Introduction to Renormalization

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First Part of the Lecture

1 Introduction

Starting with the Lagrangian of any Quantum Field Theory, we have seen in some of the previous lectures how to obtain the Feynman rules, which sufficiently describe how to do pertubation theory in that particular theory. But beyond tree level, the naive calculation of diagrams involving loops will often yield infinity, since the integrals have to be performed over the whole momentum space. Renormalization Theory deals with the systematic isolation and removing of these infinities from physical observables.

The first important insight is, that it is not the fields or the coupling constants which represent measurable quantities. Measured are cross sections, decay width, etc.. As long as we make sure that this observables are finite in the end and can be unambiguously derived from the Lagrangian, we are free to introduce new quantities, called renormalized quantities for every, so called, bare quantity. We can then go a step further and consider the bare quantities to be infinite in a way which would just cancel the infinities coming from our loop calculation. In other words, if we would be able to arrange the infinities of the bare quantities and the infinities coming from the divergent integrals to cancel each other systematically, we would be left with a finite, physically meaningful theory.

This is, roughly speaking, what Renormalization Theory does. The general procedure is now done in two different steps. At first, to give the above mentioned cancelation of infinities mathematically meaning, we need to regularize the divergent integrals. That is, we first have to make the integrals finite, e.g. by imposing a momentum cut-off. Then we are free to manipulate them. Secondly, the divergent parts of the integrals have to be absorbed into the bare quantities of the Lagrangian. We have to expect that there are different ways to treat the finite parts of the integrals, since a finite quantity can always be absorbed into an infinite one, and also that the way the divergences reside in the regularized integrals will depend on the Regularization procedure we choose. As we will work out, the freedom in the treatment of the finite parts is reflected in the existence of different renormalization schemes and also in the occurrence of scale quantities, which will have mass dimension. The fact that one choice of scale should be as good as any other and should not affect measurable quantities, will lead to the so called Renormalization Group consisting of transformations between different scales.

If there exists a consistent way to perform this procedure for a particular theory, this theory is said to be renormalizable. QED was historically the first important theory which could be showed to be renormalizable and we shall see how this can be done explicitly in next to leading order.

2 Renormalization of the QED Lagrangian

In the following we will denote the bare, unrenormalized quantities m_0, ψ_0, etc. to distinguish them from the renormalized ones.

$$\begin{aligned}\mathcal{L}_{QED} &= \bar{\psi}_0 \left(i\partial\!\!\!/ - m_0\right) \psi_0 - e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} - \frac{1}{4} F^{0\mu\nu} F_{0\mu\nu} + \mathcal{L}_{G.F.} \\ &\equiv Z_2 \bar{\psi} i\partial\!\!\!/ \psi - Z_2' m \bar{\psi}\psi - Z_1 e \bar{\psi}\gamma^\mu \psi A_\mu - Z_3 \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \mathcal{L}_{G.F.} \end{aligned} \tag{1}$$

Here $\mathcal{L}_{G.F.} = -\frac{1}{2\alpha}\left(\partial^\mu A_\mu\right)$ denotes the Faddeev and Popov Gauge Fixing term. In the following we will not take into account this term in our Renormalization procedure. For a renormalization including this term, see for example [3]. Whenever we will have to choose a gauge, we will pick the Feynman gauge, i.e. set $\alpha = 1$.

From the first and last term we see

$$\psi_0 = \sqrt{Z_2}\psi, \qquad A_{0\mu} = \sqrt{Z_3} A_\mu.$$

Similarly from the mass term:

$$\begin{aligned} Z_2' m \bar{\psi}\psi &= \frac{Z_2'}{Z_2} m \bar{\psi}_0 \psi_0 \\ \Rightarrow m &= \frac{Z_2}{Z_2'} m_0. \end{aligned}$$

We now define $m_0 = m - \delta m$ and get

$$Z_2' = Z_2 \frac{m_0}{m} = Z_2 \frac{m - \delta m}{m} = \left(1 - \frac{\delta m}{m}\right) Z_2.$$

The remaining term yields:

$$\begin{aligned} e\tfrac{Z_1}{Z_2\sqrt{Z_3}} \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} &= e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} \\ \Rightarrow e &= \tfrac{Z_2}{Z_1}\sqrt{Z_3} e_0. \end{aligned} \tag{2}$$

Note that the covariant derivative now reads $\mathcal{D}_\mu^{ren} = \partial_\mu - ie\frac{Z_1}{Z_2} A_\mu$. In order to have gauge invariance preserved, $Z_1 = Z_2$ should hold. We will find this relation in the next lecture.

Summarizing the results from above, we have:

$$Z_2' = \left(1 - \frac{\delta m}{m}\right) Z_2, \tag{3}$$

$$e = \frac{Z_2}{Z_1}\sqrt{Z_3} e_0, \tag{4}$$

$$\psi_0 = \sqrt{Z_2}\psi, \tag{5}$$

$$A_{0\mu} = \sqrt{Z_3} A_\mu. \tag{6}$$

We can now rewrite the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}_{ren} + \mathcal{L}_{counter},$$

where

$$\mathcal{L}_{ren} \equiv \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{7}$$

$$\mathcal{L}_{counter} \equiv (Z_2 - 1)\bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \delta m Z_2 \bar{\psi}\psi \tag{8}$$

$$- (Z_1 - 1)\, e\bar{\psi}\gamma^\mu\psi A_\mu - (Z_3 - 1)\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{9}$$

$$\tag{10}$$

We now introduce the interacting part of the hamiltonian:

$$H' \equiv e\bar{\psi}\gamma^\mu\psi A_\mu - \mathcal{L}_{counter}. \tag{11}$$

Note that the whole counter-term Lagrangian is treated as a perturbation.

Remark: Divergences of processes which do not have a tree level correspondence, as the photon-photon scattering (see Fig. 1) would also have to be renormalized by $\mathcal{L}_{counter}$. By naive power counting, see section 4, this diagram seems to be logarithmic divergent. It would now be hardly imaginable to absorb this divergence into a counter term, since there is no corresponding tree level process. But as it turns out when one actually does the calculation, this diagram is not divergent at all.

3 One Loop Correction to the Fermion Propagator

In this section we will look at the one loop correction to the fermion propagator. The relation to the counter-term Lagrangian will be worked out in the later sections.

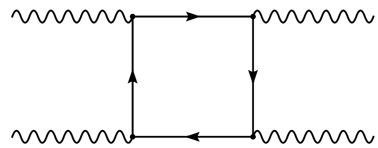


Figure 1: Box diagram of photon-photon scattering.

The scattering amplitude is given by

$$\begin{aligned} S &= \mathrm{T}\left[\exp\left(-i\int d^4x\, H'(x)\right)\right] \\ &= 1 - i\int d^4x\, H'(x) + \frac{(-i)^2}{2!}\int d^4x \int d^4x'\, \mathrm{T}\left[H'(x)H'(x)\right] + \dots\ . \end{aligned}$$

We will use Wick's theorem to evaluate the time ordered products. See (A.1-A.4) in the appendix for the definition of the field contractions. We will introduce a photon mass λ to regularize the infrared divergence in the loop integral. Fig. 2 shows the process for which we will calculate the amplitude now.

To use Wicks theorem we now have to take into account all possible contractions which correspond to this process. The factor of two in the calculation below is due to the fact that, in our case, there are actually two equivalent ways to do the contractions.

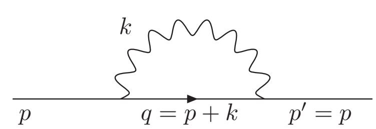


Figure 2: One loop correction to the fermion propagator.

$$
\begin{aligned}
&\langle p', s'|\, S-1\,|p,s\rangle \\
&= 2\frac{(-ie)^2}{2!}\,\langle p',s'|\int d^4x\, \mathrm{d}^4x'\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\ \bar{\psi}(x')\gamma^\nu\psi(x')A_\nu(x')\,|p,s\rangle \\
&= -e^2\int d^4x\ \int d^4x'\ e^{ip'x}\bar{u}^{s'}(p')\gamma^\mu iS_{F_{\alpha\beta}}(x-x')iD_{F_{\mu\nu}}(x-x')\gamma^\nu e^{-ipx'}u^s(p) \\
&= -e^2(i)(-i)\int d^4q d^4k\ \bar{u}^{s'}(p')\gamma^\mu\frac{\not{q}+m}{q^2-m^2+i\epsilon}g_{\mu\nu}\frac{1}{k^2-\lambda^2+i\epsilon}\gamma^\nu u^s(p) \\
&\quad\times\frac{1}{(2\pi)^8}\int d^4x\int d^4x'\ e^{ip'x}e^{-ipx'}e^{-iq(x-x')}e^{-i(-k)(x-x')}
\end{aligned}
$$

This gives us two δ-functions: $\delta(q-p-k)$ and $\delta(p'-q+k)$. Note that the sign of k depends on how we choose the momentum direction in the loop and that we choose $-k$ to be consistent with our assignments in Fig. 2. Now we perform the q-integration:

$$
\begin{aligned}
&= -e^2\bar{u}^{s'}(p')\delta(p'-p)\int d^4k\ \gamma^\mu\tfrac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\epsilon}\tfrac{1}{k^2-\lambda^2+i\epsilon}\gamma_\mu u^s(p) \\
&= (2\pi)^4\,\delta(p'-p)\ iM \equiv (2\pi)^4\,\delta(p'-p)\ \left(\bar{u}^{s'}(p')\,(-i\Sigma_2)\,u^s(p)\right).
\end{aligned}
$$

Where M denotes the matrix element which we could have obtained directly by using the Feynman rules:

$$iM = \text{sum of all connected, amputated diagrams.}$$

At this point we should stress one subtlety: We introduced a counter-term Lagrangian which will lead to additional Feynman rules. Also, we did not take into account the counter-terms in the interacting Hamiltonian we used to calculate the scattering amplitude. We shall come back to that in section 5.

Note that we in general want the calculated diagram to be part of a bigger one, so we don't include the Dirac spinors from the external lines in our definition of Σ_2. Thus we have:

$$
-i\Sigma_2(p) \equiv \frac{-e^2}{(2\pi)^4}\int d^4k\ \frac{\gamma^\mu\left(\not{p}+\not{k}+m\right)\gamma_\mu}{(p+k)^2-m^2+i\epsilon}\ \frac{1}{k^2-\lambda^2+i\epsilon}. \tag{12}
$$

A standard way to evaluate integrals like this one is to first combine the two denominators using the Feynman trick:

$$
\frac{1}{a_1a_2\ldots a_n} = (n-1)!\int_0^1\ldots\int_0^1 \frac{dx_1dx_2\ldots dx_n}{(a_1x_1+a_2x_2+\ldots+a_nx_n)^n}\ \delta\!\left(1-\sum_{i=1}^{n}x_i\right),
$$

in our case the following form is sufficient:

$$\frac{1}{a\,b} = \int_0^1 \frac{dx}{\left[a + (b-a)\,x\right]^2}. \tag{13}$$

Now let $a = k^2 - \lambda^2 + i\epsilon$ and $b = k^2 + 2pk + p^2 - m^2 + i\epsilon$, then

$$a + (b-a)\,x = k^2 - \lambda^2 + i\epsilon + \left(2pk + \lambda^2 + p^2 - m^2\right)x.$$

Thus using the Feynman trick, we can rewrite (12):

$$\begin{aligned}\Sigma_2(p) &= -i\frac{e^2}{(2\pi)^4}\int_0^1 dx \int d^4k\, \gamma^\mu \left(\not{p} + \not{k} + m\right)\gamma_\mu \\ &\quad \times \frac{1}{\left[k^2 + 2xpk + p^2x - m^2x + \lambda^2\,(x-1) + i\epsilon\right]^2} \\ &= -i\frac{e^2}{(2\pi)^4}\int_0^1 dx \int d^4k\, \frac{\gamma^\mu \left(\not{p} + \not{k} + m\right)\gamma_\mu}{\left[(k+xp)^2 - \Delta + i\epsilon\right]^2},\end{aligned} \tag{14}$$

where we have defined

$$\Delta \equiv -p^2x\,(1-x) + m^2x + \lambda^2\,(1-x)\,.$$

4 Dimensional Regularization

4.1 Superficial Degree of Divergence

We are following [1] here. In order to figure out the degree of divergence of a certain Feynman diagram, it is useful to have a rule of thumb at hand. Here we will consider a more general Lagrangian than the QED-Lagrangian, with a scalar field, a fermion field and a massless vector gauge boson field in n-dimensional space-time. Suppose we have a diagram with

- B external boson lines,
- F external fermion lines,
- L loops,
- n_i ith type vertices,

- b_i boson lines in the ith type vertex,
- f_i fermion lines in the ith type vertex,
- d_i number of derivatives in the ith type vertex,
- IB internal boson lines,
- IF internal fermion lines.

The structure of the graph gives us the relations

$$B + 2IB = \sum_i n_i b_i, \tag{15}$$

$$F + 2IF = \sum_i n_i f_i \quad \text{and} \tag{16}$$

$$L = IB + IF - \sum_i n_i + 1. \tag{17}$$

The so called superficial degree of divergence D is then given by

$$\begin{aligned} D &\equiv nL - 2IB - IF + \sum_i n_i d_i \\ &= n\left(IB + IF - \sum_i n_i + 1\right) - 2IB - IF + \sum_i n_i d_i, \end{aligned} \tag{18}$$

where we used (17) in the second step. Plugging in (15) and (16) now yields:

$$\begin{aligned} D &= \left(\frac{n-2}{2}\right)\left[\sum_i n_i b_i - B\right] + \left(\frac{n-1}{2}\right)\left[\sum_i n_i f_i - F\right] \\ &\quad + \sum_i n_i\left(d_i - n\right) + n \\ &= n - \left(\frac{n-2}{2}\right)B - \left(\frac{n-1}{2}\right)F + \sum_i n_i \delta_i, \end{aligned} \tag{19}$$

where

$$\delta_i \equiv d_i + \left(\frac{d-2}{2}\right)b_i + \left(\frac{d-1}{2}\right)f_i - n \tag{20}$$

is called the index of divergence. When $D = 0$ the integral is said to be superficial logarithmic divergent, linear divergent if $D = 1$ and quadratic divergent if $D = 2$.

Note that this is really just a rule of thumb, which for example in case of the photon-photon scattering turns out to give the wrong answer. Nevertheless the real divergence of a diagram can not be worse than the hereby estimated divergence.

4.2 The Procedure of Dimensional Regularization

In most parts we are following [2] in this section. Mathematically this section is necessarily a bit sloppy. A mathematically more profound treatment can be found in [1] and especially in [3]. The procedure of Dimensional Regularization can be summarized in the following way: Compute the Feynman diagram as an analytic function of space-time. Which is tricky, because therefore the integral expression has to be analytically continued, since the divergence will reside as a pole (as we will see) for $n = 4$ (note that in Dimensional Regularization n is a complex number). See [3, p. 115] for details here. If now n is small enough, any loop integral will finally converge. For QED, Dimensional Regularization is especially important, because it preserves the Ward Identity (in contrast to imposing a direct momentum cut-off, see [2, p. 248]). After regularization and renormalization the expressions for observables will have a well-defined limit as $n \to 4$.

The first thing we do is to go to Euclidean space by performing a Wick rotation, i.e. let $k^0 = ik_E^0$. We thus have $d^4k_E = -id^4k$. As in three dimensions, we can introduce angle coordinates and generalize our four dimensional space-time integrals in the following way:

$$\frac{d^4k_E}{(2\pi)^4} \to \frac{d^nk_E}{(2\pi)^n} = \frac{d\Omega_n}{(2\pi)^n} dk_E\, k_E^{n-1}.$$

In this notation $\Omega_n = \int d\Omega$ is the surface of the unit sphere in n dimensions. To obtain an explicit expression for this, we recall the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$$

to derive:

$$
\begin{aligned}
\pi^{\frac{n}{2}} &= \left(\int dx\, e^{-x^2}\right)^n \\
&= \int d^n x\, e^{-\sum x_i^2} \\
&= \int d\Omega_n \int_0^\infty dx\, x^{n-1} e^{-x^2} \\
&= \int d\Omega_n\, \frac{1}{2} \int_0^\infty dy\, y^{\frac{n}{2}-1} e^{-y} \\
&= \int d\Omega_n\, \frac{1}{2}\, \Gamma\left(\frac{n}{2}\right). \qquad (21)
\end{aligned}
$$

Here we used the Euler form of the Gamma function

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}.$$

See the appendix for a summary of its most important properties.

Using equation (21) we find

$$\int d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \qquad (22)$$

4.3 A Useful General Integral

Before we proceed with our calculation of Σ_2, we will study the general integral of the form

$$\int \frac{d^n k_E}{\left(k_E^2 + \Delta - i\epsilon\right)^\beta}, \qquad (23)$$

where β is an integer. At first we introduce the following trick:

$$\frac{1}{k_E^2 + \Delta - i\epsilon} = \int_0^\infty d\alpha\, e^{-\alpha\left(k_E^2 + \Delta - i\epsilon\right)}.$$

Secondly we note that

$$\frac{1}{\left(k_E^2 + \Delta - i\epsilon\right)^\beta} = \frac{(-1)^{\beta-1}}{(\beta-1)!} \left(\frac{\partial}{\partial \Delta}\right)^{\beta-1} \frac{1}{k_E^2 + \Delta - i\epsilon}.$$

Using this we obtain

$$\begin{aligned}
\int \frac{d^n k_E}{\left(k_E^2+\Delta-i\epsilon\right)^\beta} &= \frac{(-1)^{\beta-1}}{\Gamma(\beta)} \left(\frac{\partial}{\partial\Delta}\right)^{\beta-1} \int_0^\infty d\alpha \int d^n k_E \; e^{-\alpha\left(k_E^2+\Delta-i\epsilon\right)} \\
&= \frac{(-1)^{\beta-1}}{\Gamma(\beta)} \left(\frac{\partial}{\partial\Delta}\right)^{\beta-1} \int_0^\infty d\alpha \int d\Omega_n \int_0^\infty dk_E k_E^{n-1} \; e^{-\alpha\left(k_E^2+\Delta-i\epsilon\right)}.
\end{aligned}$$

Now we perform the variable substitution $t = \alpha k_E^2 \Rightarrow dk_E = \frac{dt}{2\alpha k_E}$ on the last integral:

$$\begin{aligned}
\int_0^\infty dk_E \; k_E^{n-1} e^{-\alpha k_E^2} &= \frac{1}{2\alpha} \int_0^\infty dt \; k_E^{n-2} e^{-t} \\
&= \frac{1}{2\alpha} \left(\frac{1}{\sqrt{\alpha}}\right)^{n-2} \int_0^\infty dt \; t^{\frac{n}{2}-1} e^{-t} \\
&= \frac{1}{2} \alpha^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)
\end{aligned}$$

Using this result and (22) we can go on with our calculation

$$\begin{aligned}
\int \frac{d^n k_E}{(k^2+\Delta-i\epsilon)^\beta} &= \frac{(-1)^{\beta-1}}{\Gamma(\beta)} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\partial}{\partial\Delta}\right)^{\beta-1} \int_0^\infty d\alpha \; e^{-\alpha(\Delta-i\epsilon)} \frac{1}{2} \alpha^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \\
&= \frac{(-1)^{\beta-1}}{\Gamma(\beta)} \pi^{\frac{n}{2}} \left(\frac{\partial}{\partial\Delta}\right)^{\beta-1} \int_0^\infty d\alpha \; \alpha^{-\frac{n}{2}} e^{-\alpha(\Delta-i\epsilon)} \\
&= \frac{\pi^{\frac{n}{2}}}{\Gamma(\beta)} \int_0^\infty d\alpha \; \alpha^{\beta-\frac{n}{2}-1} e^{-\alpha(\Delta-i\epsilon)} \\
&\text{Now substitute } \alpha' = \alpha\,(\Delta - i\epsilon) \Rightarrow d\alpha = \tfrac{d\alpha'}{\Delta-i\epsilon}. \\
&= \frac{\pi^{\frac{n}{2}}}{\Gamma(\beta)} \int \frac{d\alpha'}{(\Delta-i\epsilon)} \frac{1}{(\Delta-i\epsilon)^{\beta-\frac{n}{2}-1}} \alpha'^{\beta-\frac{n}{2}-1} e^{-\alpha'} \\
&= \frac{\pi^{\frac{n}{2}}}{(\Delta-i\epsilon)^{\beta-\frac{n}{2}}} \frac{\Gamma\left(\beta-\frac{n}{2}\right)}{\Gamma(\beta)} \qquad (24)
\end{aligned}$$

This result could also have been obtained more elegantly (but not as straight forward) by substituting $x = \Delta / \left(k_E^2 + \Delta\right)$ in our starting expression for the integral in (23) to massage it into a form where we can use a relation of the so called beta function

$$\mathrm{B}(\alpha,\beta) \equiv \int_0^1 dx \; x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Compare [2, p. 250].

We should note that integrals like that belong to a general class of functions, called *Passarino-Veltman functions*, which can be found in the literature.

4.4 Dirac Matrices in n Dimensions

Before we begin to work n dimensional space-time, we have to appropriately generalize our Dirac matrix identities. We define the n Dirac matrices by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$

where $g^{\mu\nu} g_{\mu\nu} = n$ in n dimensions. We will now use this definition to show:

$$\begin{aligned} \gamma^\mu \gamma_\mu &= n, \\ \gamma^\mu \gamma^\nu \gamma_\mu &= (2-n)\,\gamma^\nu, \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho} - (4-n)\,\gamma^\nu \gamma^\rho. \end{aligned} \tag{25}$$

The first identity can easily be obtained by writing

$$\{\gamma^\mu, \gamma_\mu\} = 2\gamma^\mu \gamma_\mu = 2g^\mu_{\ \mu} = 2n.$$

Using this we can also show the second identity:

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\}\,\gamma_\mu &= \gamma^\mu \gamma^\nu \gamma_\mu + \gamma^\nu n \\ &= 2g^{\mu\nu}\gamma_\mu = 2\gamma^\nu. \end{aligned}$$

And finally we use our last result to show the third identity:

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\}\,\gamma^\rho \gamma_\mu &= \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu + (2-n)\,\gamma^\nu \gamma^\rho \\ &= 2g^{\mu\nu}\gamma^\rho \gamma_\mu = 2\gamma^\rho \gamma^\nu = 4g^{\rho\nu} - 2\gamma^\nu \gamma^\rho. \end{aligned}$$

4.5 Dimensional Regularization of Σ_2

After all the preparatory work we can now go ahead and transform equation (14) for Σ_2 into n dimensions:

$$\Sigma_2(p) = \frac{-ie^2}{(2\pi)^n}\mu^{4-n}\int_0^1 dx \int d^n k\, \frac{\gamma^\mu\left(\not{p} + \not{k} + m\right)\gamma_\mu}{\left[(k+xp)^2 - \Delta + i\epsilon\right]^2} \tag{26}$$

Here we introduced the scale μ, a quantity which has mass dimension and must be introduced so that e remains dimensionless also in n dimensions:

$$e \to e\,\mu^{2-\frac{n}{2}}.$$

In the following calculation of (26), we first shift the integration variable $k' = k + xp \Rightarrow dk' = dk, \not{k} = \not{k}' - x\not{p}$ and then drop ' for notational simplicity. Then we perform a Wick rotation, use our general expression (24) and note that the $\not{k}$-term is zero by symmetry.

$$\begin{aligned}
\Sigma_2(p) &= \frac{-ie^2}{(2\pi)^n}\mu^{4-n}\int_0^1 dx \int d^n k \frac{\gamma^\mu\left(\not{k} + (1-x)\not{p} + m\right)\gamma_\mu}{\left(k^2 - \Delta + i\epsilon\right)^2} \\
&= \frac{-ie^2}{(2\pi)^n}\mu^{4-n}\int_0^1 dx\, i \int d^n k_E \frac{\gamma^\mu\left((1-x)\not{p} + m\right)\gamma_\mu}{\left(k_E^2 + \Delta - i\epsilon\right)^2} \\
&= \frac{e^2}{(2\pi)^n}\mu^{4-n}\int_0^1 dx \frac{(1-x)\gamma^\mu\not{p}\gamma_\mu + \gamma^\mu\gamma_\mu m}{(\Delta - i\epsilon)^{2-\frac{n}{2}}} \\
&= \frac{e^2}{(2\pi)^n}\mu^{4-n}\pi^{\frac{n}{2}}\Gamma\Big(2-\frac{n}{2}\Big)\int_0^1 dx \frac{(1-x)(2-n)\not{p} + nm}{(\Delta - i\epsilon)^{2-\frac{n}{2}}} \qquad (27)
\end{aligned}$$

Where in the last step we used (25). This result is usually written in the following way:

$$\Sigma_2(p) = A\big(p^2\big) + B\big(p^2\big)\ \left(\not{p} - m\right), \qquad (28)$$

where

$$A(p^2) \equiv \frac{e^2}{(\pi)^n}\mu^{4-n}\pi^{\frac{n}{2}}\Gamma\Big(2-\frac{n}{2}\Big)\int_0^1 dx\, \frac{2 - x(2-n)}{(\Delta - i\epsilon)^{2-\frac{n}{2}}}\, m \quad \text{and} \qquad (29)$$

$$B(p^2) \equiv \frac{e^2}{(\pi)^n}\mu^{4-n}\pi^{\frac{n}{2}}\Gamma\Big(2-\frac{n}{2}\Big)(2-n)\int_0^1 dx\, \frac{1-x}{(\Delta - i\epsilon)^{2-\frac{n}{2}}}. \qquad (30)$$

5 Renormalization

5.1 Summary of Last Section and Outline of the Next Steps

We should now pause for a minute and ask ourselves what we have accomplished so far. We regularized the loop integral: To cure the infra-red divergence we introduced a photon mass λ and dimensional regularization took care of the UV-divergences. Note that there exist alternative regularization procedures, too, like the Pauli-Villar regularization (discussed for example in [1, p. 45]).

In the beginning of the lecture we introduced a counter-term Lagrangian. But in our derivation of the one loop correction to the fermion propagator, we did not take into account its contribution to the interacting Hamiltonian H'. Thinking back on our motivation to introduce it, we clearly would want this contribution to cancel the divergent parts of our calculation. We therefore will in this section reinterpret our result to also contain the Feynman diagrams resulting from the full interacting Lagrangian, including also the counter term pieces. In that context it is important to note that the factors of Z in the complete Lagrangian are not defined at all, unless we exactly define how much of the *finite* parts of the loop diagrams we want them to absorb. There are different ways to do that and to choose one, means to choose a particular *renormalization scheme*. That is, renormalization schemes differ in the treatment of the *finite* parts of the loop integrals.

In this lecture we choose a *momentum subtraction* scheme (another choice would have been the Minimum Subtraction Scheme, for example).

5.2 Momentum Subtraction Schemes

In this subsection we will again follow [1]. The general idea of Momentum Subtraction Schemes is the observation that divergences of loop integrals will occur only in the first few terms of a Taylor expansion in external momenta of the Feynman diagrams. Note that by divergences we here mean properly regularized (so cut-off or dimensional dependent) quantities. An easy example shall illustrate that. Suppose we wanted to calculate the following integral:

$$\Gamma\left(p^2\right) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l-p)^2 - m^2 + i\epsilon} \, \frac{1}{l^2 - m^2 + i\epsilon}.$$

We then observe that its first derivative

$$\begin{aligned} \tfrac{\partial \Gamma\left(p^2\right)}{\partial p^2} &= \frac{1}{2p^2} p_\mu \frac{\partial}{\partial p_\mu} \Gamma\left(p^2\right) \\ &= \frac{1}{p^2} \int \frac{d^4 l}{(2\pi)^4} \frac{(l-p)\, p}{\left[(l-p)^2 - m^2 + i\epsilon\right]^2} \, \frac{1}{l^2 - m^2 + i\epsilon}, \end{aligned}$$

is finite! And all further terms of the expansion will be finite, too. Which terms of the expansion diverge depend on how badly divergent the integral is.

Clearly there is a freedom related to the expansion point (in that context sometimes called subtraction point) here. This is were naturally a scale, let's call it M, comes in: To pick a certain expansion point means, loosely speaking, to define the theory at the scale M. Later we will come back to that question when we pick the so called *renormalization conditions*. Since every expansion point is as good as every other, we can also ask how different choices of M are connected to each other. It turns out that transformations from one scale to another form a group, called the *Renormalization Group*. For details see [2, Chapter 12.2].

5.3 Back to Renormalization in QED

We would like to get a renormalized propagator of the form:

$$\frac{i}{\not{p} - m} + (\text{terms regular at } \sqrt{p^2} = +m, \text{ finite}). \tag{31}$$

Note that this form fixes the mass to be on shell (since it is precisely the pole of the propagator) and also fixes the residue to be one. Now the identity

$$\frac{\not{p} + m}{p^2 - m^2} = \frac{\not{p} + m}{\left(\not{p} + m\right)\left(\not{p} - m\right)} = \frac{1}{\not{p} - m}$$

motivates to write $\not{p} = m$ shortly for (the position of the pole at) $\sqrt{p^2} = +m$. Further we note that also $\frac{\partial}{\partial \not{p}}$ is well-defined:

$$\frac{\partial}{\partial \not{p}} = \left(\gamma^\mu\right)^{-1} \frac{\partial}{\partial p_\mu} = \gamma_\mu \frac{\partial}{\partial p_\mu},$$

where we used (25). This will allow as a more readable notation.

We will now see that (31) for the form of our propagator exactly defines two of our renormalization conditions: We are doing on (mass) shell renormalization. To see that, we expand Σ_2 around $\not{p} = m$:

$$\Sigma_2(p) = \Sigma_2\left(\not{p} = m\right) + \frac{\partial \Sigma_2}{\partial \not{p}}\Big|_{\not{p}=m} \left(\not{p} - m\right) + \Sigma_f(p)\,. \tag{32}$$

Where $\Sigma_f(p)$ should be finite when $n \to 4$ by the above argument. By construction (higher order term of a Taylor expansion) $\Sigma_f(p)$ further satisfies:

$$\begin{aligned} &\Sigma_f\left(\not{p} = m\right) = 0 \\ &\frac{\partial \Sigma_f}{\partial \not{p}}\Big|_{\not{p}=m} = 0. \end{aligned} \tag{33}$$

In a few moments we will find the renormalized propagator in one loop approximation to be

$$\frac{i}{\not{p} - m - \Sigma_f(p)}. \tag{34}$$

Using (33) we immediately find by expanding $\Sigma_f(p)$ around $\not{p} = m$

$$\frac{i}{\not{p} - m - \Sigma_f(p)} = \frac{1}{\not{p} - m}\left(1 + \frac{\partial^2 \Sigma_f}{\partial \not{p}^2}\Big|_{\not{p}=m}(\not{p} - m) + \dots\right).$$

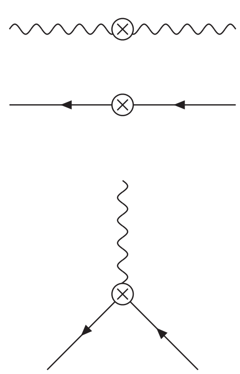
So indeed, the propagator has a pole at $\not{p} = m$ and its residue is 1.

In order to see explicitly how the counter-terms have to be chosen to cancel the divergent parts of Σ_2 in the on shell renormalization scheme (defined by (33)), we rewrite our Counter-Lagrangian for the last time:

$$\mathcal{L}_{counter} = -\frac{1}{4}\delta_3 \left(F^{\mu\nu}\right)^2 + \bar{\psi}\left(i\delta_2 \not{\partial} - \delta_m\right)\psi - e\delta_1 \bar{\psi}\gamma^\mu \psi A_\mu, \tag{35}$$

where $\delta_1 = Z_1 - 1$, $\delta_2 = Z_2 - 1$, $\delta_3 = Z_3 - 1$ and $\delta_m = Z_2 m_0 - m$ (don't confuse δ_m and δm).

Now we read off the Feynman rules. This is at first sight not straight forward to do for $-\frac{1}{4}\left(F^{\mu\nu}\right)^2$, but by integration by parts (and throwing away the surface term as usual) one obtains $-\frac{1}{2}A_\mu\left(-\partial^2 g^{\mu\nu} + \partial^\mu\partial^\nu\right)A_\nu$, where one can readily read off the corresponding Feynman Rule. All Feynman rules are shown in Figure 3. Now



$$= -i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\delta_3$$

$$= i\left(\not{p}\delta_2 - \delta_m\right)$$

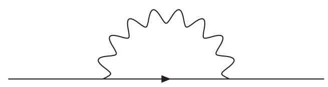
$$= -ie\gamma^\mu\delta_1$$

Figure 3: Feynman rules of the counter-terms.

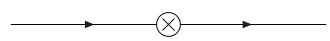
we see that for the process we calculated there is another diagram contributing:

The second one in Fig. 3. For our final result of the one loop correction of the fermion propagator we thus have to include that diagram, too: The amplitude for the one loop correction is the sum of the two diagrams, which has to be finite and which we already denoted as Σ_f, whereas Σ_2 on the other hand only contains the loop diagram:

$$-i\Sigma_f =$$



$$+$$



$$= -i\Sigma_2 - \left[(-i)\Sigma_2\big(\not p = m\big) + (-i)\,\Sigma_2'\big|_{\not p=m}\big(\not p - m\big)\right],$$

where we used the Taylor expansion from eqn. (32). Note that the right hand side of this equation is only finite, because the infinite parts (and also some of the finite parts) of the loop diagram amplitude is canceled by $\Sigma_2\big(\not p = m\big)$ and $\Sigma_2'\big|_{\not p=m}$. In the following we will have to figure out the connection between this two quantities and the constants occurring in the Feynman rules of the counter terms.

Therefore we will plug in $\not p = m$ in the last equation, use (33) and the definition of the counter-term amplitude in Fig. 3 to figure out how we have to define δ_2 and δ_m to get the desired cancelation:

$$0 = -i\Sigma_2(m) + i\,(m\delta_2 - \delta_m)\,.$$

Now we use our second renormalization condition in (33):

$$0 = -i\Sigma_2'(m) + i\delta_2.$$

The two last equations now give us

$$m\delta_2 - \delta_m = \Sigma_2(m) \quad \text{and} \tag{36}$$

$$\delta_2 = \frac{\partial\Sigma_2}{\partial\not p}\bigg|_{\not p=m}, \tag{37}$$

which exactly defines how the counter-terms have to look like in our on-shell scheme. Since we now have the Feynman rules, we can justify the form of the propagator already given in (34). It is the geometric sum over the one loop diagrams *and* the corresponding counter term diagrams displayed in Fig. 4:

$$
\begin{aligned}
&\frac{i}{\not p - m} + \frac{i}{\not p - m}\left(-i\Sigma_f\right)\frac{i}{\not p - m} + \dots \\
&= \frac{i}{\not p - m}\left[1 + \frac{\Sigma_f}{\not p - m} + \left(\frac{\Sigma_f}{\not p - m}\right)^2 + \dots\right] \\
&= \frac{i}{\not p - m - \Sigma_f}.
\end{aligned}
$$

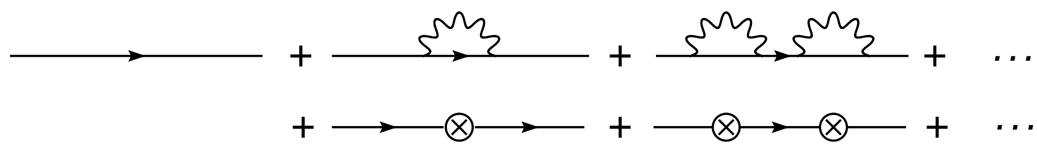


Figure 4: The one loop approximation for the renormalized propagator.

5.4 Leading Order Expression of Σ_2 for Small ϵ

We now again set $\epsilon = 4 - n$. Then using the formulas given in the appendix, the leading term of (27) for small ϵ is (here λ is set to zero):

$$
\begin{aligned}
\Sigma_2(p) = {} & \frac{1}{\epsilon}\frac{e^2}{16\pi^2}\left(\not p + 4m\right) + \frac{e^2}{8\pi^2}\left[-\frac{1}{2}\not p\,(1+\gamma) - m\,(1+2\gamma)\right. \\
& \left. + \int_0^1 dx\,\left[(1-x)\,\not p + 2m\right]\log\left(\frac{4\pi\mu^2}{p^2x\,(1-x) + m^2x}\right)\right] + O(\epsilon)
\end{aligned}
\tag{38}
$$

It is interesting to compare this result to the result one gets for regularization using the Pauli-Villars method. This method introduces a cut-off Λ and does the following replacement to regularize the integral:

$$
\frac{1}{k^2 - \lambda^2 + i\epsilon} \to \frac{1}{k^2 - \lambda^2 + i\epsilon} - \frac{1}{k^2 - \Lambda^2 + i\epsilon}
$$

We then would have obtained a result quite similar to (38):

$$
\Sigma_2(p) = \frac{\alpha}{2\pi}\int_0^1 dx\,\left[(1-x)\,\not p + 2m\right]\log\left(\frac{(1-x)\,\Lambda^2}{p^2x\,(x-1) + m^2x}\right). \tag{39}
$$

Compare [2, p. 218]. Here we can see that the logarithmic divergent terms in cut-off regularization are replaced by simple poles in Dimensional Regularization.

Second Lecture

6 Outline of the Following Procedure

In the last lecture we introduced dimensional regularization and the momentum subtraction scheme to handle the divergent parts of loop integrals. We also obtained new Feynman rules corresponding to the counter-terms in the renormalized Lagrangian. All Feynman rules for renormalized QED are summarized in the appendix. The constants in this counter-diagrams now have to be calculated order by order in perturbation theory. In the first lecture we already worked out how δ_2 and therefore $Z_2 = \delta_2 + 1$ have to be defined in our on shell renormalization scheme. Now we will proceed calculating Z_1 and Z_3.

We already stressed the dependence of the finite parts of the Z-variables on the renormalization scheme we use. The details of the renormalization scheme are expressed in the, so called, renormalization conditions. Eqns. (33) already define two of these conditions we have chosen. To evaluate Z_1 and Z_3 we will have to define two more.

Our goal in this lecture is to derive the beta function of QED which expresses the running coupling in QED. We already saw at the beginning of the last lecture, eqn. (2), the connection between the bare and the renormalized coupling constant, which for dimensional regularization we rewrite in the following way:

$$\mu^{4-n}\alpha = \frac{Z_2^2}{Z_1^2} Z_3 \alpha_0. \tag{40}$$

As we mentioned, we would like to find $Z_1 = Z_2$ in order to ensure gauge invariance. Indeed, we will now start to explicitly show this relation in first order of perturbation theory. Note that this identity holds to all orders of perturbation theory, which can be shown by using the Ward Identity. The running coupling will therefore only be dependend on Z_3.

7 The Vertex Correction

We now turn to vertex corrections in QED. On one loop level this means we look at the diagrams displayed in Fig. 5. We can use the Feynman rules given in Fig. 7 in the appendix to see that the sum of the two diagrams is equal to $-ie\gamma^\mu - ie\Lambda^\mu$, where

$$\Lambda^\mu(p',p) \equiv (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\nu \frac{i}{\not{p}' + \not{k} - m + i\epsilon} \gamma^\mu \frac{i}{\not{p} + \not{k} - m + i\epsilon} \gamma_\nu \frac{-i}{k^2 - \lambda^2 + i\epsilon}$$
$$= -ie^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{1}{\not{p}' + \not{k} - m + i\epsilon} \gamma_\nu \frac{1}{\not{p} + \not{k} - m + i\epsilon} \gamma_\mu \frac{1}{k^2 - \lambda^2 + i\epsilon}. \tag{41}$$

Note that as before the diagram is supposed to be part of a larger diagram, so

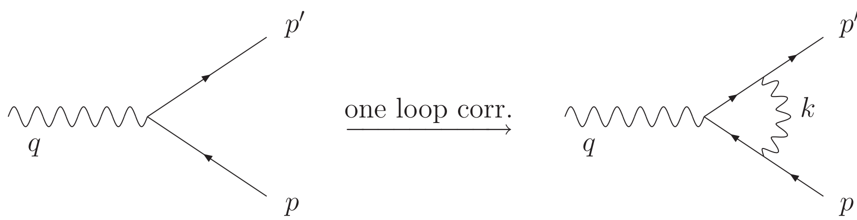


Figure 5: One loop correction of the QED vertex.

there are no external particle contributions in the amplitude.

We could now evaluate (41) with the calculational tools we derived in the first lecture. Indeed, for example for the order-α contribution to the anomalous magnetic moment of the electron, this would have to be done, since this diagram gives the strongest contribution here. See for example [3] for a detailed calculation. For our purpose we will see that this has not to be done, since we are primarily interested in finding the relation $Z_1 = Z_2$. Now in order to show this, we at first need to introduce a new renormalization condition, which will define the finite parts of Z_1:

$$\Lambda_f^\mu(p = p') = 0, \tag{42}$$

where Λ_f denotes the finite amplitude, which we obtain if we also take into account the counter term Feynman diagram:

$$-ie\Lambda_f^\mu(p,p') \equiv -ie\Lambda^\mu(p,p') - ie\gamma^\mu\delta_1, \tag{43}$$

so

$$\Lambda^\nu(p = p') = -\gamma^\nu\delta_1. \tag{44}$$

In order to proceed, we will need to know what the derivative of an inverse matrix (with respect to a real variable) is. Therefore we perform a (infinitesimal) small variation of A^{-1}, where A is a matrix:

$$\begin{aligned}\delta\big(A^{-1}\big) &= (A+\delta A)^{-1} - A^{-1} = \left[A\left(1+A^{-1}\delta A\right)\right]^{-1} A^{-1} \\ &= \left(1+A^{-1}\delta A\right)^{-1} A^{-1} - A^{-1} = \left(1-A^{-1}\delta A\right) A^{-1} - A^{-1} \\ &= -A^{-1}\delta A\, A^{-1}.\end{aligned}$$

Using this, we follow the procedure outlined in [5] and perform the following derivative:

$$\frac{\partial}{\partial p^\mu}\frac{1}{\not p - m} = -\frac{1}{\not p - m}\gamma_\mu\frac{1}{\not p - m}. \tag{45}$$

This enables us to calculate the derivative of Σ_2 as defined in eqn. (12):

$$\frac{\partial \Sigma_2}{\partial p^\nu} = \frac{ie^2}{(2\pi)^4}\int d^4k\, \gamma^\mu \frac{1}{\not p + \not k - m + i\epsilon}\gamma_\nu \frac{1}{\not p + k - m + i\epsilon}\gamma_\mu \frac{1}{k^2 - \lambda^2 + i\epsilon}. \tag{46}$$

Comparing this with (41) immediately gives us

$$\Lambda_\nu(p = p') = -\frac{\partial \Sigma_2}{\partial p^\nu}. \tag{47}$$

If we now use (44) and remember what we have already found in (37):

$$\delta_2 = \frac{\partial \Sigma_2}{\partial \not p},$$

we can derive:

$$\begin{aligned}\Lambda_\nu &= -\gamma_\nu \delta_1 \\ &= -\frac{\partial \Sigma_2}{\partial p^\nu} = -g_\nu{}^\mu\, \frac{\partial \Sigma_2}{\partial p^\mu} = -\gamma_\nu\gamma^\mu \frac{\partial \Sigma_2}{\partial p^\mu} = -\gamma_\nu \frac{\partial \Sigma_2}{\partial \not p} \\ &= -\gamma_\nu \delta_2.\end{aligned}$$

So we arrive at the important result:

$$\delta_1 = \delta_2, \tag{48}$$

or

$$Z_1 = Z_2. \tag{49}$$

Remark About Infrared Divergences

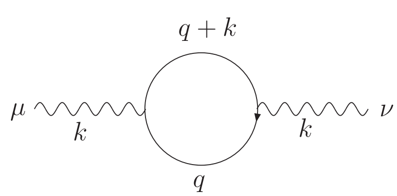
We shall now briefly discuss infrared divergences. More details could be found in [2, p. 175], for instance.

We had to introduce the photon mass λ in Λ^μ to cure the infrared divergence. But at first sight this means that the first order correction to physical observable processes like Coulomb scattering (which contain this diagram) would have a dependence on the unphysical photon mass λ. Indeed, a separate cross section correction like that would be divergent as $\lambda \to 0$.

The solution to this problem again lies in the insight that cross sections like that one are not measurable individually. We have forgotten to take into account another first order correction to processes like Coulomb scattering: Bremsstrahlung. Together with the fact that every photon detector can detect photons only down to some limiting energy, we conclude that observable is only the sum of the cross section for both processes. Now Bremsstrahlung corrections are also infrared divergent. But it turns out that the λ-dependence of the sum of the two processes cancels! So we are again left with finite (λ-independent) predictions for observables.

8 Vacuum Polarization

In this section we will calculate the last important one loop diagram: The vacuum polarization diagram, which is a correction to the photon propagator due to fermion loops:



$\equiv i\Pi^{\mu\nu}(k)$

The only Lorentz tensor in $\Pi^{\mu\nu}(q)$ can be $g^{\mu\nu}$ and $k^\mu k^\nu$. On the other hand the Ward identity tells us that $k_\mu \Pi^{\mu\nu}(k) = 0$. So we expect the following structure:

$$\Pi^{\mu\nu}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \Pi\left(k^2\right) .$$

We will soon find exactly this structure (without using the Ward identity).

Using again the Feynman rules in the appendix and noting that the closed fermion loop gives a factor of -1 and a trace over the product the Dirac matrices, we get:

$$\begin{aligned} i\Pi(k)^{\mu\nu} &= -e^2 \int \frac{d^4q}{(4\pi)^4} \mathrm{Tr}\, \gamma^\mu \frac{1}{\not{q} - m + i\epsilon} \gamma^\nu \frac{1}{\not{q} + \not{k} - m + i\epsilon} \\ &= -e^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\, \frac{\gamma^\mu \left(\not{q} + m\right) \gamma^\nu \left(\not{q} + \not{k} + m\right)}{(q^2 - m^2 + i\epsilon) \left((q+k)^2 - m^2 + i\epsilon\right)} \end{aligned}$$

We will now use the Feynman trick in the form of eqn. (13) to combine the denominators:

$$\begin{aligned} B - A &= 2qk + k^2 \\ A = (B - A)\, x &= q^2 - m^2 + (2qk + k^2)\, x + i\epsilon \\ &= (q + kx)^2 - xk^2\,(x-1) - m^2 + i\epsilon \\ &\equiv (q + xk)^2 - \Delta + i\epsilon, \end{aligned}$$

where we defined

$$\Delta \equiv xk^2\,(x-1) + m^2.$$

Noting that the trace of an odd number of Dirac matrices vanishes and going to n dimensions, we thus get:

$$\begin{aligned} i\Pi(k)^{\mu\nu} &= -e^2 \mu^{4-n} \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \mathrm{Tr}\, \frac{\gamma^\mu \not{q} \gamma^\nu \left(\not{q} + \not{k}\right) + \gamma^\mu\gamma^\nu m^2}{\left[(q + xk)^2 - \Delta + i\epsilon\right]^2} \\ &= -e^2 \mu^{4-n} \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \mathrm{Tr}\, \frac{\gamma^\mu \left(\not{q} - x\not{k}\right) \gamma^\nu \left(\not{q} + \not{k}\,(1-x)\right) + m^2 \gamma^\mu\gamma^\nu}{(q^2 - \Delta + i\epsilon)^2} \\ &= -e^2 \mu^{4-n} \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \mathrm{Tr}\, \frac{\gamma^\mu \not{q} \gamma^\nu \not{q} - x\,(1-x)\, \gamma^\mu \not{k} \gamma^\nu \not{k} + m^2 \gamma^\mu\gamma^\nu}{(q^2 - \Delta + i\epsilon)^2}, \end{aligned}$$

where we shifted the integration variable $q \to q + xk$ in the first step and used the fact that because of the antisymmetry, terms linear in q vanish in the second step. We now need to evaluate the three traces:

$$\begin{aligned} \mathrm{Tr}\, \gamma^\mu \not{q} \gamma^\nu \not{q} &= q_\rho q_\sigma\, \mathrm{Tr}\, \gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma = 4\left(2q^\mu q^\nu - q^2 g^{\mu\nu}\right) \\ \mathrm{Tr}\, \gamma^\mu \not{k} \gamma^\nu \not{k} &= 4\left(2k^\mu k^\nu - k^2 g^{\mu\nu}\right) \\ \mathrm{Tr}\, \gamma^\mu \gamma^\nu &= 4g^{\mu\nu}. \end{aligned}$$

Using this identities we arrive at:

$$\begin{aligned} i\Pi(k)^{\mu\nu} = -e^2\mu^{4-n}\int_0^1 dx \int \frac{d^nq}{(2\pi)^n}\Big[4\left(m^2+k^2(1-x)-q^2\right)\frac{g^{\mu\nu}}{\left(q^2-\Delta+i\epsilon\right)^2} \\ -8x(1-x)\frac{k^\mu k^\nu}{\left(q^2-\Delta+i\epsilon\right)^2}+8\frac{q^\mu q^\nu}{\left(q^2-\Delta+i\epsilon\right)^2}\Big]. \end{aligned}$$

To calculate the remaining integrals, we first note

$$\int\frac{d^nq}{(2\pi)^n}\frac{q^\mu q^\nu}{D}=\frac{1}{n}g^{\mu\nu}\int\frac{d^nq}{(2\pi)^n}\frac{q^2}{D}, \tag{50}$$

where D denotes a symmetric denominator. The derivation of that identity is straight forward: For $\mu \neq \nu$ the integral is zero because of the antisymmetry. Because of the Lorentz structure the result must be proportional to $g^{\mu\nu}$. The factor $1/n$ can be checked by multiplying both sides with $g_{\mu\nu}$ and using $g_{\mu\nu}g^{\mu\nu}=n$. Using our general integral from eqn. (24) we can also show:

$$\begin{aligned} \int\frac{d^nq}{(2\pi)^n}\frac{q^2}{\left(q^2-\Delta+i\epsilon\right)^2} &= -i\int\frac{d^nq_E}{(2\pi)^n}\frac{q^2+\Delta}{\left(q_E^2+\Delta-i\epsilon\right)^2}+i\Delta\int\frac{d^nq_E}{(2\pi)^n}\frac{1}{\left(q_E^2+\Delta-i\epsilon\right)^2} \\ &= -i\int\frac{d^nq_E}{(2\pi)^n}\frac{1}{\left(q_E^2+\Delta-i\epsilon\right)}+i\Delta\int\frac{d^nq_E}{(2\pi)^n}\frac{1}{\left(q_E^2+\Delta-i\epsilon\right)^2} \\ &= \frac{-i\pi^{\frac{n}{2}}}{(2\pi)^n}\frac{1}{(\Delta-i\epsilon)^{1-\frac{n}{2}}}\left[\Gamma\Big(1-\frac{n}{2}\Big)-\Gamma\Big(2-\frac{n}{2}\Big)\right] \\ &= \frac{-i\pi^{\frac{n}{2}}}{(2\pi)^n}\frac{1}{(\Delta-i\epsilon)^{1-\frac{n}{2}}}\Gamma\Big(1-\frac{n}{2}\Big)\frac{n}{2} \end{aligned} \tag{51}$$

Using this integrals we can finish our calculation:

$$\begin{aligned} i\Pi(k)^{\mu\nu} &= -4ie^2\mu^{4-n}\int_0^1 dx\frac{1}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma\left(2-\frac{n}{2}\right)}{\Delta^{2-\frac{n}{2}}} \\ &\quad\times\left[g^{\mu\nu}\left(-m^2+x(1-x)k^2\right)+g^{\mu\nu}\left(m^2+x(1-x)q^2\right)-2x(1-x)k^\mu k^\nu\right] \\ &= \left(k^2g^{\mu\nu}-k^\mu k^\nu\right)i\Pi\left(k^2\right), \end{aligned}$$

where

$$\Pi\left(k^2\right)=\frac{-8\alpha}{(4\pi)^{\frac{n}{2}-1}}\Gamma\Big(2-\frac{n}{2}\Big)\int_0^1 dx\frac{x(1-x)}{\left[k^2x(x-1)+m^2\right]^{2-\frac{n}{2}}}. \tag{52}$$

9 The Beta Function of QED

So far we have not said how we are going to renormalize our result from the last chapter. It is logarithmic divergent, so only the zero order term of a Taylor expansion is infinte. Therefore the following renormalization condition is sufficient:

$$\Pi_f\left(k^2 = -M^2\right) = 0. \tag{53}$$

Thus

$$\Pi_f\left(k^2\right) = \Pi\left(k^2\right) - \Pi\left(-M^2\right). \tag{54}$$

Meaning we make $\Pi(k^2)$ finite by subtracting at (the space-like momentum) $k^2 = -M^2$. Like in the last chapters, this defines our counter term to be

$$\delta_3 = \Pi\left(-M^2\right), \tag{55}$$

or

$$\begin{aligned} Z_3 &= 1 + \Pi\left(-M^2\right) \\ &= 1 - \frac{8\alpha}{(4\pi)^{\frac{n}{2}-1}}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx \frac{x\,(1-x)}{\left[-M^2x\,(x-1)+m^2\right]^{2-\frac{n}{2}}} \end{aligned} \tag{56}$$

Here we introduced the scale M (compare to section 5.2). This condition now defines us the renormalized coupling in the lagangian. Note that $\alpha = \frac{1}{137}$ would only hold if we chose $M^2 = 0$.

We now take the derivative of (40) with respect to the scale M and use that $Z_1 = Z_2$:

$$\begin{aligned} \tfrac{d\alpha}{dM} &= \mu^{-(4-n)}\frac{dZ_3}{dM}\,\alpha_0 \\ &= \frac{dZ_3}{dM}\,\alpha \\ &= \frac{-8\alpha}{(4\pi)^{\frac{n}{2}-1}}\Gamma\left(2-\frac{n}{2}\right)\left(-\left(2-\frac{n}{2}\right)\right)2M\int_0^1 dx\frac{x^2\,(1-x)^2}{\left[M^2x\,(1-x)+m^2\right]^{3-\frac{n}{2}}} \\ &= \frac{4M\alpha}{\pi}\int_0^1 dx\frac{x^2\,(1-x)^2}{\left[M^2x\,(1-x)+m^2\right]^{3-\frac{n}{2}}} \end{aligned} \tag{57}$$

where in the second step we used that $\mu^{4-n}\alpha \approx \alpha_0$ since Z_3 is already of order α.

For M much bigger than m we can now read off the beta function of QED:

$$\beta(\alpha) \equiv M\frac{d\alpha}{dM} = \frac{2\alpha^2}{3\pi}. \tag{58}$$

This equation can be integrated:

$$\alpha\big(M^2\big) = \frac{\alpha(M_0^2)}{1 - \frac{\alpha(M_0^2)}{3\pi}\ln\frac{M^2}{M_0^2}}. \tag{59}$$

This defines the running of the renormalized coupling constant in our renormalization scheme. Note that for example in the Minimum Subtraction (MS) scheme we get exactly the same first order approximation for the beta function. Instead of the subtraction point (which we don't have in that scheme, because there only the part proportional to $\frac{1}{\epsilon}$ of the expansion for small ϵ in the dimensional regularized loop integrals is subtracted), the renormalized coupling constant is dependent on the scale μ that you have to introduce in order to get the dimensions right, which in the MS scheme does not cancel in the expression for the beta function as it did in our case. See for example [3] for details here.

APPENDIX

Contraction Rules

The relation between contractions and propagators is given by:

$$iS_{F_{\alpha\beta}}(x-x') = \wick{\c{\psi}_\alpha(x)\c{\bar{\psi}}_\beta(x')} = i\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\left(\not{q}+m\right)_{\alpha\beta}\; e^{-iq(x-x')}}{q^2-m^2+i\epsilon}, \tag{A.1}$$

$$iD_{F_{\mu\nu}}(x-x') = \wick{\c{A}_\mu(x)\c{A}_\nu(x')} = -ig^{\mu\nu}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k^2-\lambda^2+i\epsilon}, \tag{A.2}$$

$$\wick{\c{\psi}(x')\,\c{|}p,s\rangle} = e^{-ipx'}u^s(p)\,|0\rangle\,, \tag{A.3}$$

$$\wick{\langle \c{p}',s'|\,\c{\bar{\psi}}} = \langle 0|\,e^{ip'x}\bar{u}^{s'}(p'). \tag{A.4}$$

Note that we use the Feynman gauge and that we introduced a photon mass λ to regularize infrared divergences.

The Gamma Function

The Gamma function $\Gamma(z)$ has the following important properties:

$$\Gamma(n+1) = n! \quad \forall n \in \mathbb{N},$$
$$x\,\Gamma(x) = \Gamma(x+1)\,\forall x \in \mathbb{R}^{>0}.$$

Also useful for expansions of the Gamma function is the following product representation

$$\frac{1}{\Gamma(z)} = z\,e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}, \tag{A.5}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

Fig. 6 displays the Gamma function. Note that it has a poles at $x = 0$ and at every negative integer.

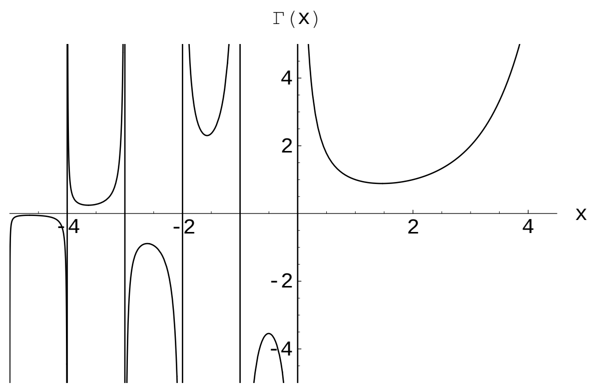


Figure 6: Plot of the Gamma function.

Leading Order Expansions

From (A.5) one can derive the leading order expansion of $\Gamma(x)$:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathrm{O}(\epsilon) \tag{A.6}$$

near $x = 0$, and

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon} - \gamma + 1 + \ldots + \frac{1}{n} + \mathrm{O}(\epsilon)\right) \tag{A.7}$$

near $x = -n$. Another expression which will occur in regularized integrals is

$$\left(\frac{1}{\Delta}\right)^{\epsilon} = 1 - \epsilon \log \Delta + \ldots . \tag{A.8}$$

Particularly useful is now the following expansion, which can be directly derived from that:

$$\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}}\left(\frac{1}{\Delta}\right)^{2-\frac{n}{2}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi) + \mathrm{O}(\epsilon)\right), \tag{A.9}$$

with $\epsilon = 4 - n$.

Feynman rules for QED

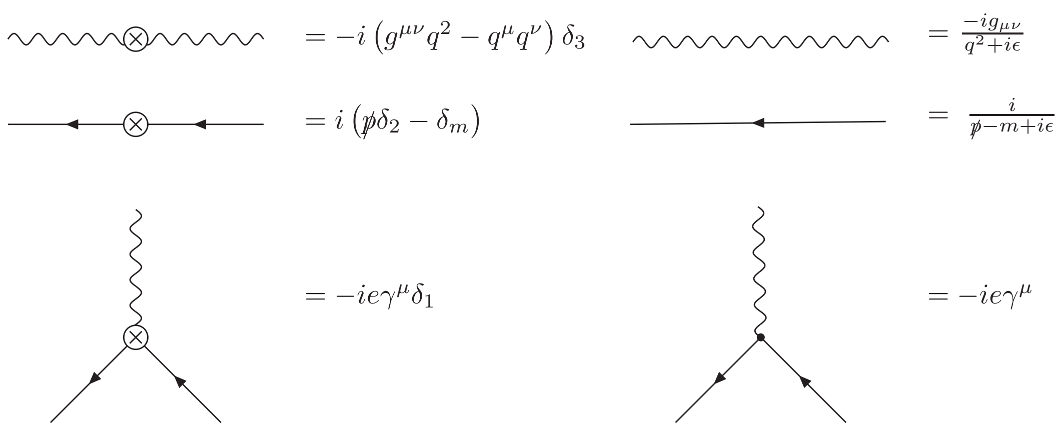


Figure 7: Feynman rules for QED in renormalized perturbation theory. Note that a closed fermion loop additionally gives a factor of -1 and a trace of a product of Dirac matrices. See [2, p. 120].

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