Tits alternatives for graph products

Ashot Minasyan
(Joint work with Yago Antolín)

University of Southampton

Group Theory Webinar, 27.09.2012

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

Background and motivation

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

Background and motivation

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

Background and motivation

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

Similar results have later been proved for other classes of groups. We were motivated by

Background and motivation

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

Similar results have later been proved for other classes of groups. We were motivated by

Theorem (Noskov-Vinberg, 2002)

Every subgroup of a finitely generated Coxeter group is either virtually abelian or large.

Background and motivation

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

Similar results have later been proved for other classes of groups. We were motivated by

Theorem (Noskov-Vinberg, 2002)

Every subgroup of a finitely generated Coxeter group is either virtually abelian or large.

Recall: a group G is large is there is a finite index subgroup $K \leqslant G$ s.t. K maps onto $\mathbb{F}_2$.

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H contains a copy of $\mathbb{F}_2$.

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H contains a copy of $\mathbb{F}_2$.

Thus Tits's result tells us that $\mathrm{GL}_n(F)$ satisfies the Tits Alternative rel. to $\mathcal{C}_{vsol}$.

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H contains a copy of $\mathbb{F}_2$.

Thus Tits's result tells us that $\mathrm{GL}_n(F)$ satisfies the Tits Alternative rel. to $\mathcal{C}_{vsol}$.

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H is large.

Various forms of Tits Alternative

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H contains a copy of $\mathbb{F}_2$.

Thus Tits's result tells us that $\mathrm{GL}_n(F)$ satisfies the Tits Alternative rel. to $\mathcal{C}_{vsol}$.

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H is large.

The thm. of Noskov-Vinberg implies that Coxeter gps. satisfy the Strong Tits Alternative rel. to $\mathcal{C}_{vab}$.

Graph products of groups

Graph products naturally generalize free and direct products.

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.

The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v \in V\Gamma} G_v$ by adding the relations

$$[a, b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u, v) \in E\Gamma.$$

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.
The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v \in V\Gamma} G_v$ by adding the relations

$$[a,b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u,v) \in E\Gamma.$$

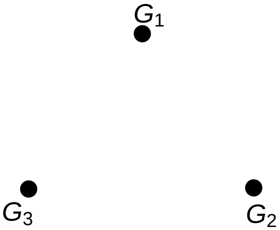


Figure : $\Gamma\mathfrak{G} \cong G_1 * G_2 * G_3$

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.
The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v \in V\Gamma} G_v$ by adding the relations

$$[a, b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u, v) \in E\Gamma.$$

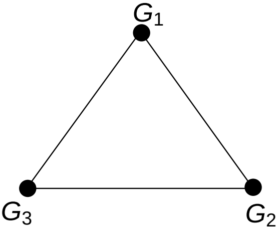


Figure : $\Gamma\mathfrak{G} \cong G_1 \times G_2 \times G_3$

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.
The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v\in V\Gamma}G_v$ by adding the relations

$$[a,b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u,v) \in E\Gamma.$$

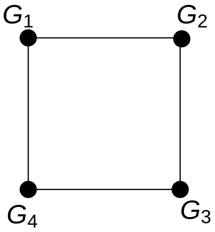


Figure : $\Gamma\mathfrak{G} \cong (G_1 * G_3) \times (G_2 * G_4)$

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.

The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v \in V\Gamma} G_v$ by adding the relations

$$[a,b] = 1 \;\forall a \in G_u, \forall b \in G_v \text{ whenever } (u,v) \in E\Gamma.$$

Basic examples of graph products are

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.
The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v \in V\Gamma} G_v$ by adding the relations

$$[a, b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u, v) \in E\Gamma.$$

Basic examples of graph products are

- right angled Artin gps. [RAAGs]

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.

The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v \in V\Gamma} G_v$ by adding the relations

$$[a, b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u, v) \in E\Gamma.$$

Basic examples of graph products are

- right angled Artin gps. [RAAGs], if all vertex gps. are $\mathbb{Z}$;

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.
The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v \in V\Gamma} G_v$ by adding the relations

$$[a, b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u, v) \in E\Gamma.$$

Basic examples of graph products are

- right angled Artin gps. [RAAGs], if all vertex gps. are $\mathbb{Z}$;
- right angled Coxeter gps.

Graph products of groups

Graph products naturally generalize free and direct products.

Let Γ be a graph and let $\mathfrak{G} = \{G_v \mid v \in V\Gamma\}$ be a family of gps.
The graph product $\Gamma\mathfrak{G}$ is obtained from the free product $*_{v \in V\Gamma} G_v$ by adding the relations

$$[a, b] = 1 \; \forall a \in G_u, \forall b \in G_v \text{ whenever } (u, v) \in E\Gamma.$$

Basic examples of graph products are

- right angled Artin gps. [RAAGs], if all vertex gps. are $\mathbb{Z}$;
- right angled Coxeter gps., if all vertex gps. are $\mathbb{Z}/2\mathbb{Z}$;

Special subgroups

If $A \subseteq V\Gamma$ and Γ_A is the full subgraph of Γ spanned by A then $\mathfrak{G}_A := \{G_v \mid v \in A\}$ generates a special subgroup G_A of $G = \Gamma\mathfrak{G}$ which is naturally isomorphic to $\Gamma_A\mathfrak{G}_A$.

Special subgroups

If $A \subseteq V\Gamma$ and Γ_A is the full subgraph of Γ spanned by A then $\mathfrak{G}_A := \{G_v \mid v \in A\}$ generates a special subgroup G_A of $G = \Gamma\mathfrak{G}$ which is naturally isomorphic to $\Gamma_A\mathfrak{G}_A$.

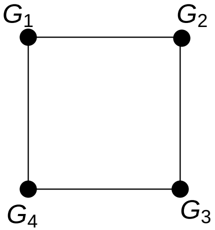


Figure : $G_{\{1,2\}} \cong G_1 \times G_2$, $G_{\{1,3\}} \cong G_1 * G_3$

Special subgroups

If $A \subseteq V\Gamma$ and Γ_A is the full subgraph of Γ spanned by A then $\mathfrak{G}_A := \{G_v \mid v \in A\}$ generates a special subgroup G_A of $G = \Gamma\mathfrak{G}$ which is naturally isomorphic to $\Gamma_A\mathfrak{G}_A$.

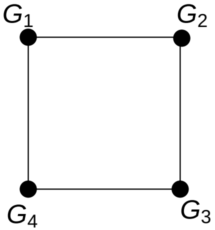


Figure : $G_{\{1,2\}} \cong G_1 \times G_2$, $G_{\{1,3\}} \cong G_1 * G_3$

There is a natural retraction $\rho_A : G = \Gamma\mathfrak{G} \to G_A$ defined by

$$\rho_A(g) = \begin{cases} g & \text{if } g \in G_u \text{ for some } u \in A \\ 1 & \text{if } g \in G_v \text{ for some } v \in V\Gamma \setminus A \end{cases}$$

Natural splittings

Graph product naturally split as amalgamated products over special subgroups:

Natural splittings

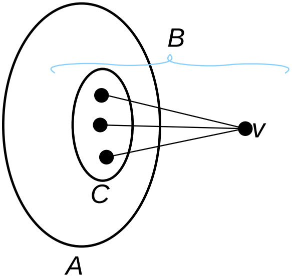
Graph product naturally split as amalgamated products over special subgroups:

$\forall v \in V\Gamma$, let $A = V\Gamma \setminus \{v\}$, $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$

Natural splittings

Graph product naturally split as amalgamated products over special subgroups:

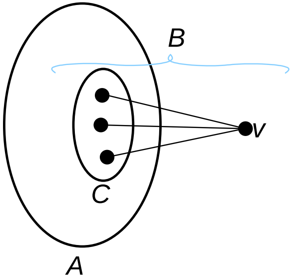
$\forall\, v \in V\Gamma$, let $A = V\Gamma \setminus \{v\}$, $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$



Natural splittings

Graph product naturally split as amalgamated products over special subgroups:

$\forall v \in V\Gamma$, let $A = V\Gamma \setminus \{v\}$, $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$



then

$$G = G_A *_{G_C} G_B \text{ and } G_B \cong G_C \times G_v.$$

Tits Alternative for graph products

Consider the following properties of the class of “small” groups $\mathcal{C}$:

Tits Alternative for graph products

Consider the following properties of the class of "small" groups $\mathcal{C}$:

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

Tits Alternative for graph products

Consider the following properties of the class of "small" groups $\mathcal{C}$:

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

Consider the following properties of the class of "small" groups $\mathcal{C}$:

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

Tits Alternative for graph products

Consider the following properties of the class of "small" groups $\mathcal{C}$:

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

Tits Alternative for graph products

Consider the following properties of the class of “small” groups $\mathcal{C}$:

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$.

Tits Alternative for graph products

Consider the following properties of the class of "small" groups $\mathcal{C}$:

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$.

Theorem A (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P4). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Tits Alternative for graph products

Consider the following properties of the class of “small” groups $\mathcal{C}$:

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$.

Theorem A (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P4). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Evidently the conditions (P0)–(P4) are necessary.

Tits Alternative for graph products

Consider the following properties of the class of "small" groups $\mathcal{C}$:

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$.

Theorem A (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P4). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Corollary

If all vertex gps. are linear then $G = \Gamma\mathfrak{G}$ satisfies the Tits Alternative rel. to $\mathcal{C}_{vsol}$.

Strong Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;

Strong Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;

(P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Strong Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;

(P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;
(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;
(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;
(P3) $\mathbb{Z} \in \mathcal{C}$;
(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;
(P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

(P5) is necessary, b/c if $L \neq \{1\}$ has no proper f.i. sbgps., then $L * L$ cannot be large.

Strong Tits Alternative for graph products

- (P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;
- (P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;
- (P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;
- (P3) $\mathbb{Z} \in \mathcal{C}$;
- (P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;
- (P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P5):

Strong Tits Alternative for graph products

- (P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;
- (P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;
- (P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;
- (P3) $\mathbb{Z} \in \mathcal{C}$;
- (P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;
- (P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P5): virt. abelian gps,

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;

(P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P5): virt. abelian gps, (virt.) polycyclic gps.,

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;

(P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P5): virt. abelian gps, (virt.) polycyclic gps., virt. nilpotent gps.,

Strong Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;

(P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P5): virt. abelian gps, (virt.) polycyclic gps., virt. nilpotent gps., (virt.) solvable gps.,

Strong Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;

(P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P5): virt. abelian gps, (virt.) polycyclic gps., virt. nilpotent gps., (virt.) solvable gps., elementary amenable gps.

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;
(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;
(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;
(P3) $\mathbb{Z} \in \mathcal{C}$;
(P4) if $\mathbb{Z}/2\mathbb{Z} \in \mathcal{C}$ then $\mathbb{D}_\infty \in \mathcal{C}$;
(P5) if $L \in \mathcal{C}$ is non-trivial and f.g. then L contains a proper f.i. sbgp.

Theorem B (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P5). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strong Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Corollary

Suppose $\mathcal{C} = \mathcal{C}_{sol-m}$ for some $m \geq 2$ or $\mathcal{C} = \mathcal{C}_{vsol-n}$ for some $n \geq 1$. Let G be a graph product of gps. from $\mathcal{C}$. Then any f.g. sbgp. of G either belongs to $\mathcal{C}$ or is large.

The Strongest Tits Alternative

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H maps onto $\mathbb{F}_2$.

The Strongest Tits Alternative

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H maps onto $\mathbb{F}_2$.

Example

The gp. $G := \langle a, b, c \mid a^2b^2 = c^2 \rangle$ is t.-f. and large but does not map onto $\mathbb{F}_2$.

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H maps onto $\mathbb{F}_2$.

Example

The gp. $G := \langle a, b, c \mid a^2b^2 = c^2 \rangle$ is t.-f. and large but does not map onto $\mathbb{F}_2$.

Example

Any residually free gp. satisfies the Strongest Tits Alternative rel. to $\mathcal{C}_{ab}$.

The Strongest Tits Alternative

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H maps onto $\mathbb{F}_2$.

Example

The gp. $G := \langle a, b, c \mid a^2 b^2 = c^2 \rangle$ is t.-f. and large but does not map onto $\mathbb{F}_2$.

Example

Any residually free gp. satisfies the Strongest Tits Alternative rel. to $\mathcal{C}_{ab}$.

Observe that if $L * L$ maps onto $\mathbb{F}_2$ then L must have an epimorphism onto $\mathbb{Z}$.

Strongest Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

Strongest Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P6) if $L \in \mathcal{C}$ is non-trivial and f.g. then L maps onto $\mathbb{Z}$.

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P6) if $L \in \mathcal{C}$ is non-trivial and f.g. then L maps onto $\mathbb{Z}$.

Theorem C (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P3) and (P6). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Strongest Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;
(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;
(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;
(P3) $\mathbb{Z} \in \mathcal{C}$;
(P6) if $L \in \mathcal{C}$ is non-trivial and f.g. then L maps onto $\mathbb{Z}$.

Theorem C (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P3) and (P6). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P3) and (P6):

Strongest Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P6) if $L \in \mathcal{C}$ is non-trivial and f.g. then L maps onto $\mathbb{Z}$.

Theorem C (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P3) and (P6). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P3) and (P6): t.-f. abelian gps,

Strongest Tits Alternative for graph products

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;

(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;

(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;

(P3) $\mathbb{Z} \in \mathcal{C}$;

(P6) if $L \in \mathcal{C}$ is non-trivial and f.g. then L maps onto $\mathbb{Z}$.

Theorem C (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P3) and (P6). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P3) and (P6): t.-f. abelian gps, t.-f. nilpotent gps,

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;
(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;
(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;
(P3) $\mathbb{Z} \in \mathcal{C}$;
(P6) if $L \in \mathcal{C}$ is non-trivial and f.g. then L maps onto $\mathbb{Z}$.

Theorem C (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P3) and (P6). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P3) and (P6): t.-f. abelian gps, t.-f. nilpotent gps, many classes of locally indicable gps.

Strongest Tits Alternative for graph products

- (P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;
- (P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;
- (P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;
- (P3) $\mathbb{Z} \in \mathcal{C}$;
- (P6) if $L \in \mathcal{C}$ is non-trivial and f.g. then L maps onto $\mathbb{Z}$.

Theorem C (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P3) and (P6). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P3) and (P6): t.-f. abelian gps, t.-f. nilpotent gps, many classes of locally indicable gps.

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

(P0) if $L \in \mathcal{C}$ and $M \cong L$ then $M \in \mathcal{C}$;
(P1) if $L \in \mathcal{C}$ and $M \leqslant L$ is f.g. then $M \in \mathcal{C}$;
(P2) if $L, M \in \mathcal{C}$ are f.g. then $L \times M \in \mathcal{C}$;
(P3) $\mathbb{Z} \in \mathcal{C}$;
(P6) if $L \in \mathcal{C}$ is non-trivial and f.g. then L maps onto $\mathbb{Z}$.

Theorem C (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P3) and (P6). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

Examples of gps. with (P0)–(P3) and (P6): t.-f. abelian gps, t.-f. nilpotent gps, many classes of locally indicable gps.

Corollary

Any non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Corollary

Any non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Applications of Theorem C

Corollary

Any non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

One can use this to recover

Theorem (Baudisch, 1981)

A 2-generator sbgp. of a RAAG is either free or free abelian.

Corollary

Any non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

One can use this to recover

Theorem (Baudisch, 1981)

A 2-generator sbgp. of a RAAG is either free or free abelian.

Combining with a result of Lyndon-Schützenberger we also get

Corollary

If G is a RAAG and $a, b, c \in G$ satisfy $a^m b^n = c^p$, for $m, n, p \geq 2$, then a, b, c pairwise commute.

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof.

Idea of the proof: modifying the embedding

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof. Let K be a RAAG and let $H \leqslant K$ be a f.g. sbgp.

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof. Let K be a RAAG and let $H \leqslant K$ be a f.g. sbgp.

First we prove that one can embed H into another RAAG G (corresponding to a finite graph Γ) s.t.

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof. Let K be a RAAG and let $H \leqslant K$ be a f.g. sbgp.

First we prove that one can embed H into another RAAG G (corresponding to a finite graph Γ) s.t.

(1) $$\rho_{\{v\}}(H) \neq \{1\} \text{ for all } v \in V\Gamma.$$

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof. Let K be a RAAG and let $H \leqslant K$ be a f.g. sbgp.

First we prove that one can embed H into another RAAG G (corresponding to a finite graph Γ) s.t.

$$\rho_{\{v\}}(H) \neq \{1\} \text{ for all } v \in V\Gamma. \tag{1}$$

Now argue by induction on $|V\Gamma|$.

Idea of the proof: modifying the embedding

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof. Let K be a RAAG and let $H \leqslant K$ be a f.g. sbgp.

First we prove that one can embed H into another RAAG G (corresponding to a finite graph Γ) s.t.

(1) $$\rho_{\{v\}}(H) \neq \{1\} \text{ for all } v \in V\Gamma.$$

Now argue by induction on $|V\Gamma|$. The case $|V\Gamma| \leq 2$ is easy.

Idea of the proof: modifying the embedding

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof. Let K be a RAAG and let $H \leqslant K$ be a f.g. sbgp.

First we prove that one can embed H into another RAAG G (corresponding to a finite graph Γ) s.t.

$$\rho_{\{v\}}(H) \neq \{1\} \text{ for all } v \in V\Gamma. \tag{1}$$

Now argue by induction on $|V\Gamma|$. The case $|V\Gamma| \leq 2$ is easy. So assume that $|V\Gamma| \geq 3$.

Idea of the proof: modifying the embedding

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof. Let K be a RAAG and let $H \leqslant K$ be a f.g. sbgp.

First we prove that one can embed H into another RAAG G (corresponding to a finite graph Γ) s.t.

$$(1) \qquad \rho_{\{v\}}(H) \neq \{1\} \text{ for all } v \in V\Gamma.$$

Now argue by induction on $|V\Gamma|$. The case $|V\Gamma| \leq 2$ is easy. So assume that $|V\Gamma| \geq 3$.

If $\rho_A(H)$ is non-abelian for some $A \subsetneqq V\Gamma$ then $H \twoheadrightarrow \rho_A(H) \twoheadrightarrow \mathbb{F}_2$ by induction, as $\rho_A(H) \leqslant G_A \cong \Gamma_A \mathfrak{G}_A$ and $|A| < |V\Gamma|$.

Idea of the proof: modifying the embedding

Corollary

Any f.g. non-abelian sbgp. of a RAAG maps onto $\mathbb{F}_2$.

Idea of the proof. Let K be a RAAG and let $H \leqslant K$ be a f.g. sbgp.

First we prove that one can embed H into another RAAG G (corresponding to a finite graph Γ) s.t.

(1) $$\rho_{\{v\}}(H) \neq \{1\} \text{ for all } v \in V\Gamma.$$

Now argue by induction on $|V\Gamma|$. The case $|V\Gamma| \leq 2$ is easy. So assume that $|V\Gamma| \geq 3$.

If $\rho_A(H)$ is non-abelian for some $A \subsetneqq V\Gamma$ then $H \twoheadrightarrow \rho_A(H) \twoheadrightarrow \mathbb{F}_2$ by induction, as $\rho_A(H) \leqslant G_A \cong \Gamma_A \mathfrak{G}_A$ and $|A| < |V\Gamma|$.
Thus we can suppose that

(2) $$\rho_A(H) \text{ is abelian for every } A \subsetneqq V\Gamma.$$

Idea of the proof: some graph theory

Γ is irreducible if the complement graph Γ^c is connected,

Idea of the proof: some graph theory

Γ is irreducible if the complement graph Γ^c is connected, where

$$V\Gamma^c := V\Gamma \text{ and } E\Gamma^c := \{(u,v) \in V\Gamma \times V\Gamma \mid (u,v) \notin E\Gamma\}.$$

Idea of the proof: some graph theory

Γ is irreducible if the complement graph Γ^c is connected, where

$$V\Gamma^c := V\Gamma \text{ and } E\Gamma^c := \{(u,v) \in V\Gamma \times V\Gamma \mid (u,v) \notin E\Gamma\}.$$

Note: Γ is reducible iff $G = \Gamma\mathfrak{G}$ splits as a direct product of two special sbgps.

Idea of the proof: some graph theory

Γ is irreducible if the complement graph Γ^c is connected, where

$$V\Gamma^c := V\Gamma \text{ and } E\Gamma^c := \{(u,v) \in V\Gamma \times V\Gamma \mid (u,v) \notin E\Gamma\}.$$

Note: Γ is reducible iff $G = \Gamma\mathfrak{G}$ splits as a direct product of two special sbgps.

If Γ is reducible then $V\Gamma = A \sqcup B$ and $G = G_A \times G_B$. Thus $H \leqslant \rho_A(H) \times \rho_B(H)$ is abelian by (2).

Idea of the proof: some graph theory

Γ is irreducible if the complement graph Γ^c is connected, where

$$V\Gamma^c := V\Gamma \text{ and } E\Gamma^c := \{(u,v) \in V\Gamma \times V\Gamma \mid (u,v) \notin E\Gamma\}.$$

Note: Γ is reducible iff $G = \Gamma\mathfrak{G}$ splits as a direct product of two special sbgps.

If Γ is reducible then $V\Gamma = A \sqcup B$ and $G = G_A \times G_B$. Thus $H \leqslant \rho_A(H) \times \rho_B(H)$ is abelian by (2).

Note: If Γ is irreducible and $|V\Gamma| \geq 2$, then $\exists\, v \in V\Gamma$ s.t. for $A := V\Gamma \setminus \{v\}$, Γ_A is also irreducible.

Idea of the proof: the Structure theorem

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$.

Idea of the proof: the Structure theorem

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$. Let $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$. Then $G = G_A *_{G_C} G_B$ as before.

Idea of the proof: the Structure theorem

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$. Let $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$. Then $G = G_A *_{G_C} G_B$ as before.

Theorem (Structure Thm.)

If G is a RAAG corresponding to a finite irreducible graph Γ with $|V\Gamma| \geq 2$ and $M \leqslant G$ then one of the following holds:

Idea of the proof: the Structure theorem

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$. Let $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$. Then $G = G_A *_{G_C} G_B$ as before.

Theorem (Structure Thm.)

If G is a RAAG corresponding to a finite irreducible graph Γ with $|V\Gamma| \geq 2$ and $M \leqslant G$ then one of the following holds:

- $\exists\, A \subsetneqq V\Gamma$ *and* $g \in G$ *s.t.* $M \subseteq gG_Ag^{-1}$;

Idea of the proof: the Structure theorem

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$. Let $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$. Then $G = G_A *_{G_C} G_B$ as before.

Theorem (Structure Thm.)

If G is a RAAG corresponding to a finite irreducible graph Γ with $|V\Gamma| \geq 2$ and $M \leqslant G$ then one of the following holds:

- $\exists\, A \subsetneqq V\Gamma$ *and* $g \in G$ *s.t.* $M \subseteq gG_Ag^{-1}$;
- $M \cong \mathbb{Z}$;

Idea of the proof: the Structure theorem

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$. Let $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$. Then $G = G_A *_{G_C} G_B$ as before.

Theorem (Structure Thm.)

If G is a RAAG corresponding to a finite irreducible graph Γ with $|V\Gamma| \geq 2$ and $M \leqslant G$ then one of the following holds:

- $\exists\, A \subsetneqq V\Gamma$ *and* $g \in G$ *s.t.* $M \subseteq gG_Ag^{-1}$;
- $M \cong \mathbb{Z}$;
- M *contains a copy of* $\mathbb{F}_2$.

Idea of the proof: the Structure theorem

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$. Let $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$. Then $G = G_A *_{G_C} G_B$ as before.

Theorem (Structure Thm.)

If G is a RAAG corresponding to a finite irreducible graph Γ with $|V\Gamma| \geq 2$ and $M \leqslant G$ then one of the following holds:

- $\exists\, A \subsetneqq V\Gamma$ *and* $g \in G$ *s.t.* $M \subseteq gG_Ag^{-1}$;
- $M \cong \mathbb{Z}$;
- *M contains a copy of* $\mathbb{F}_2$.

Applying this thm. to G_A and $M := \rho_A(H) \leqslant G_A$, we see that $M = \rho_A(H)$ is cyclic by (1).

Idea of the proof: the Structure theorem

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$. Let $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$. Then $G = G_A *_{G_C} G_B$ as before.

Theorem (Structure Thm.)

If G is a RAAG corresponding to a finite irreducible graph Γ with $|V\Gamma| \geq 2$ and $M \leqslant G$ then one of the following holds:

- $\exists\, A \subsetneqq V\Gamma$ *and* $g \in G$ *s.t.* $M \subseteq gG_Ag^{-1}$;
- $M \cong \mathbb{Z}$;
- M *contains a copy of* $\mathbb{F}_2$.

Applying this thm. to G_A and $M := \rho_A(H) \leqslant G_A$, we see that $M = \rho_A(H)$ is cyclic by (1).

It follows that $\rho_A(H) \cap hG_Ch^{-1} = \{1\}\ \forall\, h \in G_A$.

Take $v \in V\Gamma$ so that Γ_A is irreducible, where $A := V\Gamma \setminus \{v\}$. Let $C = \mathrm{link}(v)$ and $B = C \cup \{v\}$. Then $G = G_A *_{G_C} G_B$ as before.

Theorem (Structure Thm.)

If G is a RAAG corresponding to a finite irreducible graph Γ with $|V\Gamma| \geq 2$ and $M \leqslant G$ then one of the following holds:

- $\exists\, A \subsetneqq V\Gamma$ *and* $g \in G$ *s.t.* $M \subseteq gG_Ag^{-1}$;
- $M \cong \mathbb{Z}$;
- *M contains a copy of* $\mathbb{F}_2$.

Applying this thm. to G_A and $M := \rho_A(H) \leqslant G_A$, we see that $M = \rho_A(H)$ is cyclic by (1).

It follows that $\rho_A(H) \cap hG_Ch^{-1} = \{1\}\ \forall\, h \in G_A$.

Hence

$$H \cap gG_Cg^{-1} = \{1\} \quad \forall\, g \in G.$$

Idea of the proof: final step

$$H \cap gG_C g^{-1} = \{1\} \ \forall g \in G. \tag{3}$$

Idea of the proof: final step

$$(3) \qquad H \cap gG_Cg^{-1} = \{1\} \;\; \forall g \in G.$$

Recall that $G = G_A *_{G_C} G_B$.

Idea of the proof: final step

(3) $$H \cap gG_Cg^{-1} = \{1\} \;\; \forall g \in G.$$

Recall that $G = G_A *_{G_C} G_B$. By gen. Kurosh Thm. (3) $\Longrightarrow$

$$H = H_1 * \cdots * H_k * F,$$

(3) $$H \cap gG_Cg^{-1} = \{1\} \ \forall g \in G.$$

Recall that $G = G_A *_{G_C} G_B$. By gen. Kurosh Thm. (3) $\Longrightarrow$

$$H = H_1 * \cdots * H_k * F,$$

where F is free, and $H_i \leqslant g_iG_Ag_i^{-1}$ or $H_i \leqslant g_iG_Bg_i^{-1}$ for some $g_i \in G$.

Idea of the proof: final step

(3) $$H \cap gG_C g^{-1} = \{1\} \;\; \forall g \in G.$$

Recall that $G = G_A *_{G_C} G_B$. By gen. Kurosh Thm. (3) $\Longrightarrow$

$$H = H_1 * \cdots * H_k * F,$$

where F is free, and $H_i \leqslant g_i G_A g_i^{-1}$ or $H_i \leqslant g_i G_B g_i^{-1}$ for some $g_i \in G$.

Since each H_i maps onto $\mathbb{Z}$ (follows from (P6)), we deduce that H maps onto $\mathbb{Z} * \mathbb{Z} \cong \mathbb{F}_2$. $\square$