega) \cap \Lambda(H)$$

is of the first category in the space $\Lambda(H)$, hence, by a well-know theorem from topology (see, for instance, [4, Chap. XI, Thm. 10.5]),

$$\Lambda(H) \neq (G \circ \Omega) \cap \Lambda(H) \ ,$$

therefore, $\Lambda(H) \nsubseteq G \circ \Lambda(A)$ which is a contradiction to our assumptions. Thus, Case 2 is impossible. $\square$

Remark 2. Assume (in the notations of lemma 5.3) that H is non-elementary. Then the following two properties are equivalent:

1. There are no finite subsets P_1, P_2 of G such that $H \subset P_1 A P_2$;

2. On the hyperbolic boundary ∂G for any $g \in G$ the set $\big(g \circ \Lambda(A)\big) \cap \Lambda(H)$ is nowhere dense in $\Lambda(H)$.

In the proof of lemma 5.3 the condition 1 automatically puts us into the Case 2, thus 1 $\Rightarrow$ 2. Now, if the property 2 holds and the property 1 doesn't, we can find some finite subsets $P_1, P_2 \subset G$ satisfying $H \subset P_1 A P_2$. Therefore, by lemma 2.13,

$$\Lambda(H) \subset \Lambda(P_1 A P_2) = \Lambda(P_1 A) \ .$$

Hence $\Lambda(H) = \bigcup_{g \in P_1} \big(g \circ \Lambda(A) \cap \Lambda(H)\big)$ contradicting to 2 because of the fact that a finite union of nowhere dense subsets is nowhere dense in $\Lambda(H)$. Hence 2 $\Rightarrow$ 1.

Lemma 5.4. *Suppose that H is a non-elementary subgroup and Q, S are quasiconvex subsets of a hyperbolic group G. Assume that for any two finite subsets P_1, P_2 of the group G*

$$H \not\subseteq P_1 Q^{-1} Q P_2 \text{ and } H \not\subseteq P_1 S^{-1} S P_2 \ . \tag{6}$$

Then the (quasiconvex) subsets $T_1 = Q \cup S$ and $T_2 = QS$ satisfy the same property: for any $i \in \{1,2\}$ and arbitrary finite $P_1, P_2 \subset G$ one has

$$H \not\subseteq P_1 T_i^{-1} T_i P_2 \ .$$

Proof. a). Since $T_1^{-1} = Q^{-1} \cup S^{-1}$, we can use lemmas 2.13.(c) and 5.2 to obtain

$$\Lambda\left(T_1^{-1}T_1\right) = \Lambda\left(Q^{-1}Q \cup Q^{-1}S \cup S^{-1}Q \cup S^{-1}S\right) =$$
$$= \Lambda\left(Q^{-1}Q\right) \cup \Lambda\left(Q^{-1}S\right) \cup \Lambda\left(S^{-1}Q\right) \cup \Lambda\left(S^{-1}S\right) \subseteq$$
$$\subseteq \Lambda\left(Q^{-1}Q\right) \cup \Lambda(Q^{-1}) \cup G \circ \Lambda\left(S^{-1}S\right) \cup \Lambda(S^{-1}) \cup G \circ \Lambda\left(Q^{-1}Q\right) \cup \Lambda\left(S^{-1}S\right) =$$
$$= G \circ \Lambda\left(S^{-1}S\right) \cup G \circ \Lambda\left(Q^{-1}Q\right) = G \circ \Lambda\left(S^{-1}S \cup Q^{-1}Q\right)$$

(here we used the fact that if $s \in S$ then $S^{-1}s \subset S^{-1}S$, and by lemma 2.13.(d) $\Lambda(S^{-1}) = \Lambda(S^{-1}s) \subset \Lambda(S^{-1}S)$ and similarly for Q).
Thus, $G \circ \Lambda\left(T_1^{-1}T_1\right) \subseteq G \circ \Lambda\left(S^{-1}S \cup Q^{-1}Q\right)$.

The conditions (6) imply (by remark 2) that for any $g \in G$ the sets $g \circ \Lambda(Q^{-1}Q) \cap \Lambda(H)$ and $g \circ \Lambda(S^{-1}S) \cap \Lambda(H)$ are nowhere dense in $\Lambda(H)$, therefore the set $G \circ \Lambda(Q^{-1}Q \cup S^{-1}S) \cap \Lambda(H)$ is of the first category in the compact metric space $\Lambda(H)$. Consequently,

$$\Lambda(H) \not\subseteq G \circ (Q^{-1}Q \cup S^{-1}S). \tag{7}$$

Hence

$$\Lambda(H) \not\subseteq G \circ (T_1^{-1}T_1) \ .$$

The subset $T_1 \subset G$ is quasiconvex according to lemma 2.3. Therefore, to finish the proof it remains to apply lemma 5.1 to $T_1^{-1}T_1$ and then lemma 5.3 to $T_1^{-1}T_1$ and H.

b). The proof for T_2 is similar. Note that $T_2^{-1} = S^{-1}Q^{-1}$, hence, by lemma 5.2

$$\Lambda\left(T_2^{-1}T_2\right) = \Lambda\left(S^{-1}Q^{-1}QS\right) \subset \Lambda\left(S^{-1}Q^{-1}Q\right) \cup G \circ \Lambda\left(S^{-1}S\right) \ .$$

$Q^{-1}Q \subset G$ is quasiconvex by lemma 5.1 and $(Q^{-1}Q)^{-1} = Q^{-1}Q$ therefore, applying lemma 5.2 two more times we obtain

$$\Lambda\left(T_2^{-1}T_2\right) \subset \Lambda\left(S^{-1}\right) \cup G \circ \Lambda\left(Q^{-1}Q\right) \cup G \circ \Lambda\left(S^{-1}S\right) = G \circ \left(Q^{-1}Q \cup S^{-1}S\right) \ .$$

Consequently, recalling (7), one gets $\Lambda(H) \not\subseteq G \circ (T_2^{-1}T_2)$. Since T_2 is a quasiconvex subset of G (lemma 2.3), T_2 and H satisfy the needed property by lemma 5.3. $\square$

Corollary 4. *Let Q be a quasiconvex subset of a hyperbolic group G and H be a non-elementary subgroup of G. Assume, in addition, that $Q^{-1} \subset G$ is also quasiconvex. Then the following properties are equivalent:*

1. *For arbitrary finite subsets P_1, P_2 of G, $H \not\subseteq P_1 Q P_2$;*

2. *For arbitrary finite subsets P_1, P_2 of G, $H \not\subseteq P_1 Q^{-1} Q P_2$.*

Proof. Evidently, 2 implies 1. So, let's assume that 1 holds and prove 2. Since the subset Q^{-1} is quasiconvex, we are able to apply lemma 5.2 to achieve

$$\Lambda\left(Q^{-1}Q\right) \subset \Lambda\left(Q^{-1}\right) \cup G \circ \Lambda\left(Q\right) \ .$$

Thus, $G \circ \Lambda\left(Q^{-1}Q\right) \subset G \circ \left(\Lambda(Q^{-1}) \cup \Lambda(Q)\right)$. Observe that the property 1 is equivalent to $H \not\subseteq P_1 Q^{-1} P_2$ for any finite $P_1, P_2 \subset G$ (because $H^{-1} = H$). Consequently, by remark 2,

$$\Lambda(H) \not\subseteq G \circ \left(\Lambda(Q^{-1}) \cup \Lambda(Q)\right) \ , \quad \text{hence} \ \ \Lambda(H) \not\subseteq G \circ \Lambda\left(Q^{-1}Q\right) \ .$$

Now, by applying lemma 5.2, we can conclude that the property 2 holds. □

Example. We observe that the implication $1 \Rightarrow 2$ in the latter corollary may fail if Q^{-1} is not quasiconvex: let $G = F(x, y)$ be the free group with free generators x, y. Set Q to be the set of all reduced words w over the alphabet $\{x^{\pm 1}, y^{\pm 1}\}$ satisfying the property: if $k \in \mathbb{N}$ and $2^k \leq \|w\|$ then the letter on 2^k-th place in w is x. Thus,

$$Q = \{x, x^{-1}, y, y^{-1}, x^2, yx, y^{-1}x, x^3, x^2y, x^2y^{-1}, yxy,$$
$$yxy^{-1}, y^{-1}xy, y^{-1}xy^{-1}, \ldots\} \subset G \ .$$

The subset Q is quasiconvex in G since any prefix of a word from Q belongs to Q. It is not difficult to show that $G \not\subseteq P_1 Q P_2$ for any finite subsets $P_1, P_2 \subset G$; nevertheless, $G = Q^{-1}Q$ (because any reduced word is a suffix of some word from Q).

6 Small Cancellations over Hyperbolic Groups

In this section we list the results from [16] that provide the main tool for proving theorem 1.

Let $\mathcal{R}$ be a *symmetrized* set of words in an alphabet $\mathcal{A}$, i.e. if $R \in \mathcal{R}$ then R^{-1} and any cyclic permutation of R belong to $\mathcal{R}$.

Suppose G is a group with generating set $\mathcal{A}$ and $\mathcal{R}$ is a symmetrized set of words over $\mathcal{A}$. For the given constants $\varepsilon \geq 0$, $\mu > 0$, $0 < \lambda \leq 1$, $c \geq 0$, $\rho > 0$ one defines the *generalized small cancellation conditions* $C(\varepsilon, \mu, \lambda, c, \rho)$, $C_1(\varepsilon, \mu, \lambda, c, \rho)$, $C_2(\varepsilon, \mu, \lambda, c, \rho)$ and $C_3(\varepsilon, \mu, \lambda, c, \rho)$ (see [16, Chapter 4]).

Remark 3. In the definition of the generalized small cancellation conditions $C(\varepsilon, \mu, \lambda, c, \rho)$ and $C_j(\varepsilon, \mu, \lambda, c, \rho)$, $j = 1, 2, 3$, λ, c appear only in the condition on every word $R \in \mathcal{R}$ to be (λ, c)-quasigeodesic (a word R is called (λ, c)-quasigeodesic if any path in $\Gamma(G, \mathcal{A})$ labelled by R is (λ, c)- quasigeodesic). The number ρ stands for a minimal length of a relation from $\mathcal{R}$; μ has an analogous meaning to the constant in the classical small cancellation condition $C'(\mu)$; subwords of defining relations are distinguished up to factors of length at most ε.

Consider non-elementary subgroups $H_1, \dots, H_k$ (some of them may coincide) of a hyperbolic group G, elements $g_i \in H_i$ chosen according to the claim of lemma 2.11 and elements $x_{i0} \in C_G(E(H_i))\backslash E(g_i)$, $i = 1, \dots, k$. Let g_i, $x_{i0}, \dots, x_{il}$ be represented by words W_i, $X_{i0}, \dots, X_{il}$ over the alphabet $\mathcal{A}$ of minimal length, $i = 1, \dots, k$.

As the system $\mathcal{R} = \mathcal{R}_{k,l,m}(W_1, \dots, W_k, X_{10}, \dots, X_{kl}, m)$ consider all cyclic permutations of $R_i^{\pm 1}$ where

$$R_i \equiv X_{i0} W_i^m X_{i1} W_i^m \dots X_{il} W_i^m \ , \quad i = 1, 2, \dots, k \ .$$

Then we have

Lemma 6.1. ([16, Lemma 4.2]) *For the words $W_1, \dots, W_k$, $X_{10}, \dots, X_{k0}$ given above, there exist $\lambda > 0$ such that for any $\mu > 0$ there are $l \in \mathbb{N}$ and $c \geq 0$ such that for any $\varepsilon \geq 0$, $\rho > 0$ there are $m_0 \in \mathbb{N}$ and words $X_{11}, \dots, X_{kl}$ such that the system $\mathcal{R}_{k,l,m}$ satisfies $C(\varepsilon, \mu, \lambda, c, \rho)$ and $C_1(\varepsilon, \mu, \lambda, c, \rho)$-conditions if $m \geq m_0$.*

Remark 4. Moreover, from the proof of this lemma it follows that the elements $x_{i1}, \dots, x_{il} \in H_i$ can be chosen right after the choice of $l \in \mathbb{N}$ to be any elements that satisfy properties 1)-4) of lemma 2.12 for $g = g_i$ and $H = H_i$, $i = 1, 2, \dots, k$.

In this article we assume that the concepts of a *Van Kampen* (*circular*) *diagram* and a *Schupp* (*annular*) *diagram* over a group presentation are known to the reader (see, for instance, [11]). Let $\mathcal{O}$ denote the system of all relations (not only defining) in the group G. Let $\mathcal{R}$ be some symmetrized set of additional relations over the alphabet $\mathcal{A}$. The group G_1 will be defined by its presentation:

$$G_1 = \langle \mathcal{A} \parallel \mathcal{O} \cup \mathcal{R} \rangle \ . \tag{8}$$

Thus G_1 is a quotient of the group G by the subgroup $\mathcal{N} = \langle \mathcal{R}^G \rangle$ that is a normal closure of the set of elements in G represented by words from $\mathcal{R}$.

Inheriting the terminology from [16] the faces of a Van Kampen diagram with boundary labels from $\mathcal{O}$ (from $\mathcal{R}$) will be called 0-faces ($\mathcal{R}$-faces).

In [16, Chapter 5] A. Ol'shanskii also introduces *reduced* diagrams (the number of $\mathcal{R}$-faces in them can not be reduced after a finite number of certain elementary transformations; in particular, diagrams with minimal number of $\mathcal{R}$-faces are reduced). Further we will only need to know that for any word W that is trivial in the group G_1 there exists a reduced circular diagram over the presentation (8) whose boundary label is letter-to-letter equal to W. And if two words U and V are conjugate in G_1 then there is a reduced annular diagram over the presentation (8) whose boundary contours have labels (letter-to-letter) equal to U and V respectively.

Later in this paper we will consider diagrams over G and G_1 (with the presentations $G = \langle \mathcal{A} \parallel \mathcal{O} \rangle$ and (8)), so all of them will be over the alphabet $\mathcal{A}$. A path q inside such a diagram will be called (λ, c)-*quasigeodesic* if a (any) path in the Cayley graph $\Gamma(G, \mathcal{A})$ of the group G with the same label as q is (λ, c)-quasigeodesic.

The boundary $\partial\Delta$ of a diagram will be divided into at most 4 distinguished subpaths (called *sections*) each of which will be $(\lambda, c)-$quasigeodesic (for some given $\lambda > 0$, $c \geq 0$).

Suppose $\varepsilon \geq 0$ is a given number. Consider a simple closed path $o = p_1q_1p_2q_2$ in a diagram Δ over G_1, such that q_1 is a subpath of the boundary cycle of an $\mathcal{R}$-cell Π and q_2 is a subpath of a section q of $\partial\Delta$. Let Γ denote the subdiagram of Δ bounded by o. Assuming that Γ has no holes, no $\mathcal{R}$-faces and $\|p_1\|, \|p_2\| \leq \varepsilon$, it will be called an ε-*contiguity subdiagram* of Π to q. The ratio $\|q_1\|/\|\partial\Pi\|$ will be called the *contiguity degree* of Π to q and denoted (Π, Γ, q).

The following analog of Grindlinger's lemma is proved in [16, Lemma 6.6] (here we include a correction mentioned in [18]):

Lemma 6.2. *For any hyperbolic group G and any $\lambda > 0$ there is $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ and $c \geq 0$ there are $\varepsilon \geq 0$ and $\rho > 0$ with the following property:*

Let the symmetrized presentation (8) satisfy the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition. Furthermore, let Δ be a reduced circular diagram over G_1 whose boundary is decomposed into a product of (λ, c)-quasigeodesic sections $q^1, \ldots, q^r$ where $1 \leq r \leq 4$. Then, provided Δ has an $\mathcal{R}$-face, there exists an $\mathcal{R}$-face Π in Δ and disjoint ε-contiguity subdiagrams $\Gamma_1, \ldots, \Gamma_r$ (some of them may be absent) of Π to $q^1, \ldots, q^r$ respectively, such that

$$(\Pi, \Gamma_1, q^1) + \ldots + (\Pi, \Gamma_r, q^r) > 1 - 23\mu \ . \qquad (9)$$

The next lemma is an analog of the previous one for annular diagrams.

Lemma 6.3. [16, Lemma 8.1] *For any hyperbolic group G and any $\lambda > 0$ there is $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ and $c \geq 0$ there are $\varepsilon \geq 0$ and $\rho > 0$ with the following property:*

Let the symmetrized presentation (8) satisfy the $C_1(\varepsilon, \mu, \lambda, c, \rho)$-condition. Further, let Δ be a reduced annular diagram over G_1 with boundary contours $p = p_1p_2$, $q = q_1q_2$ such that p_1, p_2, q_1, q_2 are (λ, c)-quasigeodesic. Then, provided Δ has an $\mathcal{R}$-face, there exists an $\mathcal{R}$-face Π in Δ and disjoint ε-contiguity subdiagrams $\Gamma_1, \ldots, \Gamma_4$ (some of them may be absent) of Π to p_1, p_2, q_1, q_2 respectively, such that

$$(\Pi, \Gamma_1, p_1) + (\Pi, \Gamma_2, p_2) + (\Pi, \Gamma_3, q_1) + (\Pi, \Gamma_4, q_2) > 1 - 23\mu \ . \qquad (10)$$

Collecting together the claims of Lemmas 6.7,7.4 and 7.5 from [16] we obtain

Lemma 6.4. *Suppose G is a non-elementary hyperbolic group and $H'_1, \ldots, H'_{k'}$ – its non-elementary subgroups. Then for any $\lambda > 0$ there is $\mu_0 > 0$ such that* *for any $\mu \in (0, \mu_0]$ and $c \geq 0$ there is $\varepsilon \geq 0$ such that for any $N > 0$ there exists $\rho > 0$ with the following property:*

Let the symmetrized presentation (8) satisfy the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition. Then the quotient G_1 (8) is a non-elementary hyperbolic group and the images of the subgroups $H'_1, \ldots, H'_{k'}$ are non-elementary in G_1. Moreover, $W = 1$ in G_1 if and only if $W = 1$ in G for every word W with $\|W\| \leq N$.

7 Main Construction

Assume, now, that we are in the conditions of Theorem 1. The δ-hyperbolic group G is generated by the symmetrized set $\mathcal{A} = \{a'_1, \ldots, a'_r\}$. Set $s = kr$ and define $a_1, \ldots, a_s$, $\hat{H}_1, \ldots, \hat{H}_s$ as follows:

$$a_{ir+j} = a'_j,\ \hat{H}_{ir+j} = H_{i+1} \quad \text{if} \quad 1 \leq j \leq r,\ 0 \leq i \leq k-1\ ,$$

i.e. $a_1 = a'_1, \ldots, a_r = a'_r, a_{r+1} = a'_1, a_{r+2} = a'_2, \ldots$, $\hat{H}_1 = H_1, \ldots, \hat{H}_r = H_1$, $\hat{H}_{r+1} = H_2$, $\hat{H}_{r+2} = H_2, \ldots$.

Since every $\hat{H}_i$ is a G-subgroup, we can find $b_i \in \hat{H}_i$ such that $a_i b_i^{-1} \in C_G\big(E(\hat{H}_i)\big)$ (such a choice is possible because $E(\hat{H}_i) = E(G)$ and $|\hat{H}_i : K(\hat{H}_i)| = |G : K(G)|$).

For each $i = 1, 2, \ldots, s$, the subgroup $F_i = C_{\hat{H}_i}\big(E(\hat{H}_i)\big)$ has finite index in $\hat{H}_i$, hence, $\Lambda(\hat{H}_i) = \Lambda(F_i)$ by parts (c) and (d) of lemma 2.13. The set $Q^{-1}Q$ is quasiconvex by lemma 5.1, thus, according to the assumptions of theorem 1, we can apply lemma 5.3 to find the points on the boundary ∂G:

$$\alpha_i \in \Lambda(F_i) \backslash \big(G \circ \Lambda(Q^{-1}Q)\big)$$

and a sequence $\Big(y_j^{(i)}\Big)_{j \in \mathbb{N}} \subset F_i$ with $\lim\limits_{j \to \infty} y_j^{(i)} = \alpha_i,\ i = 1, 2, \ldots, s$.

The set $\{y_j^{(i)} \mid j \in \mathbb{N}\}$ is infinite, therefore, the set $\{(y_j^{(i)})^{-1} \mid j \in \mathbb{N}\}$ is also infinite, hence, by lemma 2.13 it has a limit point $\beta_i \in \partial G$. So, after passing to a subsequence, we can assume that

$$\lim_{j \to \infty} (y_j^{(i)})^{-1} = \beta_i,\ i = 1, 2, \ldots, s.$$

Using lemma 2.11 one can find an $\hat{H}_i$-suitable element $g_i \in \hat{H}_i^0$ for every $i = 1, \ldots, s$, so that the elements $g_1, \ldots, g_s$ are pairwise non-commensurable and on the Gromov boundary of the group G we have

$$\{g_i^{\infty}, g_i^{-\infty}\} \cap \{\alpha_i, \beta_i\} = \emptyset,\ i = 1, 2, \ldots, s \tag{11}$$

(recall that if for two elements of infinite order $g, h \in G$ we have $g^{\infty} = h^{\pm\infty}$ then $g^m = h^l$ for some $m, l \in \mathbb{Z}\backslash\{0\}$ – by lemma 2.7).

$\hat{H}_i$ is non-elementary, therefore $F_i \leq \hat{H}_i$ is also non-elementary, $i = 1, \ldots, s$. Now we use lemma 4.5 to obtain a non-elementary subgroup $K_i \leq F_i$ such that

$$\{\alpha_i,\ g_i^{-\infty},\ (b_i a_i^{-1}) \circ g_i^{-\infty}\} \cap \Lambda(K_i) = \emptyset \ \text{ in } \ \partial G\ ,\ i = 1, 2, \ldots, s\ . \tag{12}$$

According to (11) and (12) we can use the claim of lemma 2.14 to show that

$$C_{01i} = \sup\left\{ \left(\left(y_j^{(i)}\right)^{-1} | g_i^n \right)_{1_G} \ : \ j, n \in \mathbb{N} \right\} < \infty \ ,$$

$$C_{02i} = \sup\left\{ \left(y_j^{(i)} | g_i^{-n}\right)_{1_G} \ : \ j, n \in \mathbb{N} \right\} < \infty \ ,$$

$$C_{03i} = \sup\left\{ \left(x | y_j^{(i)}\right)_{1_G} \ : \ j \in \mathbb{N}, x \in K_i \right\} < \infty \ ,$$

$$C_{04i} = \sup\left\{ \left(g_i^{-n} | x\right)_{1_G} \ : \ n \in \mathbb{N}, x \in K_i \right\} < \infty \ \text{ and}$$

$$C_{05i} = \sup\left\{ \left(b_i a_i^{-1} g_i^{-n} | x\right)_{1_G} \ : \ n \in \mathbb{N}, x \in K_i \right\} < \infty$$

for each $i = 1, 2 \ldots, s$. Finally, define

$$C_0 = \max_{1 \le i \le s} \left\{ C_{01i}, C_{02i}, C_{03i} + |b_i a_i^{-1}|_G, C_{04i}, C_{05i} + |b_i a_i^{-1}|_G \right\} + 14\delta \ .$$

Denote $\bar{\lambda} = 1$, $\bar{c} = 0$,

$$C_1 = 12(C_0 + \delta) + \bar{c} + 1 \tag{13}$$

Let

$$\lambda = \bar{\lambda}/4 = 1/4, \ \ c = c(\bar{\lambda}, \bar{c}, C_0) \ge 0 \ \text{ be the constant from lemma 4.2} \tag{14}$$

$$\text{and let } \nu = \nu(\lambda, c) \text{ be the constant from the claim of lemma 2.1.} \tag{15}$$

By the conditions of the theorem 1, the subset Q is η-quasiconvex for some $\eta \ge 0$,

$$\text{let } \varkappa \text{ be the length of a shortest element from } Q. \tag{16}$$

Set

$$A = \{ g \in G \mid |g|_G \le 3\delta + \nu + \eta + \varkappa \} \tag{17}$$

then $card(A) < \infty$. By the construction of α_i,

$$\alpha_i \notin \bigcup_{g \in A} g \circ \Lambda(Q^{-1}Q) = \Lambda\left(\bigcup_{g \in A} gQ^{-1}Q \right) = \Lambda(AQ^{-1}Q) \ .$$

Hence, according to the lemma 2.14, one can define

$$C_3 = \max_{1 \le i \le s} \sup\{ \left(y_j^{(i)} | x\right)_{1_G} \mid j \in \mathbb{N}, x \in AQ^{-1}Q \} \ . \tag{18}$$

For every $i \in \{1, 2, \ldots, s\}$, $|y_j^{(i)}|_G \to \infty$ as $j \to \infty$, and the intersection $\{y_j^{(i)} \mid j \in \mathbb{N}\} \cap E(g_i)$ is finite by (11), therefore for some j_0 (depending on i), after setting $y_i = y_{j_0}^{(i)} \in F_i$, we will have

$$|y_i|_G > 3\delta + \nu + \eta + \varkappa + C_1 + 2C_3 \ \text{ and } \ y_i \in C_{\hat{H}_i}\big(E(\hat{H}_i)\big) \backslash E(g_i) \ . \tag{19}$$

Applying lemmas 4.3 and 4.4 we can find $n \in \mathbb{N}$ such that the elements

$$w_i = y_i g_i^n$$

have infinite order, are $\hat{H}_i$-suitable and pairwise non-commensurable when $1 \le i \le s$. Obviously, in addition, we can demand that $|g_i^n|_G, |w_i|_G > C_1$ for every i.

The subgroups $K_i \le G$ are non-elementary, hence we can find elements $c_i \in K_i \le C_{\hat{H}_i}\big(E(\hat{H}_i)\big)$ for which

$$x_{i0} \stackrel{def}{=} a_i b_i^{-1} c_i \in C_G\big(E(\hat{H}_i)\big) \backslash E(w_i) \text{ and } |x_{i0}|_G > C_1 \text{ for } i = 1, \dots, s \;. \quad (20)$$

Note that $b_i^{-1} c_i \in \hat{H}_i$ for each i.

Let $w_i \in \hat{H}_i^0$, $x_{i0} \in G$, $x_{i1}, \dots, x_{il} \in \hat{H}_i$ be represented by words W_i, X_{i0}, $X_{i1}, \dots, X_{il}$ over the alphabet $\mathcal{A}$ of minimal length, $i = 1, \dots, s$. As the system $\mathcal{R} = \mathcal{R}_{s,l,m}(W_1, \dots, W_s, X_{10}, \dots, X_{sl}, m)$ consider all cyclic permutations of $R_i^{\pm 1}$ where

$$R_i \equiv X_{i0} W_i^m X_{i1} W_i^m \dots X_{il} W_i^m \;, \quad i = 1, 2, \dots, s \;. \quad (21)$$

Lemma 7.1. *Fix $i \in \{1, \dots, s\}$. Let λ, c be the constants defined in (14) and the elements x_{ij}, $j = 1, \dots, l$, satisfy the properties:*

$$x_{ij} \in K_i \text{ and } |x_{ij}|_G > C_1 \text{ for every } j = 1, \dots, l$$

($K_i \le \hat{H}_i$ and $C_1 > 0$ are as above). Then for any $n \in \mathbb{N}$, any path q in $\Gamma(G, \mathcal{A})$ labelled by a word $R_i^{\pm n}$ is (λ, c)-quasigeodesic.

Proof. It is enough to prove this lemma for the case when q is labelled by R_i^n because if $lab(q) \equiv R_i^{-n}$ then $lab(q^{-1}) \equiv R_i^n$ and if q^{-1} is (λ, c)-quasigeodesic then so is q.

For convenience, assume that $i = 1$. Since left translations are isometries, we can suppose that $q_- = 1_G$. Set $t = (m+1)(l+1)$. The path q is a broken line $[z_0, z_1, \dots, z_{nt}]$ in the Cayley graph $\Gamma(G, \mathcal{A})$ where

$z_0 = q_- = 1_G$, $z_1 = x_{10}$, $z_2 = x_{10} w_1$, $z_3 = x_{10} w_1^2$,

. . . . . .

$z_{m+1} = x_{10} w_1^m$, $z_{m+2} = x_{10} w_1^m x_{11}$, $z_{m+3} = x_{10} w_1^m x_{11} w_1$,

. . . . . .

$z_t = x_{10} w_1^m x_{11} w_1^m \dots x_{1l} w_1^m$, $z_{t+1} = x_{10} w_1^m x_{11} w_1^m \dots x_{1l} w_1^m x_{10}$,

. . . . . .

$z_{nt} = x_{10} w_1^m \dots x_{1l} w_1^m x_{10} w_1^m \dots x_{1l} w_1^m \dots x_{10} w_1^m \dots x_{1l} w_1^m = q_+$.

By construction, $\|[z_{j-1}, z_j]\| > C_1$, $j = 1, 2, \dots, nt$. In order to apply lemma 4.2 to the path q it remains to verify that $(z_{j-1}|z_{j+1})_{z_j} \le C_0$ for every $l = 1, \dots, nt-1$. There are several types of Gromov products that appear when j changes from 1 to $nt - 1$. Below we compute them in the order of their occurrence.

Type I. $(z_0|z_2)_{z_1} = (1_G|x_{10}w_1)_{x_{10}} = (x_{10}^{-1}|w_1)_{1_G}$. Recall that $w_1 = y_1g_1^n$. By the Gromov's definition of a hyperbolic space,

$$(x_{10}^{-1}|y_1)_{1_G} \geq \min\{(x_{10}^{-1}|w_1)_{1_G}, (y_1|w_1)_{1_G}\} - \delta \ .$$

Now, we observe that

$$(x_{10}^{-1}|y_1)_{1_G} = (c_1^{-1}b_1a_1^{-1}|y_1)_{1_G} \leq (c_1^{-1}|y_1)_{1_G} + |b_1a_1^{-1}|_G \leq C_{031} + |b_1a_1^{-1}|_G.$$

Hence,

$$\min\{(x_{10}^{-1}|w_1)_{1_G}, (y_1|w_1)_{1_G}\} \leq C_{031} + |b_1a_1^{-1}|_G + \delta \leq C_0 \ . \tag{22}$$

From the geodesic triangle $1_Gy_1w_1$ in $\Gamma(G, \mathcal{A})$ we obtain

$$(y_1|w_1)_{1_G} = |y_1|_G - (1_G|w_1)_{y_1} = |y_1|_G - (y_1^{-1}|g_1^n)_{1_G} \geq C_1 - C_0 > C_0 \ .$$

Combining the latter inequality with (22) we achieve

$$(z_0|z_2)_{z_1} = (x_{10}^{-1}|w_1)_{1_G} \leq C_0.$$

Type II. $(z_1|z_3)_{z_2} = (x_{10}|x_{10}w_1^2)_{x_{10}w_1} = (w_1^{-1}|w_1)_{1_G}$.
Again, applying the definition of hyperbolicity twice, we obtain

$$(g_1^{-n}|y_1)_{1_G} \geq \min\{(g_1^{-n}|w_1)_{1_G}, (y_1|w_1)_{1_G}\} - \delta \geq$$

$$\geq \min\{(w_1^{-1}|w_1)_{1_G}, (g_1^{-n}|w_1^{-1})_{1_G}, (y_1|w_1)_{1_G}\} - 2\delta.$$

By construction, $(g_1^{-n}|y_1)_{1_G} \leq C_{021} \leq c_0 - 2\delta$. As we showed above, $(y_1|w_1)_{1_G} > C_0$.
Considering the geodesic triangle $1_Gg_1^{-n}w_1^{-1}$ we get

$$(g_1^{-n}|w_1^{-1})_{1_G} = |g^{-n}|_G - (1_G|w_1^{-1})_{g^{-n}} = |g^n|_G - (g^n|y_1^{-1})_{1_G} \geq C_1 - C_0 > C_0.$$

So, combining these inequalities, we achieve

$$(z_1|z_3)_{z_2} = (w_1^{-1}|w_1)_{1_G} \leq C_0 \ .$$

Type III. $(z_m|z_{m+2})_{z_{m+1}} = (w_1^{-1}|x_{11})_{1_G} \leq C_0$.
Type IV. $(z_{m+1}|z_{m+3})_{z_{m+2}} = (x_{11}^{-1}|w_1)_{1_G} \leq C_0$.
These two inequalities are proved in the same way as we proved the inequality for Type I (the proofs even easier since $x_{11} \in K_1$).
The last possibility is
Type V. $(z_{t-1}|z_{t+1})_{z_t} = (w_1^{-1}|x_{10})_{1_G}$.
As before, we have

$$(g_1^{-n}|x_{10})_{1_G} \geq \min\{(w_1^{-1}|x_{10})_{1_G}, (w_1^{-1}|g_1^{-n})_{1_G}\} - \delta \ .$$

$$(g_1^{-n}|x_{10})_{1_G} = (g_1^{-n}|a_1b_1^{-1}c_1)_{1_G} = (b_1a_1^{-1}g_1^{-n}|c_1)_{b_1a_1^{-1}} \leq$$

$$\leq (b_1a_1^{-1}g_1^{-n}|c_1)_{1_G} + |b_1a_1^{-1}|_G \leq C_{051} + |b_1a_1^{-1}|_G \leq C_0 - \delta \ .$$

We showed while considering the Type II, that $(g_1^{-n}|w_1^{-1})_{1_G} > C_0$. Therefore,

$$(z_{t-1}|z_{t+1})_{z_t} = (w_1^{-1}|x_{10})_{1_G} \leq C_0.$$

It is easy to see that for arbitrary $j \in \{1, 2, \ldots, nt-1\}$ the Gromov product $(z_{j-1}|z_{j+1})_{z_j}$ is equal to a Gromov product of one of the Types I-V, thus it is not larger than C_0.

Therefore, recalling that the constant C_1 was defined by formula (13), we can use the lemma 4.2 to show that the path q is (λ, c)-quasigeodesic, where $\lambda > 0$ and $c \geq 0$ are defined in (14). Q.e.d. □

Below we have an analog of the lemma 6.1 needed for our proof:

Lemma 7.2. *Suppose $W_1, \ldots, W_s$, $X_{10}, \ldots, X_{s0}$ and $\lambda > 0$, $c \geq 0$ are the words and the constants defined above. Then for any $\mu > 0$ there are $l \in \mathbb{N}$ and words $X_{11}, \ldots, X_{sl}$ (X_{ij} represents an element $x_{ij} \in \hat{H}_i$, $j = 1, \ldots, l$, $i = 1, \ldots, s$) such that for any $\varepsilon \geq 0$, $\rho > 0$ there is $m_0 \in \mathbb{N}$ such that the system $\mathcal{R}_{s,l,m}$ (21) satisfies $C(\varepsilon, \mu, \lambda, c, \rho)$ and $C_1(\varepsilon, \mu, \lambda, c, \rho)$-conditions if $m \geq m_0$.*

Proof. By lemma 6.1 there exist $\lambda' > 0$ such that for any $\mu > 0$ there are $l \in \mathbb{N}$ and $c' \geq 0$ such that for any $\varepsilon \geq 0$, $\rho > 0$ there are $m_0 \in \mathbb{N}$ and words $X_{11}, \ldots, X_{sl}$ such that the system $\mathcal{R}_{s,l,m}$ satisfies the generalized small cancellation conditions $C(\varepsilon, \mu, \lambda', c', \rho)$ and $C_1(\varepsilon, \mu, \lambda', c', \rho)$ if $m \geq m_0$.

According to the remark 4 after the formulation of lemma 6.1, lemma 2.12 and lemma 2.11, the elements $x_{i1}, \ldots, x_{is}$ can be chosen right after l, inside of the subgroup K_i, with an additional property $|x_{ij}|_G > C_1$ (the constant C_1 was defined in (13)) for every $j = 1, \ldots, l$, $i = 1, \ldots, s$.

Consider any word $R \in \mathcal{R}_{s,l,m}$. By definition, R is a subword of a word $R_i^{\pm 2}$ for some $i \in \{1, \ldots, s\}$. By lemma 7.1 the word $R_i^{\pm 2}$ is (λ, c)-quasigeodesic (where λ and c are defined in (14)), hence so is R. Taking into account remark 3, we achieve that the system $\mathcal{R}_{s,l,m}$ satisfies the conditions $C(\varepsilon, \mu, \lambda, c, \rho)$ and $C_1(\varepsilon, \mu, \lambda, c, \rho)$ if $m \geq m_0$. □

Lemma 7.3. *Let $\mathcal{R} = \mathcal{R}_{s,l,m}(W_1, \ldots, W_s, X_{10}, \ldots, X_{sl}, m)$ be the system of additional relations and $\lambda > 0$, $c \geq 0$ be the constants defined above. Then for any $\varepsilon \geq 0$ and $\xi > 0$ there exists $m_1 \in \mathbb{N}$ such that for any $m \geq m_1$ the following property holds:*

Suppose Δ is a diagram over the presentation (8) and q – a subpath of $\partial\Delta$ such that the corresponding path q' in the Cayley graph $\Gamma(G, \mathcal{A})$ of the group G with the same label as q is geodesic (in other words, $\|q\| = |elem(q)|_G$) and $elem(q) \in Q$ in G. Then for arbitrary $\mathcal{R}$-face Π of Δ and an ε-contiguity subdiagram Γ between Π and q we have

$$(\Pi, \Gamma, q) \leq \xi \ .$$

Proof. Let $\partial\Gamma = p_1 q_1 p_2 q_2$ where q_1, q_2 are subpaths of $\partial\Pi$ and q correspondingly and $\|p_1\|, \|p_2\| \leq \varepsilon$. Take arbitrary $\xi > 0$. Obviously, from the definition (21), there is $m_1 \in \mathbb{N}$ such that for any $m \geq m_1$ the inequality $\|q_1\|/\|\partial\Pi\| > \xi$

implies that the path q_1 has a subpath o labelled by the word $W_i^{\pm 1}$ for some $i \in \{1, \ldots, s\}$ and, moreover, the subpaths o_1, o_2 of q_1 (with $(o_1)_- = (q_1)_-$, $(o_1)_+ = o_-$, $(o_2)_- = o_+$, $(o_2)_- = (q_1)_+$) satisfy

$$\|o_j\| > (\varepsilon + c + \nu)/\lambda, \ j = 1, 2 \ , \tag{23}$$

(ν is chosen according to (15)).

We are going to obtain a contradiction with the definitions of elements w_i and y_i.

Since the diagram Γ contains only 0-faces (i.e. it is a diagram over the group G), we can consider the corresponding picture in $\Gamma(G, \mathcal{A})$ with a geodesic path q' starting at 1_G (its subpath q_2'), (λ, c)-quasigeodesic path q_1' (its subpaths o', o_1', o_2') and paths p_1', p_2' of lengths at most ε with $(p_1')_- = (q_2')_+$, $(p_1')_+ = (q_1')_-$, $(p_2')_- = (q_1')_+$, $(p_2')_+ = (q_2')_-$ (i.e. for every path r from Δ we construct a corresponding path r' in $\Gamma(G, \mathcal{A})$ with the same label; see Figure 1).

Pick any $z \in Q$ with $|z|_G = \varkappa$ (the constant $\varkappa$ was defined in (16)). Then $q'_+ = elem(q') = elem(q) \in Q$. Hence, since the triangles are δ-slim, one obtains

$$q' \subset \mathcal{O}_\delta([1, z] \cup [z, elem(q')]) \subset \mathcal{O}_{\delta+\varkappa}([z, elem(q')]) \subset \mathcal{O}_{\delta+\varkappa+\eta}(Q) \ .$$

Denote $u = (q_1')_-$, $v = (q_1')_+$. Then $u, v \subset \mathcal{O}_\varepsilon(q_2')$.

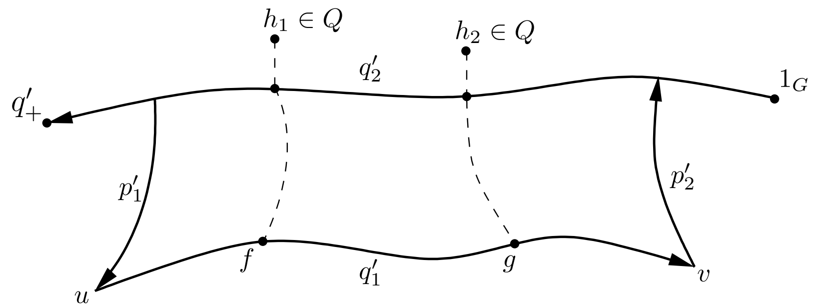

Figure 1

Now, since

$$o' \subset q_1' \subset \mathcal{O}_\nu([u, v]) \ ,$$

using (23) and lemma 2.6 we obtain

$$o' \subset \mathcal{O}_{\nu+2\delta}(q') \subset \mathcal{O}_{3\delta+\nu+\varkappa+\eta}(Q) \ .$$

Recall that $lab(o') = W_i^{\pm 1}$ by construction. So, if $lab(o') = W_i$, define the points $f = o'_-$, $g = o'_+$ and if $lab(o') = W_i^{-1}$, define $g = o'_-$ and $f = o'_+$. Thus there are elements $h_1, h_2 \in Q$ such that

$$d(f, h_1) = |f^{-1}h_1|_G \leq 3\delta + \nu + \varkappa + \eta, \ d(g, h_2) \leq 3\delta + \nu + \varkappa + \eta.$$

By the definition of w_i, we have $fy_ig_i^n = g$. According to Gromov's definition of a hyperbolic metric space, we achieve

$$(h_2|fy_i)_f \geq \min\{(h_2|g)_f, (g|fy_i)_f\} - \delta \ .$$

Observe that $(h_2|fy_i)_f = (f^{-1}h_2|y_i)_{1_G}$ and

$$x = f^{-1}h_2 = (f^{-1}h_1)h_1^{-1}h_2 \in AQ^{-1}Q$$

(the set A was defined in (17)).

$$(g|fy_i)_f = \|[f, fy_i]\| - (f|g)_{fy_i} = |y_i|_G - (y_i^{-1}|g_i^n)_{1_G} \geq$$
$$\geq |y_i|_G - C_0 > 5\delta + \nu + \varkappa + \eta + 2C_3 \ .$$

(Here we used that $C_1 - C_0 > 2\delta$.) Note that $d(f,g) \geq (g|fy_i)_f$, hence

$$(h_2|g)_f \geq \frac{1}{2}\big(d(f,g) - d(g,h_2)\big) \geq \frac{1}{2}\big((g|fy_i)_f - (3\delta + \nu + \varkappa + \eta)\big) > C_3 + \delta \ .$$

Combining the above formulas, we finally obtain

$$(h_2|fy_i)_f = (x|y_i)_{1_G} > C_3$$

contradicting to the definition (18) of C_3. Therefore, $\|q_1\|/\|\partial\Pi\| = (\Pi, \Gamma, q) \leq \xi$.

The lemma is proved. $\Box$

8 Proof of Theorem 1

Proof. The group G_1 is generated by $\phi(\mathcal{A})$, so let $|x|_{G_1}$ be the corresponding length function for elements $x \in G_1$, and let $d_1(\cdot,\cdot)$ be the corresponding metric on the Cayley graph of the group G_1. Sometimes it will be convenient for us to identify $\mathcal{A}$ and $\phi(\mathcal{A})$ for G_1, so $\Gamma(G_1, \mathcal{A})$ will be the Cayley graph of G_1. Since ϕ is a homomorphism, from the definition of the word metric it follows that

$$\forall\ x, y \in G \ \ d_1\big(\phi(x), \phi(y)\big) \leq d(x,y) \ . \tag{24}$$

Define the elements $a_1, \dots, a_s \in G$ and the subgroups $\hat{H}_1, \dots, \hat{H}_s$ as we did in the beginning of section 7. After that construct the elements g_i, y_i, w_i and x_{i0}, $i = 1, 2, \dots, s$, as described in that section. Then we can find the constants $\lambda > 0$ and $c \geq 0$ according to (14).

Suppose that W_i, $X_{i0}, \dots, X_{il}$ are shortest words in the alphabet $\mathcal{A}$ representing w_i, $x_{i0}, \dots, x_{il}$, $i = 1, \dots, s$. As the system of additional relations, consider the set

$$\mathcal{R} = \mathcal{R}_{s,l,m}(W_1, \dots, W_s, X_{10}, \dots, X_{sl}, m)$$

of all cyclic permutations of $R_i^{\pm 1}$, $i = 1, \dots, s$, established in (21).

Define the group G_1 according to (8), thus, $G_1 \cong G/\langle \mathcal{R}^G \rangle$. Let ϕ be the natural epimorphism from G to G_1.

By lemma 7.2 one can find $l, m_0 \in \mathbb{N}$ and elements $x_{ij} \in \hat{H}_i$, $j = 1, \dots, l$, $i = 1, \dots, s$, such that the group G_1 satisfies all of the conditions of lemmas 6.2 and 6.4 if $m \geq m_0$. Therefore we obtain the parts 1) and 8) of the theorem 1.

It is easy to see that the relation R_i implies $\phi(a_i z_i) = 1$ in G_1 for some $z_i \in \hat{H}_i$, hence $\phi(a_i) \in \phi(\hat{H}_i)$ for $i = 1, \dots, s$.

Due to the choice of $a_1, \dots, a_s$ and $\hat{H}_1, \dots, \hat{H}_s$ we obtain $\phi(\mathcal{A}) \subset \phi(H_j)$ for every $j \in \{1, 2, \dots, k\}$. Consequently, $G_1 = \phi(H_j)$, $j = 1, \dots, k$, so the part 3) of the theorem is proved.

Let us now prove the property 2). Let $\mu_0 > 0$, $\varepsilon \geq 0$ be chosen according to lemma 6.2. Since we can take any μ inside of the interval $(0, \mu_0]$ we can also demand that $1/(\lambda+1) < 1 - 23\mu$. Choose $\xi > 0$ in such a way that

$$\frac{1}{\lambda + 1} < 1 - 23\mu - 2\xi \ . \tag{25}$$

Denote $\theta = 1 - 23\mu - 2\xi > 0$. Then (25) implies that $(\lambda+1)\theta - 1 > 0$. Set $L_0 = \min\{\|R\| \mid R \in \mathcal{R}\}$. Evidently, L_0 depends on m and there exists $m_2 \in \mathbb{N}$ such that for any $m \geq m_2$

$$\big((\lambda+1)\theta - 1\big)L_0 > c + 4\varepsilon \ . \tag{26}$$

Now, let's apply the statement lemma 7.3 to find $m_1 = m_1(\varepsilon, \xi) \in \mathbb{N}$.

By taking any $m \geq \max\{m_0, m_1, m_2\}$ we can further assume that the claims of lemmas 6.2 and 7.3 hold together with the inequality (26).

Consider arbitrary elements $u, v \in Q$. We need to show that $d(u,v) = d_1\big(\phi(u), \phi(v)\big)$.

Observe that, by definition, $d(u,v) = |u^{-1}v|_G$, $d_1(\phi(u), \phi(v)) = |\phi(u^{-1}v)|_{G_1}$. Obviously, $|u^{-1}v|_G \geq |\phi(u^{-1}v)|_{G_1}$, so assume, by contradiction, that

$$|u^{-1}v|_G > |\phi(u^{-1}v)|_{G_1} \ . \tag{27}$$

Thus, if U, V are shortest words representing u, v in G, there is a word Z such that $U^{-1}V = Z$ in the G_1 but not in G (Z is a word of minimal length representing the element $\phi(u^{-1}v)$ in G_1).

Consider a reduced circular diagram Δ over G_1 whose boundary is labelled by the word $U^{-1}VZ^{-1}$. Let q^1, q^2, q^3 be the (geodesic) sections of the boundary $\partial\Delta$ labelled by the words U, V, Z respectively.

This diagram must contain at least one $\mathcal{R}$-face since $U^{-1}VZ^{-1} \neq 1$ in G. Therefore, by lemma 6.2 there exists an $\mathcal{R}$-face Π in Δ and ε-contiguity subdiagrams Γ_1,Γ_2,Γ_3 between Π and the sections q^1,q^2,q^3 (for our convenience, for each of the sections q^j we can choose a corresponding orientation of $\partial\Pi$, $j = 1, 2, 3$) satisfying

$$(\Pi, \Gamma_1, q^1) + (\Pi, \Gamma_2, q^2) + (\Pi, \Gamma_3, q^3) > 1 - 23\mu \ .$$

Since $elem(q^1) = u \in Q$, $elem(q^2) = v \in Q$ and $m \geq m_1$, we have $(\Pi, \Gamma_1, q^1) \leq \xi$ and $(\Pi, \Gamma_2, q^2) \leq \xi$. Hence,

$$(\Pi, \Gamma_3, q^3) > 1 - 23\mu - 2\xi = \theta \ . \tag{28}$$

Now we are going to obtain a contradiction with the choice of Z.
Let $\partial(\Gamma_3) = p_1 r_1 p_2 o_2$ where $\partial\Pi = r_1 r_2$, $q^3 = o_1 o_2 o_3$, $\|p_1\|, \|p_2\| \leq \varepsilon$ (Figure 2).

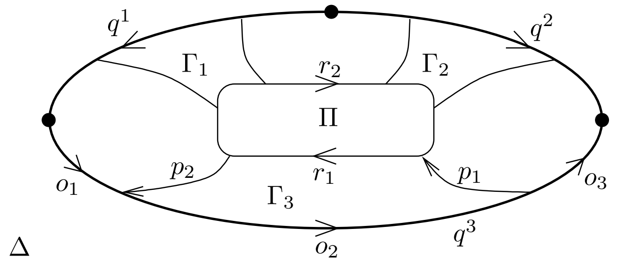

Figure 2

Let L denote the length of $\partial\Pi$. (28) implies

$$\|r_1\| > \theta L \ , \quad \|r_2\| = L - \|r_1\| < (1-\theta)L \ . \tag{29}$$

Now, since Γ_3 is a diagram over the group G, the equality

$$elem(o_2^{-1}) = elem(p_1) elem(r_1) elem(p_2)$$

holds in G. The path q^3 is geodesic, therefore, its subpath o_2 is also geodesic, thus,

$$\|o_2\| = \|o_2^{-1}\| = |elem(o_2^{-1})|_G \geq |elem(r_1)|_G - |elem(p_1)|_G - |elem(p_2)|_G \ , \text{ hence}$$

$$\|o_2\| \geq |elem(r_1)|_G - 2\varepsilon \ .$$

The path r_1 is (λ, c)-quasigeodesic as a subpath of the face contour $\partial\Pi$, therefore $|elem(r_1)|_G \geq \lambda\|r_1\| - c$. Combining the last two inequalities with (29) we obtain

$$\|o_2\| \geq \lambda\theta L - c - 2\varepsilon \ . \tag{30}$$

Consider the subdiagram Ω of Δ bounded by the closed path $p_2^{-1} r_2 p_1^{-1} o_2^{-1}$. It gives us the following equality in the group G_1:

$$elem(o_2) = elem(p_2^{-1}) \cdot elem(r_2) \cdot elem(p_1^{-1}) \ .$$

Thus,

$$\|o_2\| = |elem(o_2)|_{G_1} \leq |elem(p_2^{-1})|_{G_1} + |elem(r_2)|_{G_1} + |elem(p_1^{-1})|_{G_1} \leq$$

$$\leq \|r_2\| + 2\varepsilon \leq (1-\theta)L + 2\varepsilon \ .$$

Comparing the latter inequality with (30) we get

$$\lambda\theta L - c - 2\varepsilon \le (1-\theta)L + 2\varepsilon \ .$$

Or, equivalently,

$$\big((\lambda+1)\theta - 1\big)L \le c + 4\varepsilon \ .$$

Since $L \ge L_0$ this contradicts to the inequality (26).

Therefore, the assumption (27) was incorrect and $d(u,v) = d_1\big(\phi(u), \phi(v)\big)$ for arbitrary $u, v \in Q$. Thus $\phi|_Q$ is an isometry.

By 1) G_1 is δ_1-hyperbolic for some $\delta_1 \ge 0$. Take any ω-quasiconvex (in G) subset $S \subseteq Q$. Let's show that $\phi(S) \subset G_1$ is $(\omega + \delta_1)$-quasiconvex.

Consider arbitrary two elements $u, v \in S$ and let p be a geodesic path in $\Gamma(G, \mathcal{A})$ connecting them. Then

$$p \subset \mathcal{O}_\omega(S) \ \text{ in } \Gamma(G, \mathcal{A}) \ .$$

Let p_1 be the path in $\Gamma(G_1, \mathcal{A})$ starting at $\phi(u)$ with the same label as p. Then $(p_1)_+ = \phi(v)$ (this is equivalent to the equality $\phi(u) \cdot elem(p_1) = \phi(v)$ which follows from $u \cdot elem(p) = v$). Now, since ϕ is an isometry between S and $\phi(S)$,

$$\|p_1\| = \|p\| = d(u,v) = d_1\big(\phi(u), \phi(v)\big) \ .$$

Therefore, p_1 is a geodesic path between $\phi(u)$ and $\phi(v)$ in $\Gamma(G_1, \mathcal{A})$. (24) implies

$$p_1 \subset \mathcal{O}_\omega\big(\phi(S)\big) \ \text{ in } \Gamma(G_1, \mathcal{A}) \ .$$

The space $\Gamma(G_1, \mathcal{A})$ is δ_1-hyperbolic, hence for any geodesic path q between $\phi(u)$ and $\phi(v)$ we have $q \subset \mathcal{O}_{\delta_1}(p_1)$. Consequently,

$$q \subset \mathcal{O}_{\omega+\delta_1}\big(\phi(S)\big) \ \text{ in } \Gamma(G_1, \mathcal{A}) \ .$$

The proof of the part 2) is complete.

For the case when Q is a finite subset, the proofs in [16, Thm. 3] of the properties corresponding to 4), 5) from our theorem 1 were based on the lemma 6.3, general properties of hyperbolic groups and the fact that in a diagram over (8) with labels of boundary contours representing elements of Q in G there can not exist any "long" ε-contiguity of an $\mathcal{R}$-face to a boundary contour. The same fact is true in our case by lemma 7.3 (after an appropriate choice the parameters like in the proof of the property 2)). So, for the proofs 4), 5) the reader is referred to [16, Thm. 3].

Properties 6) and 7) do not depend on Q, thus they can be proved in the same way as they were proved in [16, Thm. 2] (we can always add a finite subset to Q: it will stay quasiconvex and the formula $(*)$ will continue to hold).

Finally, let's derive the property 9). By lemma 2.11, we can choose a G-suitable element $g \in G$. Then, by definition, $T(g) = E(G)$. Denote $S = \langle g \rangle_\infty$ – a quasiconvex subgroup of the group G. Then for any $h \in G$

$$|H_i : (H_i \cap hSh^{-1})| = \infty$$

since H_i is non-elementary for every $i = 1, \ldots, k$. Hence, according to theorem 3, H_i is not contained inside of $P_1 S^{-1} S P_2$ for arbitrary finite subsets $P_1, P_2 \subset G$. Now we can apply lemmas 2.3 and 5.4 to the union

$$Q' = Q \cup S = Q \cup \langle g \rangle_\infty$$

to show that all the requirements of theorem 1 will remain satisfied if one substitutes Q by Q' in it. Since the properties 1)-8) were already proved, we can further use them for the elements of Q'. Therefore, $ker(\phi) \cap Q' = \{1_G\}$, implying that $\phi(g)$ has infinite order in G_1.

Consider arbitrary $x \in E(G_1)$. Then, in particular, $x \in E\big(\phi(g)\big)$. By definition, there exists $n \in \mathbb{N}$ such that $x(\phi(g))^n x^{-1} = (\phi(g))^{\pm n}$.

If $x\phi(g)^n x^{-1} = \phi(g)^{-n}$ then by the part 4) the elements $g^n, g^{-n} \in Q'$ must be conjugate in G which fails because $E(g) = E^+(g)$. Hence, $x(\phi(g))^n x^{-1} = (\phi(g))^n$, i.e. $x \in C_{G_1}\big(\phi(g^n)\big)$.

Since $g^n \in Q'$, one can apply the part 5) to find $y \in C_G(g^n)$ with $\phi(y) = x$. $g \in G$ is G-suitable, therefore $C_G(g^n) \leq E(g) = T(g) \times \langle g \rangle$. G_1 is non-elementary, therefore the subgroup $E(G_1) \leq G_1$ is finite, thus x has a finite order in G_1. It follows that y has a finite order in G, because, otherwise, we would get $y^{l_1} = g^{l_2}$ for some $l_1, l_2 \in \mathbb{Z} \backslash \{0\}$ and $x^{l_1} = \phi(y^{l_1}) = \phi(g^{l_2})$ where $\phi(g^{l_2})$ has an infinite order in G_1. Consequently, $y \in T(g) = E(G)$ and

$$x = \phi(y) \in \phi\big(E(G)\big) \ .$$

The proof of the theorem is finished. □

9 Constructing Simple Quotients

Lemma 9.1. *Suppose N is an infinite normal subgroup of a hyperbolic group G and K is a quasiconvex subgroup of G such that $|G : K| = \infty$. Then for arbitrary $h \in G$, $|N : (N \cap hKh^{-1})| = \infty$.*

Proof. Since a conjugate to a quasiconvex subgroup of infinite index is again a quasiconvex subgroup of infinite index, it is enough to consider the case when $h = 1_G$. Assume, by the contrary, that $|N : (N \cap K)| < \infty$. Then there exist elements $h_1, \ldots, h_n \in N$ such that $N \subseteq Kh_1 \cup \ldots \cup Kh_n$. Applying lemmas 2.16 and 2.13 we achieve

$$\Lambda(G) = \Lambda(N) \subseteq \Lambda(Kh_1 \cup \ldots \cup Kh_n) = \Lambda(K) \ .$$

Hence, by lemma 2.18, $G \subset K \cdot P = \bigcup_{p \in P} Kp$ for some finite subset P of G, which implies that $|G : K| < \infty$ – a contradiction to our conditions. □

Lemma 9.2. *If H is a non-elementary subgroup of a hyperbolic group G then $E(H)$ coincides with the subgroup $K = \bigcap_{\alpha \in \Lambda(H)} St_G(\{\alpha\}) \leq G$.*

Proof. Indeed,

$$K \subseteq \bigcap_{h\in H^0} St_G(\{h^\infty\}) = \bigcap_{h\in H^0} E^+(h) \subseteq E(H) \ ,$$

i.e. $K \subseteq E(H)$. Now, since $E(H)$ is a finite subgroup normalized by H, for every $x \in E(H)$ and $\alpha \in \Lambda(H)$ we can find a sequence of elements $y_i \in H$, $i \in \mathbb{N}$, with $\lim_{i\to\infty} y_i = \alpha$. Denote $Y = \{y_i \mid i \in \mathbb{N}\} \subset H$. Then $\Lambda(Y) = \{\alpha\}$, $xY \subset Y \cdot E(H)$ and by lemma 2.13 we achieve

$$x \circ \{\alpha\} = x \circ \Lambda(Y) = \Lambda(xY) \subset \Lambda\big(YE(H)\big) =$$

$$= \Lambda(\bigcup_{z\in E(H)} Yz) = \bigcup_{z\in E(H)} \Lambda(Yz) = \bigcup_{z\in E(H)} \Lambda(Y) = \{\alpha\}.$$

Thus, $x \in K$ which implies $E(H) \subseteq K$. □

Lemma 9.3. *Assume that A is a non-elementary normal subgroup of a subgroup H in a hyperbolic group G. Then $E(A) = E(H)$.*

Proof. According to lemma 9.2 and lemma 2.16 we have

$$E(A) = \bigcap_{\alpha\in\Lambda(A)} St_G(\{\alpha\}) = \bigcap_{\alpha\in\Lambda(H)} St_G(\{\alpha\}) = E(H) \ .$$

□

Proof of corollary 1. First, since $E(G)$ is the maximal finite normal subgroup of G, we can consider the quotient $\hat{G} = G/E(G)$. Obviously, the natural homomorphism $\psi : G \to \hat{G}$ is a quasiisometry between G and $\hat{G}$, therefore $\hat{G}$ is a non-elementary hyperbolic group without non-trivial finite normal subgroups ([5, Ch. 5, Thm. 2.12]). Consequently, $E(\hat{G}) = \{1_{\hat{G}}\}$.

Now, consider the free product $F = \hat{G} * H$. F is hyperbolic as a free product of hyperbolic groups ([5, Ch. 1, Exercise 4.34]) and non-elementary. Identify $\hat{G}$ and H with their canonical copies inside of F. Evidently, we have $E(\hat{G}) = E(F) = \{1_F\}$ in F, hence $\hat{G}$ is a G-subgroup of F. By lemma 2.10 one can find an element $g \in \hat{G} \leq F$ of infinite order. Then

$$\langle g\rangle \cap H = \{1_F\} \ \text{ in } F.$$

As it follows from the normal forms of elements of a free product, the subgroup H is undistorted in F, hence, by lemma 3.1, H is a quasiconvex subgroup of F. Define the quasiconvex subset $Q \subset F$ by $Q = H \cup \langle g\rangle$. Obviously, no non-trivial element of $\hat{G}$ is conjugate to an element of H in F, therefore, according to theorem 3, we can apply theorem 1 to obtain a non-elementary hyperbolic quotient G_1 of F and an epimorphism $\phi_0 : F \to G_1$ that is surjective on $\hat{G}$, injective on Q, $\phi_0(H)$ is quasiconvex in G_1 and

$$E(G_1) = \phi_0\big(E(F)\big) = \{1_{G_1}\} \ , \tag{31}$$

$$\langle \phi_0(g) \rangle \cap \phi_0(H) = \{1_{G_1}\} \ . \tag{32}$$

In particular, $\phi_0(H) \cong H$.

Let the $\{\chi_j \mid j \in \mathbb{N}\}$ denote the set of all non-trivial conjugacy classes of elements in the group G_1. Let N_1 be the normal subgroup of G_1 generated by χ_1. Observe that (31) implies that N_1 is infinite, consequently, it is non-elementary (because $\Lambda(N_1) = \Lambda(G) = \partial G$ according to lemma 2.16 and this set is uncountable, but the limit set of an infinite elementary subgroup consists of only two points).

By lemma 9.3 $E(N_1) = E(G_1)$ is trivial, hence, N_1 is a G-subgroup of the group G_1. Denote $g_1 = \phi_0(g) \in G_1$, $H_1 = \phi_0(H) \le G_1$, $Q_1 = \langle g_1 \rangle \cup H_1$. The order of g_1 in the group G_1 is infinite, hence (32) implies that $|G_1 : H_1| = \infty$. Therefore, $|N_1 : (N_1 \cap h\langle g_1 \rangle h^{-1})| = \infty$ and $|N_1 : (N_1 \cap hH_1h^{-1})| = \infty$ for any $h \in G_1$ (by lemma 9.1). Thus, by theorem 3, we can apply theorem 1 again and achieve a non-elementary hyperbolic quotient G_2 of G_1 together with an epimorphism $\phi_1 : G_1 \to G_2$ satisfying $\phi_1(N_1) = G_2$, ϕ_1 is injective on Q_1, $H_2 = \phi_1(H_1)$ is a quasiconvex subgroup of G_2, $E(G_2) = \{1_{G_2}\}$ and $\langle g_2 \rangle \cap H_2 = \{1_{G_2}\}$ where $g_2 = \phi_1(g_1)$.

Now, let $j_1 = 1$ and $j_2 > j_1$ be the smallest index such that $\phi_1(\chi_{j_2})$ is non-trivial in G_2. Set $N_2 = \langle \phi_1(\chi_{j_2}) \rangle \lhd G_2$. We can apply the same argument as before to get a non-elementary hyperbolic quotient G_3 of G_2 with the natural epimorphism $\phi_2 : G_2 \to G_3$ satisfying the properties we need (as above). And so on.

Thus, we obtain an infinite sequence of epimorphisms

$$G \stackrel{\psi}{\to} \hat{G} \stackrel{\phi_0}{\to} G_1 \stackrel{\phi_1}{\to} G_2 \stackrel{\phi_2}{\to} \dots$$

where each epimorphism ϕ_i is injective on the image of $\phi_{i-1}(H)$, $i \in \mathbb{N}$.

Denote by M the corresponding inductive limit of non-elementary hyperbolic groups. Then M is a quotient of G. As it is evident from the construction, M is a simple group and the group H is isomorphically embedded into M. So, the corollary is proved. $\square$

Proof of corollary 2. Let $A_1, A_2, A_3, \dots$ be an enumeration of all non-elementary hyperbolic groups and $B_1, B_2, B_3, \dots$ – an enumeration of all hyperbolic groups (there are countably many of them since every hyperbolic group is finitely presented [1]). Denote $\hat{A}_i = A_i/E(A_i)$, $i = 1, 2, \dots$.

Set $F_1 = \hat{A}_1 * B_1$. Then, applying theorem 1, we can obtain a non-elementary hyperbolic group G_1 and an epimorphism $\phi_0 : F \to G_1$ that is surjective on $\hat{A}_1$ and injective on B_1 (as before, we can demand that $\phi_0(B_1)$ is quasiconvex in G_1, $|G_1 : \phi_0(B_1)| = \infty$ and $E(G_1) = \{1_{G_1}\}$).

Again, let the $\{\chi_j \mid j \in \mathbb{N}\}$ be the set of all non-trivial conjugacy classes of elements in the group G_1, $N_1 = \langle \chi_1 \rangle \lhd G_1$. By theorem 1 we obtain a (non-elementary hyperbolic) quotient $\hat{G}_1$ with the natural epimorphism $\psi_1 : G_1 \to \hat{G}_1$ that is surjective on N_1 and injective on the image $\phi_0(B_1)$ of B_1 in G_1.

Next, define $F_2 = \hat{G}_1 * \hat{A}_2 * B_2$. Let G_2 be a non-elementary hyperbolic quotient of F_2 such that the natural epimorphism $\phi_1 : F_2 \to G_2$ is surjective on the subgroups $\hat{G}_1, \hat{A}_2 \le F_2$ and injective on B_2 and the image of $B_1 \le G_1 \le F_2$.

Now, let $j_1 = 1$ and $j_2 > j_1$ be the smallest index such that the image of χ_{j_2} (under the composition $\phi_1 \circ \psi_1$) is non-trivial in G_2. Let $N_2 = \langle (\phi_1 \circ \psi_1)(\chi_{j_2}) \rangle \lhd G_2$. Then we can find an epimorphism $\psi_2 : G_2 \to \hat{G}_2$ onto a non-elementary hyperbolic group $\hat{G}_2$ that is surjective on N_2 and injective on the images of B_1, B_2.

And so on. Thus we achieve a sequence of epimorphisms

$$\hat{G}_1 \to \hat{G}_2 \to \hat{G}_3 \to \dots \ .$$

Let M the corresponding inductive limit of these groups. As it follows from the construction, M satisfies all the properties required. □

10 Thrifty Embeddings

Observe that since any elementary torsion-free group is cyclic, maximal elementary subgroups are malnormal in torsion-free non-elementary hyperbolic groups.

Remark 5. Let H be a malnormal subgroup of a group G, $g \in G$. Then

(a) The conjugate subgroup $gHg^{-1} \leq G$ is also malnormal;

(b) If $K \leq G$ is an infinite subgroup and $|K : (K \cap gHg^{-1})| < \infty$ then $g^{-1}Kg \leq H$;

(c) For any $h \in H \backslash \{1_G\}$, $C_G(h) \leq H$;

(d) If $f \in G$ and $fHf^{-1} \cap gHg^{-1} \neq \{1_G\}$ then $fHf^{-1} = gHg^{-1}$.

For the proof of corollary 3 we will need the following auxiliary lemma:

Lemma 10.1. *Suppose G is a group and H, A, B are its subgroups. Assume that H and B are malnormal in G, $H \cap gBg^{-1} = \{1_G\}$ for any $g \in G$ and there is an isomorphism $\tau : A \to B$. Then the natural image of H in the HNN-extension*

$$G_1 = \langle G, t \mid tAt^{-1} = B \rangle \stackrel{def}{=} \langle G, t \mid tat^{-1} = \tau(a),\ a \in A \rangle$$

is malnormal.

Proof. Identify G and H with their canonical images in G_1. Assume that there exists $w \in G_1 \backslash H$ and non-trivial elements $x, y \in H$ such that $wxw^{-1} = y$. Then we can write

$$w = u_0 t^{\epsilon_1} u_1 t^{\epsilon_2} \cdot \ldots \cdot t^{\epsilon_{n-1}} u_{n-1} t^{\epsilon_n} u_n \quad \text{in } G_1 \quad , \tag{33}$$

where $u_0, u_n \in G$, $u_1, \dots, u_{n-1} \in G \backslash \{1_G\}$, $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$, and this representation is reduced (i.e. it contains no occurences of the form tut^{-1} or $t^{-1}vt$ where $u \in A$, $v \in B$).

Observe that $n \geq 1$ since $w \notin G$ (by malnormality of H in G) and

$$u_0 t^{\epsilon_1} \cdot \ldots \cdot t^{\epsilon_n} u_n x u_n^{-1} t^{-\epsilon_n} \cdot \ldots \cdot t^{-\epsilon_2} u_0^{-1} y^{-1} = 1_{G_1} \ . \tag{34}$$

By Britton's lemma ([11]) the left-hand side in (34) is not reduced, hence $u_n x u_n^{-1}$ belongs to A or B. But this element is a conjugate of $x \in H$ therefore, according to the assumptions of the lemma, it has to be in A and $\epsilon_n = 1$. Consequently, $t^{\epsilon_n} u_n x u_n^{-1} t^{-\epsilon_n} = v \in B \backslash \{1_G\}$. Since no element of B is conjugate to the element $y \in H$ in the group G, the number n from the representation (33) must be at least 2 and

$$wxw^{-1}y^{-1} \stackrel{G_1}{=} u_0 t^{\epsilon_1} \cdot \ldots \cdot t^{\epsilon_{n-1}} u_{n-1} v u_{n-1}^{-1} t^{-\epsilon_{n-1}} \cdot \ldots \cdot t^{-\epsilon_2} u_0^{-1} y^{-1} = 1_{G_1} \ .$$

Applying Britton's lemma again, we get that the element $u_{n-1} v u_{n-1}^{-1}$ either belongs to A (and $\epsilon_{n-1} = 1$) or to B (and $\epsilon_{n-1} = -1$). So, if it is in A, then $t^{\epsilon_{n-1}} u_{n-1} v u_{n-1}^{-1} t^{-\epsilon_{n-1}} \in B$ and n has to be at least 3; thus we can proceed as before. This process will end after finitely many steps because each time we eliminate a $t^{\pm 1}$-element from the representation (33) of w. Therefore, we can assume that $u_{n-1} v u_{n-1}^{-1} \in B$ and $\epsilon_{n-1} = -1$. But the subgroup B was malnormal in G and $v \in B \backslash \{1_G\}$, hence $u_{n-1} \in B$. Hence $t^{\epsilon_{n-1}} u_{n-1} t^{\epsilon_n} \equiv t^{-1} u_{n-1} t \in A$ which contradicts to our assumption that the right-hand side of (33) is reduced.

The lemma is proved. □

Lemma 10.2. ([12, Thm. 3],[10, Cor. 1]) *Let G be a hyperbolic group with isomorphic infinite elementary subgroups A and B, and let τ be an isomorphism from A to B. The HNN-extension $G_1 = \langle G, t \mid tat^{-1} = \tau(a), a \in A \rangle$ of G with associated subgroups A and B is hyperbolic if and only if the following two conditions hold:*
1) *either A or B is a maximal elementary subgroup of G;*
2) *for all $g \in G$ the subgroup $gAg^{-1} \cap B$ is finite.*

Lemma 10.3. ([10, Thm. 4]) *Let the HNN-extension $G_1 = \langle G, t \mid tAt^{-1} = B \rangle$ be hyperbolic with A quasiconvex in G_1. Then G is quasiconvex in G_1.*

Remark 6. Suppose G_1 is a hyperbolic group and $H \leq G \leq G_1$. If H is quasiconvex in G and G is quasiconvex in G_1 then H is quasiconvex in G_1.

This follows from lemma 3.1 and the observation that an undistorted subgroup of an undistorted subgroup is undistorted in the entire group.

We are now ready to give the

Proof of corollary 3. Consider the free product $F = G * H$. Then F is a non-elementary torsion-free hyperbolic group, G is a G-subgroup of F and H is quasiconvex in F (because it is undistorted). H is non-trivial by the assumptions of the corollary, hence there is an element $y \in H$ of infinite order. Pick any $f \in G \backslash \{1_F\}$ and set $x = fyf^{-1} \in F$. From normal forms of elements of the free product F it follows that H is malnormal in F, $gHg^{-1} \cap G = \{1_F\}$ for any $g \in F$ and the infinite cyclic subgroup of F generated by x has trivial intersection with H. Denote $Q = \langle x \rangle \cup H$ – a quasiconvex subset of F.

By theorems 3 and 1 there exists a non-elementary hyperbolic quotient G_0 of F and an epimorphism $\psi_0 : F \to G_0$ with the properties 1) – 9) from the claim of theorem 1. Thus $\psi_0(G) = G_0$, ψ_0 is injective on Q, G_0 is torsion-free (by the property 7)), $\psi_0(H)$ is quasiconvex in G_0, $\psi_0(x) \in (G_0)^0$ and $\psi_0(H) \cap \langle \psi_0(x) \rangle = \{1_{G_0}\}$.

Suppose for some non-trivial $z \in G_0$ there are non-trivial $a, b \in H$ such that $z\psi_0(a)z^{-1} = \psi_0(b)$. By property 4) from the claim of theorem 1, there exists an element $u \in F$ such that $uau^{-1} = b$. H was malnormal in F, therefore $u \in H$ and $z^{-1}\psi_0(u)\psi_0(a)\psi_0(u)^{-1}z = \psi_0(a)$, i.e. $z^{-1}\psi_0(u) \in C_{G_0}\big(\psi_0(a)\big)$. Then, according to property 5), there is $v \in C_G(a)$ satisfying $\psi_0(v) = z^{-1}\psi_0(u)$. Also, by remark 5, $v \in H$. Thus, $z = \psi_0(u)\psi_0(v)^{-1} \in \psi_0(H)$, i.e. $\psi_0(H)$ is malnormal in G_0.

Enumerate all non-trivial elements of the group G_0: $g_1, g_2, \ldots$, and all its two-generated non-elementary subgroups: $K_1, K_2, \ldots$.

The group M will be constructed as an inductive limit of groups G_i, $i = 0, 1, \ldots$. Assume, the non-elementary hyperbolic torsion-free quotient G_{i-1} of G_0 has already been constructed, $i \geq 1$, and it satisfies the following properties: the natural epimorphism $\pi_{i-1} : G_0 \to G_{i-1}$ ($\pi_0 = id_{G_0} : G_0 \to G_0$) is injective on $\psi_0(H) \cup \langle \psi_0(x) \rangle$, the image of $\psi_0(H)$ is quasiconvex and malnormal in G_{i-1}; images of the elements $g_1, \ldots, g_{i-1}$ are conjugate in G_{i-1} to some elements from $\pi_{i-1}\big(\psi_0(H)\big)$, and images of the subgroups $K_1, \ldots, K_{i-1}$ either coincide with G_{i-1} or are conjugate in G_{i-1} to a subgroup of $\pi_{i-1}\big(\psi_0(H)\big)$, or are elementary.

Let us now construct the group G_i. Consider the element $\pi_{i-1}(g_i) \in G_{i-1}$. To simplify the notation, identify H and $\pi_{i-1}\big(\psi_0(H)\big)$. If $\pi_{i-1}(g_i)$ is conjugate in G_{i-1} to an element from H then set $F_i = G_{i-1}$.

If not, then the element $\pi_{i-1}(g_i)$ has infinite order in G_{i-1} and the maximal elementary subgroup $B = E\big(\pi_{i-1}(g_i)\big)$ is infinite cyclic (because G_{i-1} is torsion-free) and malnormal in G_{i-1}. Part (b) of remark 5 implies that $H \cap gBg^{-1} = \{1_{G_{i-1}}\}$ for any $g \in G_{i-1}$. Denote by $A \leq G_{i-1}$ the infinite cyclic subgroup of H generated by the element y chosen in the beginning of the proof. Then we can construct an HNN-extension

$$F_i = \langle G_{i-1}, t \mid tAt^{-1} = B \rangle \ .$$

According to lemmas 10.2 and 10.3 and basic properties of HNN-extensions (see [11, Ch. IV]), F_i is a torsion-free non-elementary hyperbolic group and the natural image of G_{i-1} is quasiconvex in it. By lemma 10.1 and remark 6, H is malnormal and quasiconvex in F_i. Note that the latter implies that $|G_{i-1} : (G_{i-1} \cap gHg^{-1})| = \infty$ for any $g \in F_i$ because, otherwise, by part (b) of remark 5, $G_{i-1} \leq gHg^{-1}$ and since $H \leq G_{i-1}$ is non-trivial, part (d) of the same remark would claim that $G_{i-1} \leq gHg^{-1} = H$. This leads to a contradiction with the fact that $x \in G_{i-1} \backslash H$.

By construction, $\pi_{i-1}(g_i)$ is conjugate to some element of H in F_i.

Now consider the subgroup $\pi_{i-1}(K_i) \leq G_{i-1} \leq F_i$. If this subgroup is elementary or conjugate to a subgroup of H in F_i, then we apply theorems 3 and 1 to obtain a torsion-free non-elementary hyperbolic group G_i and an epimorphism $\psi_i : F_i \to G_i$ such that $\psi_i(G_{i-1}) = G_i$, ψ_i is injective on $H \cup \langle x \rangle$, $\psi_i(H)$ is quasiconvex and malnormal (as before) in G_i. Then $\psi_i\big(\pi_{i-1}(K_i)\big) \leq G_i$ is either elementary or conjugate to a subgroup of $\psi_i(H)$ in G_i.

Thus, we can assume that $\pi_{i-1}(K_i)$ is non-elementary and not conjugate with a subgroup of H in F_i. Then, by remark 5,

$$|\pi_{i-1}(K_i) : (\pi_{i-1}(K_i) \cap gHg^{-1})| = \infty \quad \text{for any } g \in F_i,$$

hence we can use theorems 3 and 1 to get an epimorphism ψ_i of F_i onto a non-elementary torsion-free hyperbolic group G_i satisfying the following conditions: $\psi_i\big(\pi_{i-1}(K_i)\big) = G_i$ (consequently, $\psi_i(G_{i-1}) = G_i$), ψ_i is injective on $H \cup \langle x\rangle$ and $\psi_i(H)$ is quasiconvex and malnormal in G_i.

Thus, we have constructed the group G_i for every $i = 0, 1, 2, \ldots$.

Set $M = \lim_{\rightarrow}(G_i, \psi_{i+1})$. It remains to prove that M satisfies the properties required. There is a natural epimorphism $\pi : G_0 \to M$. Note that if a word w is trivial in M, then (by the definition of an inductive limit) w is trivial in G_i for some i, hence M is torsion-free, π is injective on H, $\pi(x) \neq 1_M$ and $\pi(H) \cap \langle \pi(x)\rangle = \{1_M\}$ (we identify H and x with their images in G_0). Therefore $\pi(H)$ is a proper subgroup of M and, since the image of H was malnormal in each G_i, $\pi(H)$ will be malnormal in M.

Denote $P = \pi(H) \leq M$ and assume that L is a proper non-trivial subgroup of M. Then there exists $a \in L \backslash \{1_M\}$. Suppose that for every $b \in L$ there exists $g_b \in M$ such that $g_b\langle a, b\rangle g_b^{-1} \leq P$. Set $g = g_{1_M}$ and pick an arbitrary $b \in L$. Then

$$a \in g_b^{-1} P g_b \cap g^{-1} P g \neq \{1_M\} \ .$$

Therefore, applying remark 5, we get $g_b^{-1} P g_b = g^{-1} P g$, thus $b \in g^{-1} P g$ for any $b \in L$, hence $gLg^{-1} \leq P$. So, if L is not conjugate to a subgroup of P then there should exist $b \in L$ such that the subgroup $\langle a, b\rangle \leq L$ is not conjugate to any subgroup from P. Choose arbitrary elements $c, d \in G_0$ with $\pi(c) = a$, $\pi(d) = b$. Then $\pi(\langle c, d\rangle) = \langle a, b\rangle$ and the image of $\langle c, d\rangle$ in G_i is not conjugate to a subgroup of the (corresponding) image of H for all i. Thus, this image is non-elementary (i.e. non-cyclic) in G_i for all i (since every cyclic subgroup will eventually be conjugate to some cyclic subgroup from an image of H). Consequently, $\langle c, d\rangle = K_j$ for some $j \in \mathbb{N}$ and the homomorphism $\pi_j : G_0 \to G_j$ will be surjective on K_j. It follows that $\langle a, b\rangle = \pi(\langle c, d\rangle) = M$ – a contradiction with the condition $L \neq M$. So, we showed that any proper subgroup L of M is conjugate to some subgroup of P.

Finally, if $N \lhd M$ and $N \neq M$ then, applying the above, we obtain an element $g \in M$ such that $N = gNg^{-1} \leq P$. But this implies that $N = \{1_M\}$ because P is malnormal. Thus, M is simple. $\square$

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