The SQ-universality and residual properties of relatively hyperbolic groups

G. Arzhantseva A. Minasyan * D. Osin †

Abstract

In this paper we study residual properties of relatively hyperbolic groups. In particular, we show that if a group $G$ is non-elementary and hyperbolic relative to a collection of proper subgroups, then $G$ is SQ-universal.

1 Introduction

The notion of a group hyperbolic relative to a collection of subgroups was originally suggested by Gromov [9] and since then it has been elaborated from different points of view [3, 6, 5, 21]. The class of relatively hyperbolic groups includes many examples. For instance, if $M$ is a complete finite-volume manifold of pinched negative sectional curvature, then $\pi_1(M)$ is hyperbolic with respect to the cusp subgroups [3, 6]. More generally, if $G$ acts isometrically and properly discontinuously on a proper hyperbolic metric space $X$ so that the induced action of $G$ on $\partial X$ is geometrically finite, then $G$ is hyperbolic relative to the collection of maximal parabolic subgroups [3]. Groups acting on $CAT(0)$ spaces with isolated flats are hyperbolic relative to the collection of flat stabilizers [13]. Algebraic examples of relatively hyperbolic groups include free products and their small cancellation quotients [21], fully residually free groups (or Sela’s limit groups) [4], and, more generally, groups acting freely on $\mathbb{R}^n$-trees [10].

The main goal of this paper is to study residual properties of relatively hyperbolic groups. Recall that a group $G$ is called SQ-universal if every countable group can be embedded into a quotient of $G$ [25]. It is straightforward to see that any SQ-universal group contains an infinitely generated free subgroup. Furthermore, since the set of all finitely generated groups is uncountable and every single quotient of $G$ contains (at most) countably many finitely generated subgroups, every SQ-universal group has uncountably many non-isomorphic quotients. Thus the property of being SQ-universal may, in a very rough sense, be considered as an indication of "largeness" of a group.

The first non-trivial example of an SQ-universal group was provided by Higman, Neumann and Neumann [11], who proved that the free group of rank 2 is SQ-universal. Presently

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many other classes of groups are known to be SQ-universal: various HNN-extensions and amalgamated products \[7, 15, 24\], groups of deficiency 2 \[2\], most \(C(3)\&T(6)\)-groups \[12\], etc. The SQ-universality of non-elementary hyperbolic groups was proved by Olshanskii in \[19\]. On the other hand, for relatively hyperbolic groups, there are some partial results. Namely, in \[8\] Fine proved the SQ-universality of certain Kleinian groups. The case of fundamental groups of hyperbolic 3-manifolds was studied by Ratcliffe in \[23\].

In this paper we prove the SQ-universality of relatively hyperbolic groups in the most general settings. Let a group \(G\) be hyperbolic relative to a collection of subgroups \(\{H_\lambda\}_{\lambda \in \Lambda}\) (called peripheral subgroups). We say that \(G\) is properly relatively hyperbolic relative to \(\{H_\lambda\}_{\lambda \in \Lambda}\) (or \(G\) is a PRH group for brevity), if \(H_\lambda \neq G\) for all \(\lambda \in \Lambda\). Recall that a group is elementary, if it contains a cyclic subgroup of finite index. We observe that every non-elementary PRH group has a unique maximal finite normal subgroup denoted by \(E_G(G)\) (see Lemmas \[4.3\] and \[3.3\] below).

**Theorem 1.1.** Suppose that a group \(G\) is non-elementary and properly relatively hyperbolic with respect to a collection of subgroups \(\{H_\lambda\}_{\lambda \in \Lambda}\). Then for each finitely generated group \(R\), there exists a quotient group \(Q\) of \(G\) and an embedding \(R \hookrightarrow Q\) such that:

1. \(Q\) is properly relatively hyperbolic with respect to the collection \(\{\psi(H_\lambda)\}_{\lambda \in \Lambda} \cup \{R\}\) where \(\psi: G \to Q\) denotes the natural epimorphism;

2. For each \(\lambda \in \Lambda\), we have \(H_\lambda \cap \ker(\psi) = H_\lambda \cap E_G(G)\), that is, \(\psi(H_\lambda)\) is naturally isomorphic to \(H_\lambda/(H_\lambda \cap E_G(G))\).

In general, we can not require the epimorphism \(\psi\) to be injective on every \(H_\lambda\). Indeed, it is easy to show that a finite normal subgroup of a relatively hyperbolic group must be contained in each infinite peripheral subgroup (see Lemma \[4.4\]). Thus the image of \(E_G(G)\) in \(Q\) will have to be inside \(R\) whenever \(R\) is infinite. If, in addition, the group \(R\) is torsion-free, the latter inclusion implies \(E_G(G) \leq \ker(\psi)\). This would be the case if one took \(G = F_2 \times \mathbb{Z}/(2\mathbb{Z})\) and \(R = \mathbb{Z}\), where \(F_2\) denotes the free group of rank 2 and \(G\) is properly hyperbolic relative to its subgroup \(\mathbb{Z}/(2\mathbb{Z}) = E_G(G)\).

Since any countable group is embeddable into a finitely generated group, we obtain the following.

**Corollary 1.2.** Any non-elementary PRH group is SQ-universal.

Let us mention a particular case of Corollary \[1.2\]. In \[7\] the authors asked whether every finitely generated group with infinite number of ends is SQ-universal. The celebrated Stallings theorem \[26\] states that a finitely generated group has infinite number of ends if and only if it splits as a nontrivial HNN-extension or amalgamated product over a finite subgroup. The case of amalgamated products was considered by Lossov who provided the positive answer in \[15\]. Corollary \[1.2\] allows us to answer the question in the general case. Indeed, every group with infinite number of ends is non-elementary and properly relatively hyperbolic, since the action of such a group on the corresponding Bass-Serre tree satisfies Bowditch’s definition of relative hyperbolicity \[9\].

**Corollary 1.3.** A finitely generated group with infinite number of ends is SQ-universal.
The methods used in the proof of Theorem 1.1 can also be applied to obtain other results:

**Theorem 1.4.** Any two finitely generated non-elementary PRH groups $G_1, G_2$ have a common non-elementary PRH quotient $Q$. Moreover, $Q$ can be obtained from the free product $G_1 \ast G_2$ by adding finitely many relations.

In [18] Olshanskii proved that any non-elementary hyperbolic group has a non-trivial finitely presented quotient without proper subgroups of finite index. This result was used by Lubotzky and Bass [11] to construct representation rigid linear groups of non-arithmetic type thus solving in negative the Platonov Conjecture. Theorem 1.4 yields a generalization of Olshanskii’s result.

**Definition 1.5.** Given a class of groups $G$, we say that a group $R$ is *residually incompatible with* $G$ if for any group $A \in G$, any homomorphism $R \to A$ has a trivial image.

If $G$ and $R$ are finitely presented groups, $G$ is properly relatively hyperbolic, and $R$ is residually incompatible with a class of groups $G$, we can apply Theorem 1.4 to $G_1 = G$ and $G_2 = R \ast R$. Obviously, the obtained common quotient of $G_1$ and $G_2$ is finitely presented and residually incompatible with $G$.

**Corollary 1.6.** Let $G$ be a class of groups. Suppose that there exists a finitely presented group $R$ that is residually incompatible with $G$. Then every finitely presented non-elementary PRH group has a non-trivial finitely presented quotient group that is residually incompatible with $G$.

Recall that there are finitely presented groups having no non-trivial recursively presented quotients with decidable word problem [16]. Applying the previous corollary to the class $G$ of all recursively presented groups with decidable word problem, we obtain the following result.

**Corollary 1.7.** Every non-elementary finitely presented PRH group has an infinite finitely presented quotient group $Q$ such that the word problem is undecidable in each non-trivial quotient of $Q$.

In particular, $Q$ has no proper subgroups of finite index. The reader can easily check that Corollary 1.6 can also be applied to the classes of all torsion (torsion-free, Noetherian, Artinian, amenable, etc.) groups.

### 2 Relatively hyperbolic groups

We recall the definition of relatively hyperbolic groups suggested in [21] (for equivalent definitions in the case of finitely generated groups see [3, 5, 6]). Let $G$ be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a fixed collection of subgroups of $G$ (called *peripheral subgroups*), $X$ a subset of $G$. We say that $X$ is a *relative generating set* of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ if $G$ is generated by $X$ together with the union of all $H_\lambda$ (for convenience, we always assume that $X = X^{-1}$). In this situation the group $G$ can be considered as a quotient of the free product

$$F = (\ast_{\lambda \in \Lambda} H_\lambda) \ast F(X),$$

(1)
where \( F(X) \) is the free group with the basis \( X \). Suppose that \( R \) is a subset of \( F \) such that the kernel of the natural epimorphism \( F \to G \) is a normal closure of \( R \) in the group \( F \), then we say that \( G \) has relative presentation

\[
\langle X, \{ H_\lambda \}_{\lambda \in \Lambda} \mid R = 1, R \in R \rangle.
\]

If sets \( X \) and \( R \) are finite, the presentation (2) is said to be relatively finite.

**Definition 2.1.** We set

\[
\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} (H_\lambda \setminus \{1\}).
\]

A group \( G \) is relatively hyperbolic with respect to a collection of subgroups \( \{ H_\lambda \}_{\lambda \in \Lambda} \), if \( G \) admits a relatively finite presentation (2) with respect to \( \{ H_\lambda \}_{\lambda \in \Lambda} \) satisfying a linear relative isoperimetric inequality. That is, there exists \( C > 0 \) satisfying the following condition. For every word \( w \) in the alphabet \( X \cup \mathcal{H} \) representing the identity in the group \( G \), there exists an expression

\[
w = F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i
\]

with the equality in the group \( F \), where \( R_i \in R \), \( f_i \in F \), for \( i = 1, \ldots, k \), and \( k \leq C \|w\| \), where \( \|w\| \) is the length of the word \( w \). This definition is independent of the choice of the (finite) generating set \( X \) and the (finite) set \( R \) in (2).

For a combinatorial path \( p \) in the Cayley graph \( \Gamma(G, X \cup \mathcal{H}) \) of \( G \) with respect to \( X \cup \mathcal{H} \), \( p_- \), \( p_+ \), \( l(p) \), and \( \text{Lab}(p) \) will denote the initial point, the ending point, the length (that is, the number of edges) and the label of \( p \) respectively. Further, if \( \Omega \) is a subset of \( G \) and \( g \in \langle \Omega \rangle \leq G \), then \( |g|_\Omega \) will be used to denote the length of a shortest word in \( \Omega^{\pm 1} \) representing \( g \).

Let us recall some terminology introduced in [21]. Suppose \( q \) is a path in \( \Gamma(G, X \cup \mathcal{H}) \).

**Definition 2.2.** A subpath \( p \) of \( q \) is called an \( H_\lambda \)-component for some \( \lambda \in \Lambda \) (or simply a component) of \( q \), if the label of \( p \) is a word in the alphabet \( H_\lambda \setminus \{1\} \) and \( p \) is not contained in a bigger subpath of \( q \) with this property.

Two components \( p_1, p_2 \) of a path \( q \) in \( \Gamma(G, X \cup \mathcal{H}) \) are called connected if they are \( H_\lambda \)-components for the same \( \lambda \in \Lambda \) and there exists a path \( c \) in \( \Gamma(G, X \cup \mathcal{H}) \) connecting a vertex of \( p_1 \) to a vertex of \( p_2 \) such that \( \text{Lab}(c) \) entirely consists of letters from \( H_\lambda \). In algebraic terms this means that all vertices of \( p_1 \) and \( p_2 \) belong to the same coset \( gH_\lambda \) for a certain \( g \in G \). We can always assume \( c \) to have length at most 1, as every nontrivial element of \( H_\lambda \) is included in the set of generators. An \( H_\lambda \)-component \( p \) of a path \( q \) is called isolated if no distinct \( H_\lambda \)-component of \( q \) is connected to \( p \). A path \( q \) is said to be without backtracking if all its components are isolated.

The next lemma is a simplification of Lemma 2.27 from [21].

**Lemma 2.3.** Suppose that a group \( G \) is hyperbolic relative to a collection of subgroups \( \{ H_\lambda \}_{\lambda \in \Lambda} \). Then there exists a finite subset \( \Omega \subseteq G \) and a constant \( K \geq 0 \) such that
the following condition holds. Let \( q \) be a cycle in \( \Gamma(G, X \cup \mathcal{H}) \), \( p_1, \ldots, p_k \) a set of isolated \( H_\lambda \)-components of \( q \) for some \( \lambda \in \Lambda \), \( g_1, \ldots, g_k \) elements of \( G \) represented by labels \( \text{Lab}(p_1), \ldots, \text{Lab}(p_k) \) respectively. Then \( g_1, \ldots, g_k \) belong to the subgroup \( \langle \Omega \rangle \leq G \) and the word lengths of \( g_i \)'s with respect to \( \Omega \) satisfy the inequality

\[
\sum_{i=1}^{k} |g_i|_{\Omega} \leq Kl(q).
\]

3 Suitable subgroups of relatively hyperbolic groups

Throughout this section let \( G \) be a group which is properly hyperbolic relative to a collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \), \( X \) a finite relative generating set of \( G \), and \( \Gamma(G, X \cup \mathcal{H}) \) the Cayley graph of \( G \) with respect to the generating set \( X \cup \mathcal{H} \), where \( \mathcal{H} \) is given by \([3]\). Recall that an element \( g \in G \) is called hyperbolic if it is not conjugate to an element of some \( H_\lambda, \lambda \in \Lambda \). The following description of elementary subgroups of \( G \) was obtained in \([20]\).

Lemma 3.1. Let \( g \) be a hyperbolic element of infinite order of \( G \). Then the following conditions hold.

1. The element \( g \) is contained in a unique maximal elementary subgroup \( E_G(g) \) of \( G \), where

\[
E_G(g) = \{ f \in G : f^{-1}g^n f = g^{\pm n} \text{ for some } n \in \mathbb{N} \}.
\]

2. The group \( G \) is hyperbolic relative to the collection \( \{H_\lambda\}_{\lambda \in \Lambda} \cup \{E_G(g)\} \).

Given a subgroup \( S \leq G \), we denote by \( S^0 \) the set of all hyperbolic elements of \( S \) of infinite order. Recall that two elements \( f, g \in G^0 \) are said to be commensurable (in \( G \)) if \( f^k \) is conjugated to \( g^l \) in \( G \) for some non-zero integers \( k \) and \( l \).

Definition 3.2. A subgroup \( S \leq G \) is called suitable, if there exist at least two non-commensurable elements \( f_1, f_2 \in S^0 \), such that \( E_G(f_1) \cap E_G(f_2) = \{1\} \).

If \( S^0 \neq \emptyset \), we define

\[
E_G(S) = \bigcap_{g \in S^0} E_G(g).
\]

Lemma 3.3. If \( S \leq G \) is a non-elementary subgroup and \( S^0 \neq \emptyset \), then \( E_G(S) \) is the maximal finite subgroup of \( G \) normalized by \( S \).

Proof. Indeed, if a finite subgroup \( M \leq G \) is normalized by \( S \), then \( |S : C_S(M)| < \infty \) where \( C_S(M) = \{ g \in S : g^{-1}xg = x, \forall x \in M \} \). Formula \([5]\) implies that \( M \leq E_G(g) \) for every \( g \in S^0 \), hence \( M \leq E_G(S) \).

On the other hand, if \( S \) is non-elementary and \( S^0 \neq \emptyset \), there exist \( h \in S^0 \) and \( a \in S^0 \setminus E_G(h) \). Then \( a^{-1}ha \in S^0 \) and the intersection \( E_G(a^{-1}ha) \cap E_G(h) \) is finite. Indeed if \( E_G(a^{-1}ha) \cap E_G(h) \) were infinite, we would have \( (a^{-1}ha)^n = h^k \) for some \( n, k \in \mathbb{Z}\setminus\{0\} \), which would contradict to \( a \notin E_G(h) \). Hence \( E_G(S) \leq E_G(a^{-1}ha) \cap E_G(h) \) is finite. Obviously, \( E_G(S) \) is normalized by \( S \) in \( G \).
The main result of this section is the following

**Proposition 3.4.** Suppose that a group $G$ is hyperbolic relative to a collection $\{H_\lambda\}_{\lambda \in \Lambda}$ and $S$ is a subgroup of $G$. Then the following conditions are equivalent.

1. $S$ is suitable;
2. $S^0 \neq \emptyset$ and $E_G(S) = \{1\}$.

Our proof of Proposition 3.4 will make use of several auxiliary statements below.

**Lemma 3.5** (Lemma 4.4, [20]). For any $\lambda \in \Lambda$ and any element $a \in G \setminus H_\lambda$, there exists a finite subset $F_\lambda = F_\lambda(a) \subseteq H_\lambda$ such that if $h \in H_\lambda \setminus F_\lambda$, then $ah$ is a hyperbolic element of infinite order.

It can be seen from Lemma 3.1 that every hyperbolic element $g \in G$ of infinite order is contained inside the elementary subgroup

$$E^+_G(g) = \{f \in G : f^{-1}g^n f = g^n \text{ for some } n \in \mathbb{N}\} \leq E_G(g),$$

and $|E_G(g) : E^+_G(g)| \leq 2$.

**Lemma 3.6.** Suppose $g_1, g_2 \in G^0$ are non-commensurable and $A = \langle g_1, g_2 \rangle \leq G$. Then there exists an element $h \in A^0$ such that:

1. $h$ is not commensurable with $g_1$ and $g_2$;
2. $E_G(h) = E^+_G(h) \leq \langle h, E_G(g_1) \cap E_G(g_2) \rangle$. If, in addition, $E_G(g_j) = E^+_G(g_j)$, $j = 1, 2$, then $E_G(h) = E^+_G(h) = \langle h \rangle \times (E_G(g_1) \cap E_G(g_2))$.

**Proof.** By Lemma 3.1, $G$ is hyperbolic relative to the collection of peripheral subgroups $C_1 = \{H_\lambda\}_{\lambda \in \Lambda} \cup \{E_G(g_1)\} \cup \{E_G(g_2)\}$. The center $Z(E^+_G(g_j))$ has finite index in $E^+_G(g_j)$, hence (possibly, after replacing $g_j$ with a power of itself) we can assume that $g_j \in Z(E^+_G(g_j))$, $j = 1, 2$. Using Lemma 3.5 we can find an integer $n_1 \in \mathbb{N}$ such that the element $g_3 = g_2g_1^{n_1} \in A$ is hyperbolic relatively to $C_1$ and has infinite order. Applying Lemma 3.1 again, we achieve hyperbolicity of $G$ relative to $C_2 = C_1 \cup \{E_G(g_3)\}$. Set $H' = \bigcup_{H \in C_2} (H \setminus \{1\})$.

Let $\Omega \subset G$ be the finite subset and $K > 0$ the constant chosen according to Lemma 2.3 (where $G$ is considered to be relatively hyperbolic with respect to $C_2$). Using Lemma 3.5 two more times, we can find numbers $m_1, m_2, m_3 \in \mathbb{N}$ such that

$$g_i^{m_i} \notin \{y \in \langle \Omega \rangle : |y|_\Omega \leq 21K\}, \quad i = 1, 2, 3,$$

and $h = g_1^{m_1} g_3^{m_3} g_2^{m_2} \in A$ is a hyperbolic element (with respect to $C_2$) and has infinite order. Indeed, first we choose $m_1$ to satisfy (6). By Lemma 3.5 there is $m_3$ satisfying [6], so that $g_1^{m_1} g_3^{m_3} \in A^0$. Similarly $m_2$ can be chosen sufficiently big to satisfy (6) and $g_1^{m_1} g_3^{m_3} g_2^{m_2} \in A^0$. In particular, $h$ will be non-commensurable with $g_j$, $j = 1, 2$ (otherwise, there would exist $f \in G$ and $n \in \mathbb{N}$ such that $f^{-1}h^nf \in E(g_j)$, implying $h \in fE(g_j)f^{-1}$ by Lemma 3.1 and contradicting the hyperbolicity of $h$).
Consider a path $q$ labelled by the word $(g_1^{m_1}g_3^{m_3}g_2^{m_2})^l$ in $\Gamma(G, X \cup \mathcal{H}')$ for some $l \in \mathbb{Z} \setminus \{0\}$, where each $g_i^{m_i}$ is treated as a single letter from $\mathcal{H}'$. After replacing $q$ with $q^{-1}$, if necessary, we assume that $l \in \mathbb{N}$. Let $p_1, \ldots, p_{3l}$ be all components of $q$; by the construction of $q$, we have $l(p_j) = 1$ for each $j$. Suppose not all of these components are isolated. Then one can find indices $1 \leq s < t \leq 3l$ and $i \in \{1, 2, 3\}$ such that $p_s$ and $p_t$ are $E_G(g_i)$-components of $q$, $(p_t)_-$ and $(p_s)_+$ are connected by a path $r$ with $\text{Lab}(r) \in E_G(g_i)$, $l(r) \leq 1$, and $(t - s)$ is minimal with this property. To simplify the notation, assume that $i = 1$ (the other two cases are similar). Then $p_{s+1}p_{s+2}\ldots p_{t-1}r$, and there are exactly $(t - s)/3 \geq 1$ of them. Applying Lemma 2.3 we obtain $g_i^{m_i} \in \langle \Omega \rangle$ and

$$\frac{t - s}{3} g_i^{m_i} |_{\Omega} \leq K(t - s).$$

Hence $|g_i^{m_i}|_{\Omega} \leq 3K$, contradicting (6). Therefore two distinct components of $q$ can not be connected with each other; that is, the path $q$ is without backtracking.

To finish the proof of Lemma 3.6 we need an auxiliary statement below. Denote by $\mathcal{W}$ the set of all subwords of words $(g_1^{m_1}g_3^{m_3}g_2^{m_2})^l$, $l \in \mathbb{Z}$ (where $g_i^{\pm m_i}$ is treated as a single letter from $\mathcal{H}'$). Consider an arbitrary cycle $o = rqr'q'$ in $\Gamma(G, X \cup \mathcal{H}')$, where $\text{Lab}(q), \text{Lab}(q') \in \mathcal{W}$; and set $C = \max\{l(r), l(r')\}$. Let $p$ be a component of $q$ (or $q'$). We will say that $p$ is regular if it is not an isolated component of $o$. As $q$ and $q'$ are without backtracking, this means that $p$ is either connected to some component of $q'$ (respectively $q$), or to a component of $r$, or $r'$.

Lemma 3.7. In the above notations

(a) if $C \leq 1$ then every component of $q$ or $q'$ is regular;

(b) if $C \geq 2$ then each of $q$ and $q'$ can have at most $15C$ components which are not regular.

Proof. Assume the contrary to (a). Then one can choose a cycle $o = rqr'q'$ with $l(r), l(r') \leq 1$, having at least one $E(g_i)$-isolated component on $q$ or $q'$ for some $i \in \{1, 2, 3\}$, and such that $l(q) + l(q')$ is minimal. Clearly the latter condition implies that each component of $q$ or $q'$ is an isolated component of $o$. Therefore $q$ and $q'$ together contain $k$ distinct $E(g_i)$-components of $o$ where $k \geq 1$ and $k \geq \lceil l(q)/3 \rceil + \lceil l(q')/3 \rceil$. Applying Lemma 2.3 we obtain $g_i^{m_i} \in \langle \Omega \rangle$ and $k|g_i^{m_i}|_{\Omega} \leq K(l(q) + l(q') + 2)$, therefore $|g_i^{m_i}|_{\Omega} \leq 11K$, contradicting the choice of $m_i$ in (6).

Let us prove (b). Suppose that $C \geq 2$ and $q$ contains more than $15C$ isolated components of $o$. We consider two cases:

Case 1. No component of $q$ is connected to a component of $q'$. Then a component of $q$ or $q'$ can be regular only if it is connected to a component of $r$ or $r'$. Since $q$ and $q'$ are without backtracking, two distinct components of $q$ or $q'$ can not be connected to the same component of $r$ (or $r'$). Hence $q$ and $q'$ together can contain at most $2C$ regular components. Thus there is an index $i \in \{1, 2, 3\}$ such that the cycle $o$ has $k$ isolated $E(g_i)$-components, where $k \geq \lceil l(q)/3 \rceil + \lceil l(q')/3 \rceil - 2C \geq \lceil 5C \rceil - 2C > 2C > 3$. By Lemma 2.3 $g_i^{m_i} \in \langle \Omega \rangle$ and $k|g_i^{m_i}|_{\Omega} \leq K(l(q) + l(q') + 2C)$, hence

$$|g_i^{m_i}|_{\Omega} \leq K \frac{3\lceil l(q)/3 \rceil + 3\lceil l(q')/3 \rceil + 1 + 2C}{\lceil l(q)/3 \rceil + \lceil l(q')/3 \rceil - 2C} \leq K \left(3 + \frac{6 + 8C}{2C}\right) \leq 9K.$$
contradicting the choice of \( m_i \) in (6).

**Case 2.** The path \( q \) has at least one component which is connected to a component of \( q' \). Let \( p_1, \ldots, p_{l(q)} \) denote the sequence of all components of \( q \). By part (a), if \( p_s \) and \( p_t \), \( 1 \leq s \leq t \leq l(q) \), are connected to components of \( q' \), then for any \( j, s \leq j \leq t, p_j \) is regular. We can take \( s \) (respectively \( t \)) to be minimal (respectively maximal) possible. Consequently \( p_1, \ldots, p_{s-1}, p_{t+1}, \ldots, p_{l(q)} \) will contain the set of all isolated components of \( o \) that belong to \( q \).

Without loss of generality we may assume that \( s - 1 \geq 15C/2 \). Since \( p_s \) is connected to some component \( p'_s \) of \( q' \), there exists a path \( v \) in \( \Gamma(G, X \cup \mathcal{H}') \) satisfying \( v_- = (p_s)_-, v_+ = p'_s \), \( \text{Lab}(v) \in \mathcal{H}', l(v) = 1 \). Let \( q \) (respectively \( q' \)) denote the subpath of \( q \) (respectively \( q' \)) from \( q_- \) to \( (p_s)_- \) (respectively from \( p'_s \) to \( q'_+ \)). Consider a new cycle \( \bar{o} = r\bar{q}vq' \). Reasoning as above, we can find \( i \in \{1, 2, 3\} \) such that \( \bar{o} \) has \( k \) isolated \( E(g_i) \)-components, where \( k \geq \lceil l(q)/3 \rceil + \lceil l(q')/3 \rceil - C - 1 \geq 15C/6 - C - 1 > C - 1 \geq 1 \). Using Lemma 2.3 we get \( g_i'^{m_i} \in \langle \Omega \rangle \) and \( k|g_i'^{m_i}|_\Omega \leq K(l(q) + l(q') + C + 1) \). The latter inequality implies \( |g_i'^{m_i}|_\Omega \leq 21K \), yielding a contradiction in the usual way and proving (b) for \( q \). By symmetry this property holds for \( q' \) as well.

Continuing the proof of Lemma 3.6 consider an element \( x \in E_G(h) \). According to Lemma 3.1, there exists \( l \in \mathbb{N} \) such that

\[
xh^l x^{-1} = h^\epsilon,
\]

where \( \epsilon = \pm 1 \). Set \( C = |x|_{X \cup \mathcal{H}'} \). After raising both sides of (7) in an integer power, we can assume that \( l \) is sufficiently large to satisfy \( l > 32C + 3 \).

Consider a cycle \( o = rqr'q' \) in \( \Gamma(G, X \cup \mathcal{H}') \) satisfying \( r_- = q'_+ = 1, r_+ = q_- = x, q_+ = q'_- = xh^l x^{-1}, \text{Lab}(q) = (g_1^{m_1} g_3^{m_3} g_2^{m_2})^l, \text{Lab}(q') = (g_1^{m_1} g_3^{m_3} g_2^{m_2})^{-\epsilon}, l(q) = l(q') = 3l, l(r) = l(r') = C \).

Let \( p_1, p_2, \ldots, p_3l \) and \( p'_1, p'_2, \ldots, p'_{3l} \) be all components of \( q \) and \( q' \) respectively. Thus, \( p_3, p_6, p_9, \ldots, p_{3l} \) are all \( E_G(g_2) \)-components of \( q \). Since \( l > 17C \) and \( q \) is without backtracking, by Lemma 3.7 there exist indices \( 1 \leq s, s' \leq 3l \) such that the \( E_G(g_2) \)-component \( p_s \) of \( q \) is connected to the \( E_G(g_2) \)-component \( p_{s'} \) of \( q' \). Without loss of generality, assume that \( s \leq 3l/2 \) (the other situation is symmetric). There is a path \( u \) in \( \Gamma(G, X \cup \mathcal{H}') \) with \( u_- = (p_s)_-, u_+ = (p_s)_+ \), \( \text{Lab}(u) \in E_G(g_2) \) and \( l(u) \leq 1 \). We obtain a new cycle \( o' = up_{s+1} \ldots p_{3l}r'_1 \ldots p'_{s'-1} \) in the Cayley graph \( \Gamma(G, X \cup \mathcal{H}') \). Due to the choice of \( s \) and \( l \), the same argument as before will demonstrate that there are \( E_G(g_2) \)-components \( p_s, p_{s'} \) of \( q, q' \) respectively, which are connected and \( s < s' \leq 3l, 1 \leq l' < s' \) (in the case when \( s > 3l/2 \), the same inequalities can be achieved by simply renaming the indices correspondingly).

It is now clear that there exist \( i \in \{1, 2, 3\} \) and connected \( E_G(g_i) \)-components \( p_t, p'_t \) of \( q, q' \) (\( s < t < 3l, 1 < t' < s' \)) such that \( t > s \) is minimal. Let \( v \) denote a path in \( \Gamma(G, X \cup \mathcal{H}') \) with \( v_- = (p_t)_-, v_+ = (p_t')_+ \), \( \text{Lab}(v) \in E_G(g_i) \) and \( l(v) \leq 1 \). Consider a cycle \( o'' \) in \( \Gamma(G, X \cup \mathcal{H}') \) defined by \( o'' = up_{s+1} \ldots p_{t-1}v p_{t'+1} \ldots p'_{s'-1} \). By part a) of Lemma 3.7 \( p_{s+1} \) is a regular component of the path \( p_{s+1} \ldots p_{t-1} \) in \( o'' \) (provided that \( t - 1 \geq s + 1 \)). Note that \( p_{s+1} \) can not be connected to \( u \) or \( v \) because \( q \) is without backtracking, hence it must be connected to a component of the path \( p'_{t'+1} \ldots p'_{s'-1} \). By the choice of \( t \), we have
$t = s + 1$ and $i = 1$. Similarly, $t' = s' - 1$. Thus $p_{s+1} = p_t$ and $p_{s'-1} = p_{t'}$ are connected $E_G(g_1)$-components of $q$ and $q'$. 

In particular, we have $\epsilon = 1$. Indeed, otherwise we would have $\text{Lab}(p_{s'-1}) \equiv g_3^{m_3}$ but $g_3^{m_3} \notin E_G(g_1)$. Therefore $x \in E_G^+_1(h)$ for any $x \in E_G(h)$, consequently $E_G(h) = E_G^+_1(h)$. 

Observe that $u_- = v_+$ and $u_+ = v_-$, hence $\text{Lab}(u)$ and $\text{Lab}(v)^{-1}$ represent the same element $z \in E_G(g_2) \cap E_G(g_1)$. By construction, $x = h^\alpha z h^\beta$ where $\alpha = (3l - s')/3 \in \mathbb{Z}$, and $\beta = -s/3 \in \mathbb{Z}$. Thus $x \in \langle h, E_G(g_1) \cap E_G(g_2) \rangle$ and the first part of the claim 2 is proved. 

Assume now that $E_G(g_j) = E_G^+_1(g_j)$ for $j = 1, 2$. Then $h = g_1^{m_1} g_2^{n_1} g_3^{m_2}$ belongs to the centralizer of the finite subgroup $E_G(g_1) \cap E_G(g_2)$ (because of the choice of $g_1, g_2$ above). Consequently $E_G(h) = \langle h \rangle \times (E_G(g_1) \cap E_G(g_2))$. \hfill $\square$

**Lemma 3.8.** Let $S$ be a non-elementary subgroup of $G$ with $S^0 \neq \emptyset$. Then 

(i) there exist non-commensurable elements $h_1, h'_1 \in S^0$ with $E_G(h_1) \cap E_G(h'_1) = E_G(S)$; 

(ii) $S^0$ contains an element $h$ such that $E_G(h) = \langle h \rangle \times E_G(S)$. 

**Proof.** Choose an element $g_1 \in S^0$. By Lemma 3.1, $G$ is hyperbolic relative to the collection $\mathcal{C} = \{H_\lambda\}_{\lambda \in \Lambda} \cup \{E_G(g_1)\}$. Since the subgroup $S$ is non-elementary, there is $a \in S \setminus E_G(g_1)$, and Lemma 3.5 provides us with an integer $n \in \mathbb{N}$ such that $g_2 = ag_1^n \in S$ is a hyperbolic element of infinite order (now, with respect to the family of peripheral subgroups $\mathcal{C}$). In particular, $g_1$ and $g_2$ are non-commensurable and hyperbolic relative to $\{H_\lambda\}_{\lambda \in \Lambda}$. 

Applying Lemma 3.6, we find $h_1 \in S^0$ (with respect to the collection of peripheral subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$) with $E_G(h_1) = E_G^+_1(h_1)$ such that $h_1$ is not commensurable with $g_j, j = 1, 2$. Hence, $g_1$ and $g_2$ stay hyperbolic after including $E_G(h_1)$ into the family of peripheral subgroups (see Lemma 3.1). This allows to construct (in the same manner) one more element $h_2 \in \langle g_1, g_2 \rangle \leq S$ which is hyperbolic relative to $\{H_\lambda\}_{\lambda \in \Lambda} \cup E_G(h_1))$ and satisfies $E_G(h_2) = E_G^+_1(h_2)$. In particular, $h_2$ is not commensurable with $h_1$. 

We claim now that there exists $x \in S$ such that $E_G(x^{-1} h_2 x) \cap E_G(h_1) = E_G(S)$. By definition, $E_G(S) \subset E_G(x^{-1} h_2 x) \cap E_G(h_1)$. To obtain the inverse inclusion, arguing by the contrary, suppose that for each $x \in S$ we have 

$$E_G(x^{-1} h_2 x) \cap E_G(h_1)) \setminus E_G(S) \neq \emptyset. \quad (8)$$

Note that if $g \in S^0$ with $E_G(g) = E_G^+_1(g)$, then the set of all elements of finite order in $E_G(g)$ form a finite subgroup $T(g) \leq E_G(g)$ (this is a well-known property of groups, all of whose conjugacy classes are finite). The elements $h_1$ and $h_2$ are not commensurable, therefore 

$$E_G(x^{-1} h_2 x) \cap E_G(h_1)) = T(x^{-1} h_2 x) \cap T(h_1) = x^{-1} T(h_2) x \cap T(h_1).$$

For each pair of elements $(b, a) \in D = T(h_2) \times (T(h_1) \setminus E_G(S))$ choose $x = x(b, a) \in S$ so that $x^{-1} bx = a$ if such $x$ exists; otherwise set $x(b, a) = 1$. 

The assumption $(8)$ clearly implies that $S = \bigcup_{(b, a) \in D} x(b, a) C_S(a)$, where $C_S(a)$ denotes the centralizer of $a$ in $S$. Since the set $D$ is finite, a well-know theorem of B. Neumann
[17] implies that there exists $a \in T(h_1) \setminus E_G(S)$ such that $|S : C_S(a)| < \infty$. Consequently, $a \in E_G(g)$ for every $g \in S^0$, that is, $a \in E_G(S)$, a contradiction.

Thus, $E_G(xh_2x^{-1}) \cap E_G(h_1) = E_G(S)$ for some $x \in S$. After setting $h'_1 = x^{-1}h_2x \in S^0$, we see that elements $h_1$ and $h'_1$ satisfy the claim (i). Since $E_G(h'_1) = x^{-1}E_G(h_2)x$, we have $E_G(h'_1) = E_G^+(h'_1)$. To demonstrate (ii), it remains to apply Lemma 3.6 and obtain an element $h \in \langle h_1, h'_1 \rangle \leq S$ which has the desired properties.

Proof of Proposition 3.4. The implication (1) $\Rightarrow$ (2) is an immediate consequence of the definition. The inverse implication follows directly from the first claim of Lemma 3.8 ($S$ is non-elementary as $S^0 \neq \emptyset$ and $E_G(S) = \{1\}$).

4 Proofs of the main results

The following simplification of Theorem 2.4 from [22] is the key ingredient of the proofs in the rest of the paper.

**Theorem 4.1.** Let $U$ be a group hyperbolic relative to a collection of subgroups $\{V_\lambda\}_{\lambda \in \Lambda}$, $S$ a suitable subgroup of $U$, and $T$ a finite subset of $U$. Then there exists an epimorphism $\eta: U \to W$ such that:

1. The restriction of $\eta$ to $\bigcup_{\lambda \in \Lambda} V_\lambda$ is injective, and the group $W$ is properly relatively hyperbolic with respect to the collection $\{\eta(V_\lambda)\}_{\lambda \in \Lambda}$.

2. For every $t \in T$, we have $\eta(t) \in \eta(S)$.

Let us also mention two known results we will use. The first lemma is a particular case of Theorem 1.4 from [21] (if $g \in G$ and $H \leq G$, $H^g$ denotes the conjugate $g^{-1}Hg \leq G$).

**Lemma 4.2.** Suppose that a group $G$ is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Then

(a) For any $g \in G$ and any $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, the intersection $H^g_\lambda \cap H_\mu$ is finite.

(b) For any $\lambda \in \Lambda$ and any $g \notin H_\lambda$, the intersection $H^g_\lambda \cap H_\lambda$ is finite.

The second result can easily be derived from Lemma 3.5.

**Lemma 4.3** (Corollary 4.5, [20]). Let $G$ be an infinite properly relatively hyperbolic group. Then $G$ contains a hyperbolic element of infinite order.

**Lemma 4.4.** Let the group $G$ be hyperbolic with respect to the collection of peripheral subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ and let $N \triangleleft G$ be a finite normal subgroup. Then

1. If $H_\lambda$ is infinite for some $\lambda \in \Lambda$, then $N \leq H_\lambda$;

2. The quotient $\bar{G} = G/N$ is hyperbolic relative to the natural image of the collection $\{H_\lambda\}_{\lambda \in \Lambda}$.
Proof. Let $K_\lambda, \lambda \in \Lambda$, be the kernel of the action of $H_\lambda$ on $N$ by conjugation. Since $N$ is finite, $K_\lambda$ has finite index in $H_\lambda$. On the other hand $K_\lambda \leq H_\lambda \cap H_\lambda^g$ for every $g \in N$. If $H_\lambda$ is infinite this implies $N \leq H_\lambda$ by Lemma 4.2.

To prove the second assertion, suppose that $G$ has a relatively finite presentation (2) with respect to the free product $F$ defined in (1). Denote by $\hat{X}$ and $\hat{H}_\lambda$ the natural images of $X$ and $H_\lambda$ in $\hat{G}$. In order to show that $\hat{G}$ is relatively hyperbolic, one has to consider it as a quotient of the free product $\hat{F} = \langle \varphi \lambda : \lambda \in \Lambda \rangle / \hat{F}(X)$. As $G$ is a quotient of $F$, we can choose some finite preimage $M \subset F$ of $N$. For each element $f \in M$, fix a word in $X \cup \mathcal{H}$ which represents it in $F$ and denote by $\mathcal{S}$ the (finite) set of all such words. By the universality of free products, there is a natural epimorphism $\psi : F \to \hat{F}$ mapping $X$ onto $\hat{X}$ and each $H_\lambda$ onto $\hat{H}_\lambda$. Define the subsets $\mathcal{R}$ and $\mathcal{S}$ of words in $\hat{X} \cup \mathcal{H}$ (where $\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} (\hat{H}_\lambda \setminus \{1\})$) by $\mathcal{R} = \psi(\mathcal{R})$ and $\mathcal{S} = \varphi(\mathcal{S})$. Then the group $G$ possesses the relatively finite presentation

$$\langle \hat{X}, \{\hat{H}_\lambda \lambda \in \Lambda \mid \hat{R} = 1, \hat{R} \in \mathcal{R}; \hat{S} = 1, \hat{S} \in \mathcal{S} \rangle.$$  \hspace{1cm} (9)

Let $\psi : F \to G$ denote the natural epimorphism and $D = \max\{\|s\| : s \in \mathcal{S}\}$. Consider any non-empty word $\bar{w}$ in the alphabet $\hat{X} \cup \hat{H}$ representing the identity in $\hat{G}$. Evidently we can choose a word $w$ in $X \cup \mathcal{H}$ such that $\bar{w} = \varphi(w)$ and $\|w\| = \|\bar{w}\|$. Since $\ker(\psi) \cdot M$ is the kernel of the induced homomorphism from $F$ to $\hat{G}$, we have $w = \varphi(v) \cdot u$ where $u \in \mathcal{S}$ and $v$ is a word in $X \cup \mathcal{H}$ satisfying $v = g_1$ and $\|v\| \leq \|w\| + D$. Since $G$ is relatively hyperbolic there is a constant $C \geq 0$ (independent of $v$) such that

$$v = \varphi \left( \prod_{i=1}^{k} f_{i}^{-1} R_{i}^\pm f_{i} \right),$$

where $R_i \in \mathcal{R}$, $f_i \in F$, and $k \leq C\|v\|$. Set $\bar{R}_i = \varphi(R_i) \in \mathcal{R}$, $\bar{f}_i = \varphi(f_i) \in \mathcal{F}$, $i = 1, 2, \ldots, k$, and $\bar{R}_{k+1} = \varphi(u) \in \mathcal{S}$, $\bar{f}_{k+1} = 1$. Then

$$\bar{w} = \varphi \left( \prod_{i=1}^{k+1} \bar{f}_{i}^{-1} \bar{R}_{i}^\pm \bar{f}_{i} \right),$$

where

$$k + 1 \leq C\|v\| + 1 \leq C(\|w\| + D) + 1 \leq C\|\bar{w}\| + CD + 1 \leq (C + CD + 1)\|\bar{w}\|.$$ 

Thus, the relative presentation (9) satisfies a linear isoperimetric inequality with the constant $(C + CD + 1)$. \hfill \square

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Observe that the quotient of $G$ by the finite normal subgroup $N = E_G(G)$ is obviously non-elementary. Hence the image of any finite $H_\lambda$ is a proper subgroup of $G/N$. On the other hand, if $H_\lambda$ is infinite, then $N \leq H_\lambda \leq G$ by Lemma 4.4 hence its image is also proper in $G/N$. Therefore $G/N$ is properly relatively hyperbolic with respect to the collection of images of $H_\lambda$, $\lambda \in \Lambda$ (see Lemma 1.4). Lemma 3.3 implies $E_{G/N}(G/N) = \{1\}$. Thus, without loss of generality, we may assume that $E_G(G) = 1$. 

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It is straightforward to see that the free product $U = G \ast R$ is hyperbolic relative to the collection $\{H_\lambda\}_{\lambda \in \Lambda} \cup \{R\}$ and $E_{G \ast R}(G) = E_G(G) = 1$. Note that $G^0$ is non-empty by Lemma 4.3. Hence $G$ is a suitable subgroup of $G \ast R$ by Proposition 3.4. Let $Y$ be a finite generating set of $R$. It remains to apply Theorem 4.1 to $U = G \ast R$, the obvious collection of peripheral subgroups, and the finite set $Y$.

To prove Theorem 1.4 we need one more auxiliary result which was proved in the full generality in [21] (see also [6]):

**Lemma 4.5** (Theorem 2.40, [21]). Suppose that a group $G$ is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda} \cup \{S_1, \ldots, S_m\}$, where $S_1, \ldots, S_m$ are hyperbolic in the ordinary (non-relative) sense. Then $G$ is hyperbolic relative to $\{H_\lambda\}_{\lambda \in \Lambda}$.

**Proof of Theorem 1.4.** Let $G_1, G_2$ be finitely generated groups which are properly relatively hyperbolic with respect to collections of subgroups $\{H_{1\lambda}\}_{\lambda \in \Lambda}$ and $\{H_{2\mu}\}_{\mu \in M}$ respectively. Denote by $X_i$ a finite generating set of the group $G_i$, $i = 1, 2$. As above we may assume that $E_{G_1}(G_1) = E_{G_2}(G_2) = \{1\}$. We set $G = G_1 \ast G_2$. Observe that $E_G(G_i) = E_{G_i}(G_i) = \{1\}$ and hence $G_i$ is suitable in $G$ for $i = 1, 2$ (by Lemma 4.3 and Proposition 3.4).

By the definition of suitable subgroups, there are two non-commensurable elements $g_1, g_2 \in G_2^0$ such that $E_G(g_1) \cap E_G(g_2) = \{1\}$. Further, by Lemma 3.1, the group $G$ is hyperbolic relative to the collection $\mathfrak{H} = \{H_{1\lambda}\}_{\lambda \in \Lambda} \cup \{H_{2\mu}\}_{\mu \in M} \cup \{E_G(g_1), E_G(g_2)\}$. We now apply Theorem 4.1 to the group $G$ with the collection of peripheral subgroups $\mathfrak{P}$, the suitable subgroup $G_1 \leq G$, and the subset $T = X_2$. The resulting group $W$ is obviously a quotient of $G_1$.

Observe that $W$ is hyperbolic relative to (the image of) the collection $\{H_{1\lambda}\}_{\lambda \in \Lambda} \cup \{H_{2\mu}\}_{\mu \in M}$ by Lemma 4.5. We would like to show that $G_2$ is a suitable subgroup of $W$ with respect to this collection. To this end we note that $\eta(g_1)$ and $\eta(g_2)$ are elements of infinite order as $\eta$ is injective on $E_G(g_1)$ and $E_G(g_2)$. Moreover, $\eta(g_1)$ and $\eta(g_2)$ are not commensurable in $W$. Indeed, otherwise, the intersection $(\eta(E_G(g_1)))^g \cap \eta(E_G(g_2))$ is infinite for some $g \in G$ that contradicts the first assertion of Lemma 4.2. Assume now that $g \in E_W(\eta(g_i))$ for some $i \in \{1, 2\}$. By the first assertion of Lemma 3.1, $(\eta(g_i^m))^g = \eta(g_i^m)$ for some $m \neq 0$. Therefore, $(\eta(E_G(g_i)))^g \cap \eta(E_G(g_i))$ contains $\eta(g_i^m)$ and, in particular, this intersection is infinite. By the second assertion of Lemma 4.2 this means that $g \in \eta(E_G(g_i))$. Thus, $E_W(\eta(g_i)) = \eta(E_G(g_i))$. Finally, using injectivity of $\eta$ on $E_G(g_1) \cup E_G(g_2)$, we obtain

$$E_W(\eta(g_1)) \cap E_W(\eta(g_2)) = \eta(E_G(g_1)) \cap \eta(E_G(g_2)) = \eta(E_G(g_1) \cap E_G(g_2)) = \{1\}.$$

This means that the image of $G_2$ is a suitable subgroup of $W$.

Thus we may apply Theorem 4.1 again to the group $W$, the subgroup $G_2$ and the finite subset $X_1$. The resulting group $Q$ is the desired common quotient of $G_1$ and $G_2$. The last property, which claims that $Q$ can be obtained from $G_1 \ast G_2$ by adding only finitely many relations, follows because $G_1 \ast G_2$ and $G$ are hyperbolic with respect to the same family of peripheral subgroups and any relatively hyperbolic group is relatively finitely presented. \qed
References


G. Arzhantseva, Université de Genève, Section de Mathématiques, 2-4 rue du Lièvre, Case postale 64, 1211 Genève 4, Switzerland

Email: Goulnara.Arjantseva@math.unige.ch

A. Minasyan, Université de Genève, Section de Mathématiques, 2-4 rue du Lièvre, Case postale 64, 1211 Genève 4, Switzerland

Email: aminasyan@gmail.com

D. Osin, NAC 8133, Department of Mathematics, The City College of the City University of New York, Convent Ave. at 138th Street, New York, NY 10031, USA

Email: denis.osin@gmail.com