# Chapter 5

# Path Integrals in Quantum Mechanics

## 5.1 Introduction

The first thing you need to know about "path integral quantisation" is that it is sometimes called "functional methods". Essentially path integral quantisation is an alternative formulation of quantum mechanics which was developed by Feynman in 1948, based on some early ideas by Dirac in 1933. The Schrodinger formulation of quantum mechanics, which we reviewed in section 1.2, is based on the Hamiltonian approach, i.e. we take the Hamiltonian from classical mechanics and promote the generalised positions and momenta to the status of operators. We used this approach to quantise the KG field, which we regarded as the continuum limit of a system of coupled oscillators. By contrast Feynman's path integral approach to quantum mechanics is based on the Lagrangian approach, and uses the action $S$ which is related to the Lagrangian by Eq.1.1.2. We have already seen in Eq.1.2.10 that in quantum mechanics, knowledge of the wavefunction at time $t_1$, together with the propagator $< q_2|U(t_2,t_1)|q_1 >$, is sufficient to determine the wavefunction at some later time $t_2$, providing that the integral in Eq.1.2.10 can be performed. Providing we know the propagator, we don't need Schrodinger's equation at all. What Feynman's path integral approach does is give the propagator directly, as a sum over all paths between the two points $q_i, t_i$ and $q_f, t_f$, where each path is ascribed an amplitude $e^{iS[q(t)]}$ where $S$ is in units of $h/(2\pi)$. Thus schematically,

$$< q_2|U(t_2,t_1)|q_1 >= const. \sum_{allpaths} e^{iS[q(t)]} \tag{5.1.1}$$

Notice that the sum is over all paths, not just the classical path. Thus a classical ball falling under gravity between two spacetime points is, from the quantum point of view, taking all possible paths across the Universe, rather like a Star Trek transporter. Classically the Newtonian trajectory is preferred not because the other paths are not taken, but because all the amplitudes associated with the other paths tend to accurately cancel out in the sum. We are left only

with the classical path and paths which are very close to the classical path, where very close means that the action is equal to the classical action to an accuracy of $h/(2\pi)$. The classical path is of course the one which extremises the action, which is why paths very close to it do not cancel and this provides an understanding or derivation of the principle of least action in the classical limit.

The path integral approach to quantum mechanics clearly has great aethetic value. It answers deep philosophical questions like "which of the two slits does the particle really pass through in the two slit experiment?" with "the particle takes both paths through the slits". In fact the particle really takes two infinite classes of paths. However, from the point of view of quantum field theory, you may wonder why we are bothering to consider this approach when the standard or "canonical" approach to quantisation seems perfectly serviceable. Why are we wasting time considering an alternative approach to quantum field theory when we already have achieved our goal of quantising the KG field? The answer is twofold. Firstly the path integral approach based on the notion of a propagator provides a more intuitive understanding of Feynman diagrams and Feynman rules which are the nuts and bolts of scattering and decay calculations. Indeed, Feynman himself introduced these diagrams from the point of view of the path integral approach, and there is no higher recommendation than that. The second reason is that the Hamiltonian formalism of canonical quantisation does not manifestly preserve Lorentz invariance, while the Lagrangian formalism of the path integral approach does. When we consider more complicated field theories such as QED, and gauge theories in general, it will prove to be technically convenient to maintain Lorentz invariance in a manifest way all through the calculations. Thus, in the following QFT courses, we will rely almost exclusively on path integral methods – so it is as well to learn these methods as early as possible.

## 5.2 The Point to Point Transition Amplitude

Our starting point is the expression for the propagator in the Heisenberg basis as given in Eq.5.2.1

$$< q_f|U(t_f,t_i)|q_i >=< q_f,t_f|q_i,t_i > \tag{5.2.1}$$

which shows very clearly that the propagator is just the amplitude that the particle is at some final (f) point $q_f, t_f$ given that it was at the initial (i) point $q_i, t_i$ where $t_f > t_i$. In the Schrodinger picture the propagator is

$$< q_f|U(t_f,t_i)|q_i >=< q_f|e^{-iH(t_f-t_i)}|q_i > \tag{5.2.2}$$

For convenience we also reproduce Eq.1.2.10 which shows how the propagator may be used to determine the future value (at $t_f > t_i$) of the wavefunction,

$$\psi(q_f,t_f) = \int dq_i iG(q_f,t_f;q_i,t_i)\psi(q_i,t_i) \tag{5.2.3}$$

where we have defined

$$iG(q_f, t_f; q_i, t_i) \equiv < q_f|U(t_f, t_i)|q_i > \tag{5.2.4}$$

again valid only for $t_f > t_i$. The function $G(q_f, t_f; q_i, t_i)$ is known as the Green's function. It has the property that if the initial wavefunction at $t_i = 0$ is just a delta function at $q_i = 0$, $\psi(q_i, t_i) = \delta(q_i)$, then the subsequent wavefunction is just equal to $i$ times the Green's function

$$\psi(q_f, t_f) = iG(q_f, t_f; 0, 0) \tag{5.2.5}$$

Eq.5.2.5 is only valid for $t_f > t_i$, but can be written in a form valid for all times as

$$\theta(t_f - t_i)\psi(q_f, t_f) = \int dq_i iG_(q_f, t_f; q_i, t_i)\psi(q_i, t_i) \tag{5.2.6}$$

where the Green's function is defined to be zero for $t_f < t_i$,

$$G_(q_f, t_f; q_i, t_i) \equiv 0, \quad for \;\; t_f < t_i$$

and is called a retarded Green's function or propagator.

The name of the game in this approach is therefore to find the retarded propagator. We do this by converting $< q_f, t_f|q_i, t_i >$ to a path integral as follows.

Let us begin by dividing the time interval between $t_i$ and $t_f$ into two, with $t_1$ as the intermediate time and $q_1$ as the intermediate point in space. Inserting a complete set of Heisenberg states at time $t_1$ we have,

$$< q_f, t_f|q_i, t_i >= \int dq_1 < q_f, t_f|q_1, t_1 >< q_1, t_1|q_i, t_i > \tag{5.2.7}$$

which corresponds to the product of the propagator from $q_i, t_i$ to $q_1, t_1$ with the propagator from $q_1, t_1$ to $q_f, t_f$, integrated over all possible positions $q_1$ at the intermediate time $t_1$. Mathematically Eq.5.2.7 displays a sort of group property of propagators. Physically we are performing a very crude sum over paths between the inital and final points which allow the particle to be anywhere at the intermediate time $t_1$. Of course we haven't gained anything yet because the expression still involves a product of propagators of the type we wish to evaluate. Nevertheless Eq.5.2.2 shows that there may be some merit in continuing the process of dividing up the time interval into smaller and smaller steps. If the time steps $\tau$ are small enough then we can write,

$$e^{-iH\tau} = 1 - iH\tau + O(H\tau 2)^2 \tag{5.2.8}$$

where $H\tau$, in units of $h/(2\pi)$, is small compared to unity.

Motivated by the above discussion, let us now divide the time interval between $t_i$ and $t_f$ into $n + 1$ equal steps $\tau = (t_f - t_i)/(n + 1)$. Inserting a

complete set of Heisenberg position eigenstates at each time step we have the obvious generalisation of Eq.5.2.7,

$$< q_f, t_f|q_i, t_i >= \int \ldots \int dq_1 \ldots dq_n < q_f, t_f|q_n, t_n >< q_n, t_n| \ldots |q_1, t_1 >< q_1, t_1|q_i, t_i > \tag{5.2.9}$$

Physically we are now performing a more refined sum over all paths by allowing the particle to be in any position at each of the specified time slices. Even though the time slices may be infinitesimally close together the particle is allowed to be infinitely far apart between the two times, so this cannot be regarded as a sum over trajectories in the usual sense (these "paths" are strictly called "Markov chains").

Now consider one of the propagators between $q_j, t_j$ and $q_{j+1}, t_{j+1}$,

$$< q_{j+1}, t_{j+1}|q_j, t_j >=< q_j|e^{-iH\tau}|q_j > \tag{5.2.10}$$

Using Eq.5.2.8 we have,

$$\begin{aligned} < q_{j+1}, t_{j+1}|q_j, t_j > &= < q_j|1 - iH\tau|q_j > \\ &= \delta(q_{j+1} - q_j) - i\tau < q_{j+1}|H|q_j > \end{aligned} \tag{5.2.11}$$

In the first term the Dirac delta function can be written as

$$\delta(q_{j+1} - q_j) = \frac{1}{2\pi} \int dp e^{ip(q_{j+1} - q_j)} \tag{5.2.12}$$

In the second term the Hamiltonian matrix element can be written

$$< q_{j+1}|H|q_j >=< q_{j+1}|\frac{p^2}{2m}|q_j > + < q_{j+1}|V(q)|q_j > \tag{5.2.13}$$

where we have assumed a simple kinetic plus potential form for the Hamiltonian. We now indulge in a little operator expungation. Inserting complete sets of momentum eigenfuctions in the first (kinetic) term and remembering that $p$ is an operator gives,

$$\begin{aligned} < q_{j+1}|\frac{p^2}{2m}|q_j > &= \int \int dp dp' < q_{j+1}|p' >< p'|\frac{p^2}{2m}|p >< p|q_j > \\ &= \int \int \frac{dp dp'}{2\pi} e^{i(p'q_{j+1} - pq_j)} \frac{p^2}{2m} \delta(p - p') \\ &= \int \frac{dp}{2\pi} e^{ip(q_{j+1} - q_j)} \frac{p^2}{2m} \end{aligned} \tag{5.2.14}$$

where $p^2$ in the last line is a number, and we have used the plane wave result,

$$< q|p >= \frac{1}{\sqrt{2\pi}} e^{ipq} \tag{5.2.15}$$

Similarly the operators may be expunged from the potential term, which may also be expressed as a momentum integral,

$$\begin{aligned} < q_{j+1}|V(q)|q_j > &= V(\overline{q_j}) < q_{j+1}|q_j > \\ &= V(\overline{q_j}) \int \frac{dp}{2\pi} e^{ip(q_{j+1} - q_j)} \end{aligned} \tag{5.2.16}$$

where we define the average value as

$$\overline{q_j} \equiv \frac{1}{2}(q_{j+1} + q_j) \tag{5.2.17}$$

Putting Eqs.5.2.14 and 5.2.16 together yields the Hamiltonian matrix element with all operators expunged,

$$\begin{aligned} < q_{j+1}|H|q_j > &= < q_{j+1}|\frac{p^2}{2m}|q_j > + < q_{j+1}|V(q)|q_j > \\ &= \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)}(\frac{p^2}{2m} + V(\overline{q_j})) \\ &= \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)} H(p, \overline{q_j}) \end{aligned} \tag{5.2.18}$$

Inserting Eqs.5.2.18 and 5.2.12 into 5.2.11

$$\begin{aligned} < q_{j+1}, t_{j+1}|q_j, t_j > &= < q_j|1 - iH\tau|q_j > \\ &= \delta(q_{j+1} - q_j) - i\tau < q_{j+1}|H|q_j > \\ &= \frac{1}{2\pi}\int dp e^{ip(q_{j+1}-q_j)} - i\tau \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)} H(p, \overline{q_j}) \\ &= \frac{1}{2\pi}\int dp e^{ip(q_{j+1}-q_j)}(1 - i\tau H(p_j, \overline{q_j})) \\ &= \frac{1}{2\pi}\int dp e^{ip(q_{j+1}-q_j)} e^{-iH(p_j,\overline{q_j})\tau} \\ &= \frac{1}{2\pi}\int dp_j e^{i((p_j(q_{j+1}-q_j) - H(p_j,\overline{q_j})\tau)} \end{aligned} \tag{5.2.19}$$

where we have used Eq.5.2.8 in reverse, have written the dummy momentum variable as $p_j$ to remind us that we are doing this near the time $t_j$. Eq.5.2.19 is an expression for the propagator between $q_j, t_j$ and $q_{j+1}, t_{j+1}$ which only involves an integration over c-numbers (no q-numbered operators), but is only valid in the small $\tau$ limit. The full propagator in Eq.5.2.9 is merely a product of such little propagators,

$$< q_f, t_f|q_i, t_i > = n \overset{lim}{\rightarrow} \infty \int \prod_{j=1}^{n} dq_j \prod_{j=0}^{n} \frac{dp_j}{2\pi} e^{i\sum_{j=0}^{n}((p_j(q_{j+1}-q_j) - H(p_j,\overline{q_j})\tau)} \tag{5.2.20}$$

with $q_0 = q_i$ and $q_{n+1} = q_f$. In the continuum limit we write $q_i = q(t_i)$ and $q_f = q(t_f)$ and Eq.5.2.20 may be written in symbollic form as

$$< q_f, t_f|q_i, t_i > = \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q} - H(p,q))\right)} \tag{5.2.21}$$

We stress again that this expression only involves c-numbers. Eq.5.2.21 means no more and no less than Eq.5.2.20, but is easier to write down. In practical calculations we must return to 5.2.20.

We can expunge the Hamiltonian by performing the momentum integrations in Eq.5.2.20, using the standard Gaussian integral,

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c} \tag{5.2.22}$$

Using this result for each of the little propagators yields the full propagator,

$$\begin{aligned} < q_f, t_f|q_i, t_i > &= n \stackrel{lim}{\rightarrow} \infty \int \prod_{j=1}^{n} dq_j \prod_{j=0}^{n} \frac{dp_j}{2\pi} e^{i\sum_{j=0}^{n}((p_j(q_{j+1}-q_j)-\frac{p_j^2}{2m}\tau - V(\overline{q_j})\tau)} \\ &= n \stackrel{lim}{\rightarrow} \infty (\frac{m}{2\pi i\tau})^{\frac{n+1}{2}} \int \prod_{j=1}^{n} dq_j e^{i\tau\sum_{j=0}^{n}(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\tau}\right)^2 - V(\overline{q_j})} \end{aligned} \tag{5.2.23}$$

The schematic for the continuum limit is thus,

$$< q_f, t_f|q_i, t_i >= N \int \mathcal{D}q(t) e^{i\int_{t_i}^{t_f} dtL(q,\dot{q})} = N \int \mathcal{D}q(t) e^{iS[q(t)]} \tag{5.2.24}$$

where $N$ is the constant of proportionality in the continuum limit. We have proven our original assertion that the propagator is just equal to the sum over all possible paths of a bunch of phases, where the phase of each path is given by $e^{iS[q(t)]}$. Even more importantly we have given a practical way of calculating the sum over all paths of these phases, as the continuum limit of a lot of time slices, where the particle is allowed to be infinitely far apart in its coordinates between infinitesimally close time slices. In no way are these to be imagined as possible classical paths - in fact they are not paths at all since over an infinitesimal time the particle is allowed to travel from one end of the Universe to another which makes no physical sense for a path. Perhaps we should not call this the path integral approach since the word path is a misnomer. A better name would be the Markov chain approach or even the Star Trek transporter approach, but I shall stick with tradition and use the conventional name!

## 5.3 Imaginary Time

In the previous section we considered the point to point transition amplitude [1] and showed how it could be expressed as a path integral. In this section we shall *analytically continue* the time coordinate $t$ from the real axis to the complex plane:

$$t \rightarrow z$$

In other words we shall take all the results obtained in the previous section and make the above analytic continuation. In particular we shallbe interested in analytic continuation of time from the real axis to the imaginary axis:

$$t \rightarrow -i\tau$$

[1] i.e. the amplitude that given the particle is at a point $q_i$ at time $t_i$ that it will be found at the point $q_f$ at time $t_f$ where $t_f > t_i$.

where $\tau$ is a real number. In other words we set $t = -i\tau$ in all the previous results. This implies that

$$x^2 - c^2t^2 = x^2 + c^2\tau^2 \tag{5.3.1}$$

which is familiar in relativity as Euclidean space.

The whole of quantum mechanics can be re-formulated in imaginary time just by replacing $t$ by $-i\tau$ everywhere. The Schrodinger equation becomes:

$$i\frac{\partial}{\partial t}|\psi(t)>= H|\psi(t)> \longrightarrow -\frac{\partial}{\partial \tau}|\psi(\tau)>= H|\psi(\tau)>$$

However the energy eigenfunctions and eigenvalues do not change since $H$ is time independent:

$$H|E_n>= E_n|E_n> \longrightarrow H|E_n>= E_n|E_n>$$

We shall exploit this fact. The time translation operator becomes:

$$U(t) = e^{-iHt} \longrightarrow U(\tau) = e^{-H\tau}$$

The point to point transition amplitude becomes

$$< q_f|U(t_f, t_i)|q_i > \longrightarrow < q_f|U(\tau_f, \tau_i)|q_i >$$

Note that the position eigenstates $|q>$ do not change. Its path integral evaluation

$$< q_f|U(t_f, t_i)|q_i >= N \int \mathcal{D}q(t) e^{i\int_{t_i}^{t_f} dtL}$$

becomes

$$< q_f|U(\tau_f, \tau_i)|q_i >= N \int \mathcal{D}q(\tau) e^{-\int_{\tau_i}^{\tau_f} d\tau L_E}$$

where the measure

$$N \int \mathcal{D}q(t) = n \stackrel{lim}{\rightarrow} \infty \left(\frac{m}{2\pi i\delta t}\right)^{\frac{n+1}{2}} \int \prod_{j=1}^{n} dq_j$$

becomes the measure

$$N \int \mathcal{D}q(\tau) = n \stackrel{lim}{\rightarrow} \infty \left(\frac{m}{2\pi \delta \tau}\right)^{\frac{n+1}{2}} \int \prod_{j=1}^{n} dq_j$$

Thus the derivation proceeds exactly as before, dividing up $\tau$ into small slices, $\delta\tau$ etc. The Euclidean Lagrangian is defined as

$$L_E \equiv \frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)$$

which is the negative of the Lagrangian in Euclidean space, causing the potential to appear in an upside down way.

A useful feature of the Euclidean formulation is that for large Euclidean times any initial state evolves into the ground state, providing it is not orthogonal to it. In particular if the particle is in a position eigenstate $|q_i>$ at time $\tau_i$ then at a much later time $\tau_f >> \tau_i$ it will have evolved totally into the ground state $|E_0>$. To prove this we write $\tau = \tau_f - \tau_i$ and consider

$$\tau \stackrel{lim}{\rightarrow} \infty < q_f|U(\tau_f, \tau_i)|q_i >$$

$$= \tau \stackrel{lim}{\rightarrow} \infty \sum_n < q_f|E_n >< E_n|q_i > e^{-E_n \tau}$$

$$\approx < q_f|E_0 >< E_0|q_i > e^{-E_0 \tau} = \psi_0(q_f)\psi_0^*(q_i)e^{-E_0 \tau}$$

and the result follows.

If we set $q_i = q_f$ then the above result gives us the ground state wavefunction (or rather its modulus squared) in terms of a limit of the propagator and the energy $E_0$.

$$\psi_0(q)\psi_0^*(q) = \tau \stackrel{lim}{\rightarrow} \infty \frac{< q|U(\tau_f, \tau_i)|q >}{e^{-E_0 \tau}}$$

The normalisation condition

$$\int_{-\infty}^{\infty} \psi_0(q)\psi_0^*(q)dq = 1$$

allows us to eliminate the exponential term since it implies,

$$e^{-E_0 \tau} \approx \int_{-\infty}^{\infty} dq < q|U(\tau_f, \tau_i)|q >$$

hence we can write

$$\psi_0(q)\psi_0^*(q) = \tau \stackrel{lim}{\rightarrow} \infty \frac{< q|U(\tau_f, \tau_i)|q >}{\int_{-\infty}^{\infty} dq < q|U(\tau_f, \tau_i)|q >}$$

It is worth pointing out that it is not necessary to analytically continue to imaginary time to obtain useful results. Suppose that we analytically continue,

$$t \rightarrow e^{-i\delta}\tilde{t}$$

where $\tilde{t}$ is a real variable and $\delta$ is a real constant. For example if we set $\delta = \pi/2$ then we can identify $\tilde{t}$ as $\tau$ considered above. Clearly if $\delta = 0$ then $\tilde{t}$ is identical to $t$, so we see that by adjusting $\delta$ we can analytically continue results from real time to imaginary time and back again. In fact a give choice of delta corresponds to a rotation of the real time axis through an angle $\delta$ in the complex plane. Of course for general $\delta$ the Schrodinger equation looks a little odd, but the important point is that the energy eigenvalues and eigenvectors are fixed since the Hamiltonian has not explicit time dependence and so the energy eigenvalue problem is unchanged for any choice of $\delta$. The important

point is that a non-zero choice of $\delta$ is sufficient to allow the ground state to be isolated since in real time we have

$$t \overset{lim}{\rightarrow} \infty < q_f|U(t_f,t_i)|q_i >$$

$$= t \overset{lim}{\rightarrow} \infty \sum_n < q_f|E_n >< E_n|q_i > e^{-iE_nt}$$

and we see that after analytic continuation

$$e^{-iE_nt} \rightarrow e^{-iE_n\tilde{t}\cos\delta}e^{-E_n\tilde{t}\sin\delta}$$

so that the ground state is isolated provided $0 < \delta < \pi$ for real positive $\tilde{t}$. For such $\delta$ we have,

$$\tilde{t} \overset{lim}{\rightarrow} \infty < q_f|U(\tilde{t_f},\tilde{t_i})|q_i >$$

$$\approx < q_f|E_0 >< E_0|q_i > e^{-iE_0\tilde{t}\cos\delta}e^{-E_0\tilde{t}\sin\delta}$$

and again the higher energy states are exponentially damped for large times, leaving only the ground state in the sum.

Of course if we choose $\delta = 0$ then the approximation is no longer valid, but we may choose a small positive value of $\delta$,

$$\delta = \epsilon, \quad 0 < \epsilon \ll 1$$

providing we consider sufficiently large values of $\tilde{t}$. In this case the orientation of the complex time axis is very close to the real axis and we shall be rather sloppy and drop the tilde to write,

$$t \overset{lim}{\rightarrow} \infty < q_f|U(t_f,t_i)|q_i >$$

$$= t \overset{lim}{\rightarrow} \infty < q_f|E_0 >< E_0|q_i > e^{-iE_0t}e^{-E_0t\epsilon}$$

Although we are being sloppy we must always remember that the lhs is not strictly the physical point to point transition amplitude, but an analytic continuation of it slightly off the real axis. For example in the case of the harmonic oscillator propagator considered above we must analytically continue this result according to

$$t \rightarrow (1 - i\epsilon)t \tag{5.3.2}$$

then take $\epsilon \rightarrow 0$ to avoid any ambiguities.

## 5.4 Transition Amplitudes With an External Driving Force

Now let us return to real time (at least for the moment!) and consider adding an external time dependent driving force to the particle so that the Lagrangian becomes

$$L \rightarrow L + J(t)q(t) \tag{5.4.1}$$

where $J(t)$ is the driving force. For example if the particle is in a harmonic oscillator potential then the new term coresponds to forced harmonic motion, where $J(t)$ may be a sinusoidally time dependent driving term for example.

Why are we considering such a driving term? The answer has to do with quantum field theory where the field is equivalent to a continnum of coupled harmonic oscillators, as we have seen. Consider the field to be an electromagnetic field for example, whose quanta are photons. Photons may be created or destroyed by atoms which in some sense act as a souce or sink of photons. From the point of view of the e.m. field the electron in the atom acts like an external driving force for the oscillators of the e.m. field which may excite or de-excite the e.m. field oscillators, corresponding to photon emmission or absorption. In fact the electron is itself described by a field, and it is the interaction of the electron field with the e.m. field which is responsible, but we may choose to consider the e.m. field only, and describe the effects of the interaction by an external term driving the photon field oscillators. The idea of an external driving term is therefore a useful way of describing the creation or destruction of field quanta, where the driving term acts locally on all the field oscillators. In this field context the oscillator driving term becomes a spacetime dependent driving term and we have in the case of the KG field

$$J(t)q(t) \rightarrow J(x)\phi(x) \tag{5.4.2}$$

where $x$ is the spacetime point. In this case $J(x)$ is referred to as a field *source*.

Having motivated the consideration of a driving term, let us now suppose it is only operative between the two times $t$ and $t'$ where $t < t'$. For example at time $t$ an electron in an atom may begin to drive the e.m. field and cause a photon to be created. The photon may later be absorbed by another atom such that its aborption is complete at time $t'$. The process of emission and absorption therefore takes place over a time interval between $t$ and $t'$, and this is the interval over which we may consider the external driving force to be operative. Suppose that $T$ is an earlier time than $t$ and $T'$ is a later time than $t'$, and let us consider the point to point transition amplitude of a single particle.

We shall now see that it is possible to relate such a point to point transition amplitude to the *ground state to ground state* transition amplitude in the presence of the external driving force. This is the analogue of finding the ground state wavefunction in the absence of the driving force as considered in the previous section. As in the previous section we shall need to analytically continue to complex time in order to achieve this.

The amplitude that a particle is at the point $Q'$ at time $T'$ given that it was at $Q$ at time $T$, and taking account of the driving term $J$ between $t$ and $t'$ where $T < t < t' < T'$ is then (in real time),

$$< Q', T'|Q, T >^J = N \int \mathcal{D}q(t) e^{i \int_T^{T'} dt L(q,\dot{q}) + Jq} \tag{5.4.3}$$

where we have put a superscript $J$ on the lhs to remind us about the driving term dependence of the matrix element. We now insert a complete set of Heisenberg

position states at times $t$ and $t'$ at which the driving term switches on and off,

$$< Q', T'|Q, T >^J = \int dqdq' < Q', T'|q', t' >< q', t'|q, t >^J < q, t|Q, T > \quad (5.4.4)$$

where we have a superscript $J$ only on the middle matrix element. Now the two matrix elements which do not depend on $J$ can each be treated as in the previous section:

$$< Q'|U(T', t')|q' > = \sum_n < Q'|E_n >< E_n|q' > e^{-iE_n(T'-t')}$$

$$< q|U(t, T)|Q > = \sum_n < q|E_n >< E_n|Q > e^{-iE_n(t-T)}$$

In each case the ground state is isolated by analytically continuing to complex time and taking the time differences to be large. An $\epsilon$ rotation of the time axis as in Eq.5.3.2 yields the results

$$< Q'|U(T', t')|q' > \approx < Q'|E_0 >< E_0|q' > e^{-iE_0(T'-t')} e^{-E_0(T'-t')\epsilon}$$

$$< q|U(t, T)|Q > \approx < q|E_0 >< E_0|Q > e^{-iE_0(t-T)} e^{-E_0(t-T)\epsilon}$$

valid for very large (infinite) time differences and as before we use a slightly sloppy notation for the lhs's which must strictly be evaluated slightly off the real axis if any ambiguity arises. Inserting these results Eq.5.4.4 gives,

$$< Q', T'|Q, T >^J \approx \int dqdq' < Q'|E_0 >< E_0|q' > e^{-iE_0(T'-t')} e^{-E_0(T'-t')\epsilon}$$

$$\times < q', t'|q, t >^J < q|E_0 >< E_0|Q > e^{-iE_0(t-T)} e^{-E_0(t-T)\epsilon}$$

Thus we find

$$< Q', T'|Q, T >^J \approx < Q'|E_0 >< E_0|Q > e^{-iE_0(T'-T)} e^{-E_0(T'-T)\epsilon}$$

$$\times \int dqdq' < E_0|q', t' > \times < q', t'|q, t >^J < q, t|E_0 >$$

Or, rearranging,

$$\int dqdq' < E_0|q', t' >< q', t'|q, t >^J < q, t|E_0 >$$

$$\approx \frac{< Q', T'|Q, T >^J}{< Q'|E_0 >< E_0|Q > e^{-iE_0(T'-T)} e^{-E_0(T'-T)\epsilon}}$$

The LHS is just the amplitude that a particle in the ground state wavefunction at time $t$ will be found in the ground state wavefunction at time $t'$. [2] The result is only true in the limit

$$T' \to \infty, \quad T \to -\infty$$

[2] Actually it is not strictly this because we must always remember that our times have been analytically continued off the real axis. But this is only by a small amount $\epsilon$, so unless ambiguities arise we can think of the LHS as indicated.

Taking the subsequent limits

$$t' \to \infty, \quad t \to -\infty$$

allows $J$ to act over all times. We write the *ground state to ground state* transition amplitude in this limit as

$$< 0, \infty | 0, -\infty >^J = \int dq dq' < E_0 | q', \infty >< q', \infty | q, -\infty >^J < q, -\infty | E_0 >$$

This amplitude would of course be unity in the absence of the external driving force $J$, by energy conservation. We write the *point to point* transition amplitude in the above limit in the presence of the driving force as

$$< Q', \infty | Q, -\infty >^J \tag{5.4.5}$$

The evaluation of the above matrix element requires some care. The staring point for its evaluation is Eq.5.4.3 for $< Q', T' | Q, T >^J$ for real finite times. We must then analytically continue off the real axis, and take the limit of large times to obtain the desired amplitude. With these provisos we write,

$$< Q', \infty | Q, -\infty >^J = N \int \mathcal{D}q(t) e^{i \int_{-\infty}^{\infty} dt L(q,\dot{q}) + Jq} \tag{5.4.6}$$

where in the above expression we should write $t \to (1 - i\epsilon)t$. Providing we stay off the real time axis we see a nice relationship between the point to point amplitude and the ground state to ground state amplitude $< 0, \infty | 0, -\infty >^J$,

$$< 0, \infty | 0, -\infty >^J = \frac{< Q', \infty | Q, -\infty >^J}{< Q' | E_0 >< E_0 | Q > e^{-iE_0(T'-T)} e^{-E_0(T'-T)\epsilon}} \tag{5.4.7}$$

The denominator is just the product of two wavefunctions; it is a constant independent of $J$. It is conventional to define the RHS of the above expressiion as the functional $Z[J]$,

$$Z[J] \equiv \frac{< Q', \infty | Q, -\infty >^J}{< Q' | E_0 >< E_0 | Q > e^{-iE_0(T'-T)} e^{-E_0(T'-T)\epsilon}} \tag{5.4.8}$$

Then we have simply

$$Z[J] =< 0, \infty | 0, -\infty >^J \tag{5.4.9}$$

where

$$Z[J] = \frac{N \int \mathcal{D}q(t) e^{i \int_{-\infty}^{\infty} dt L(q,\dot{q}) + Jq}}{< Q' | E_0 >< E_0 | Q > e^{-iE_0(T'-T)} e^{-E_0(T'-T)\epsilon}} \tag{5.4.10}$$

Up to a constant, $Z[J]$ is just the point to point transition amplitude, in the presence of $J$ in the limit of large time differences, and evaluated off the real time axis. In field theory $Z[J]$ is known as the *generating functional*.

## 5.5 Expectation Values of Heisenberg Position Operators

Let us again return to real time but instead of considering the amplitude $< q_f, t_f | q_i, t_i >$ between the Heisenberg position states $< q_f, t_f |$ and $| q_i, t_i >$ we consider the matrix element

$$< q_f, t_f | q(t_{n_1}) | q_i, t_i > \tag{5.5.1}$$

where $q(t_{n_1})$ is a Heisenberg position operator and $t_f > t_{n_1} > t_i$.

Let us now proceed as in Eq.5.2.9 and choose $t_{n_1}$ to be one of the times $t_1 \cdots t_n$. Then the matrix element in Eq.5.5.1 becomes

$$< q_f, t_f | q(t_{n_1}) | q_i, t_i >= \int \ldots \int dq_1 \ldots dq_n < q_f, t_f | q_n, t_n >< q_n, t_n | q_{n-1}, t_{n-1} >$$

$$\ldots < q_{n_1}, t_{n_1} | q(t_{n_1}) | q_{n_1-1}, t_{n_1-1} > \ldots | q_1, t_1 >< q_1, t_1 | q_i, t_i >$$

We can now write

$$< q_{n_1}, t_{n_1} | q(t_{n_1}) | q_{n_1-1}, t_{n_1-1} >= q(t_{n_1}) < q_{n_1}, t_{n_1} | q_{n_1-1}, t_{n_1-1} >$$

where $q(t_{n_1})$ is henceforth a number not an operator. The rest of the argument follows that below Eq.5.2.9, and we obtain

$$< q_f, t_f | q(t_1) | q_i, t_i >= \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} q(t_1) e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q} - H(p,q))\right)} \tag{5.5.2}$$

where as before the RHS involves no operators.

Next consider the matrix element

$$< q_f, t_f | q(t_{n_1}) q(t_{n_2}) | q_i, t_i > \tag{5.5.3}$$

where $q(t_{n_1}), q(t_{n_2})$ are Heisenberg position operators and $t_f > t_{n_1} > t_{n_2} > t_i$.

$$< q_f, t_f | q(t_{n_1}) q(t_{n_2}) | q_i, t_i >= \int \ldots \int dq_1 \ldots dq_n < q_f, t_f | q_n, t_n >$$

$$\ldots < q_{n_1}, t_{n_1} | q(t_{n_1}) | q_{n_1-1}, t_{n_1} - 1 > \ldots < q_{n_2}, t_{n_2} | q(t_{n_2}) | q_{n_2-1}, t_{n_2} - 1 > \ldots < q_1, t_1 | q_i, t_i >$$

We obtain

$$< q_f, t_f | q(t_1) q(t_2) | q_i, t_i >= \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} q(t_1) q(t_2) e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q} - H(p,q))\right)} \tag{5.5.4}$$

valid for $t_1 > t_2$. If however $t_2 > t_1$ then the RHS is equal to

$$< q_f, t_f | q(t_2) q(t_1) | q_i, t_i > .$$

In general then the RHS is equal to the time ordered product of the operators,

$$< q_f, t_f | T(q(t_1) q(t_2)) | q_i, t_i >= \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} q(t_1) q(t_2) e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q} - H(p,q))\right)} \tag{5.5.5}$$

where time ordering was defined earlier (earlier times are to the right).

The result generalises to,

$$< q_f, t_f|T(q(t_1)q(t_2)\cdots q(t_n))|q_i, t_i >= N\int \mathcal{D}q(t)q(t_1)q(t_2)\cdots q(t_n)e^{i\int_{t_i}^{t_f} dtL} \tag{5.5.6}$$

Now we recall from Eq.5.4.10 that

$$Z[J] \propto N\int \mathcal{D}q(t)e^{i\int_{-\infty}^{\infty} dtL(q,\dot{q})+Jq} \tag{5.5.7}$$

where the RHS is evaluated off the real time axis. The *functional derivative* (defined in the appendix) of $Z[J]$ with respect to $J$ is

$$\frac{\delta Z[J]}{\delta J(t_1)} \propto iN\int \mathcal{D}q(t)q(t_1)e^{i\int_{-\infty}^{\infty} dtL(q,\dot{q})+Jq} \tag{5.5.8}$$

and generally

$$\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)} \propto i^n N\int \mathcal{D}q(t)q(t_1)\cdots q(t_n)e^{i\int_{-\infty}^{\infty} dtL(q,\dot{q})+Jq} \tag{5.5.9}$$

If we set $J = 0$ we obtain,

$$\left[\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right]_{J=0} \propto i^n N\int \mathcal{D}q(t)q(t_1)\cdots q(t_n)e^{i\int_{-\infty}^{\infty} dtL(q,\dot{q})} \tag{5.5.10}$$

If we compare the RHS of Eq.5.5.10 to Eq.5.5.6 we see that there is a strong similarity. If we analytically continue time in Eq.5.5.6 in the same way as in the definition of $Z[J]$ and insert complete sets of energy eigenstates, then we see that as usual the ground state is isolated,

$$\begin{aligned} &< q_f, t_f|T(q(t_1)q(t_2)\cdots q(t_n))|q_i, t_i > \\ &= \sum_{n,m} < q_f|E_n >< E_n|e^{-iE_n t_f}T(q(t_1)q(t_2)\cdots q(t_n))|E_m >< E_m|q_i > e^{iE_n t_i} \\ &\approx < q_f|E_0 > e^{-iE_0 t_f} < E_0|q_i > e^{iE_0 t_i} < E_0|T(q(t_1)q(t_2)\cdots q(t_n))|E_0 > \end{aligned}$$

The wavefunction factor which arises is the same constant as in Eq.5.4.10 and they both cancel to leave,

$$i^n < 0|T(q(t_1)q(t_2)\cdots q(t_n))|0 > = \left[\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right]_{J=0} \tag{5.5.11}$$

where we have written $|E_0 >= |0 >$. Eq.5.5.11 relates the ground state expectation value of time ordered products of Heisenberg coordinate operators to some quantity on the RHS which can be calculated. We saw in Eq.3.6.12 that in canonical quantum field theory the S-matrix boils down to calculating the vacuum expectation value of a time ordered product of field operators in the Heisenberg picture, so we can see that an expression such as in Eq.5.5.11 when generalised to field theory could be useful.

# 5.6 Appendix

## 5.6.1 Gaussian Integration

**Integral $\alpha > 0$:**

$$I_0(\alpha) = \int_{-\infty}^{\infty} dx e^{-\alpha x^2} \tag{5.6.1}$$

This is called a Gaussian integral.

**Result:**

$$I_0(\alpha) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} \tag{5.6.2}$$

**Proof:**

$$I_0^2 = \int_{-\infty}^{\infty} dx e^{\alpha x^2} \int_{-\infty}^{\infty} dy e^{\alpha y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dy e^{-\alpha(x^2+y^2)}$$

In polar coordinates $r, \theta$ we can perform the integral.

**Integral ($\alpha > 0$):**

$$I_0(\alpha, \beta) = \int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} \tag{5.6.3}$$

**Result:**

$$I_0(\alpha, \beta) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{\frac{\beta^2}{4\alpha}} \tag{5.6.4}$$

**Proof:**

We complete the square

$$I_0(\alpha, \beta) = \int_{-\infty}^{\infty} dx e^{-\alpha(x - \frac{\beta}{2\alpha})^2 + \frac{\beta^2}{4\alpha^2}}$$

Change variables to $x - \frac{\beta}{2\alpha}$ and we use the previous result.

**Integral:**

$$I_2(\alpha) = \int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2} \tag{5.6.5}$$

**Result:**

$$I_2(\alpha) = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} \tag{5.6.6}$$

**Proof:**

Differentiate $I_0(\alpha)$ with respect to $\alpha$. Or differentiate $I_0(\alpha, \beta)$ with respect to $\beta$ then set $\beta = 0$. The second method shows that $I_1(\alpha) = 0$.

In general we can find $I_{2n}(\alpha)$ this way for any $n$. The odd powers of $x$ in the integrand $I_{2n+1}(\alpha)$ lead to a zero result (odd integrand).

Note that all these results are also valid for $\alpha$ and $\beta$ being complex numbers providing $Re(\alpha) > 0$. Next consider the result:

$$\int_0^\infty dr e^{-\alpha r} = \frac{1}{\alpha} \tag{5.6.7}$$

By applying the operator

$$\left(\frac{-d}{d\alpha}\right)^n$$

we obtain

$$\int_0^\infty dr r^n e^{-\alpha r} = \frac{n!}{\alpha^{n+1}} \tag{5.6.8}$$

Setting $\alpha = 1$ and replacing $n$ by $z - 1$ where $z$ is a complex number defines the gamma function,

$$\Gamma(z) = \int_0^\infty dr r^{z-1} e^{-r} \tag{5.6.9}$$

For real positive integer $z$ we have for example $\Gamma(z) = (z-1)!$.

Last but no no means least we come to what some people have described as the most important integral in the world! You will soon see why.

**Integral:**

$$Z_0 = \int_{-\infty}^\infty \prod_{i=1}^n dq_i e^{-\sum_{i,j=1}^n q_i K_{ij} q_j + \sum_{k=1}^n J_k q_k} \tag{5.6.10}$$

This is an $n$ dimensional integral and $K_{ij}$ is an $n \times n$ symmetric matrix, while $J_k$ is an $n$ component column vector. Amazingly the above integral can be done for all invertible $K_{ij}$!

**Result:**

$$Z_0 = \frac{\pi^{n/2}}{\sqrt{det(K)}} e^{+\frac{1}{4}\sum_{i,j=1}^n J_i (K^{-1})_{ij} J_j} \tag{5.6.11}$$

**Proof:**

The strategy is to reduce $Z_0$ to a product of $n$ do-able integrals of the type $I_0(\alpha, \beta)$. To do this we diagonalise $K$ as follows. In matrix notation:

$$q^T K q = q^T U^T U K U^T U q = q'^T K' q'$$

where $U^T = U^{-1}$ is an orthogonal matrix (the complex analogue being a unitary matrix) and $T$ of course means transpose, and

$$q' = Uq, \quad K' = UKU^T = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

The fact that $K$ is symmetric (Hermitian) means that we can diagonalise it by a single orthogonal (Unitary) matrix $U$. Now change integration variables to $q'_i$ with a trivial Jacobian of $detU = 1$,

$$Z_0 = \int \prod_{i=1}^n dq'_i e^{-\sum_{j=1}^n q'_i \lambda_j q'_j + \sum_{k,j=1}^n (J_k U_{jk}) q'_j} \tag{5.6.12}$$

The integral is now reduced to $n$ products of $I_0(\alpha, \beta)$ and the result involves in the denominator $\prod_i \sqrt{\lambda_i}$. Now clearly

$$det(K) = det(K') = \prod_i \lambda_i$$

so we get a factor of $\sqrt{det(K)}$ in the denominator. In addition we have a product of $e^{\frac{\beta_i^2}{4\alpha_i}}$ type factors where

$$\beta_i = (JU^T)_i$$

and

$$\alpha_i = \lambda_i = (K')_{ii}$$

Thus the exponent consists of the sum

$$\sum_i \frac{(JU^T)_i (JU^T)_i}{4\lambda_i}$$

It is straightforward to see that this sum is equal to

$$\frac{1}{4} J^T K^{-1} J = \frac{1}{4} J^T U^T K'^{-1} U J$$

so the result is proved.

### 5.6.2 Functionals

In mathematics a *function* maps a number into a different number, i.e. $f(x)$ maps a number $x$ into another number $y = f(x)$.

A *function of a function* again maps a number into a different number, i.e. $F(f(x))$ maps a number $x$ into another number $z = F(y)$ where $y = f(x)$.

A *functional* on the other hand maps a function into a number, i.e. $F[f(x)]$ maps the entire function $f(x)$ into a number $F$. e.g.

$$F[f(x)] = \int_0^1 dx f(x)$$

is a functional; for each choice of function $f(x)$ the result will be a different number.

The action $S$ is clearly a functional since it is defined as

$$S[x(t)] = \int_{t_i}^{t_f} dt L(x(t))$$

i.e. for each path $x(t)$ there is a number $S$.

Quantities like the path integral

$$< q_f, t_f | q_i, t_i >= N \int \mathcal{D}q(t) e^{iS[q(t)]} \tag{5.6.13}$$

are *functional integrals*: the integrand is a functional and the integration is performed over all functions $q(t)$ to yield a numerical result.

The other aspect of functional calculus is *functional differentiation*. This is the operation of taking the derivative of a functional $F[f(x)]$ with respect to the function $f(y)$, and is defined by

$$\frac{\delta F[f(x)]}{\delta f(y)} \equiv \epsilon \overset{lim}{\rightarrow} 0 \; \frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon} \tag{5.6.14}$$

## 5.7 Problems Set 5

1. Using Gaussian integration show explicitly that Eq.5.2.23 follows from Eq.5.2.20.

2. By performing the Gaussian integrals in the discretised case, derive the transition amplitude for a free non-relativistic particle of mass $m$:

$$< q_f, t_f | q_i, t_i >= \theta(t_f - t_i) \left( \frac{m}{i(t_f - t_i)} \right)^{\frac{1}{2}} exp \left[ \frac{im(x_f - x_i)^2}{2(t_f - t_i)} \right]$$

3. Show that for the harmonic oscillator we have in real time

$$< q_f | U(t_f, t_i) | q_i >= A(t) exp \left( -\frac{im\omega}{2 \sin \omega t} [(q_f^2 + q_i^2) \cos \omega t - 2 q_f q_i)] \right)$$

where $A(t)$ is some function of $t = t_f - t_i$.

4. Analytically continue the result in Q.3 to imaginary time and show

$$< q_f | U(\tau_f, \tau_i) | q_i >= A(\tau) exp \left( -\frac{m\omega}{2 \sinh \omega \tau} [(q_f^2 + q_i^2) \cosh \omega \tau - 2 q_f q_i)] \right)$$

As $\tau \to \infty$ show that the rhs becomes proportional to a product of ground state wavefunctions.

[This example demonstrates that analytic continuation is a useful device or trick to obtain physically meaningful results, even though the lhs $< q_f | U(\tau_f, \tau_i) | q_i >$ does not have a direct physical interpretation as the point to point transition amplitude, at least not in our universe where time is not imaginary.]

5. Given following functionals $F[f]$ find their first and second functional derivatives with respect to $f(y)$ and $f(z)$:

$$\frac{\delta F[f]}{\delta f(y)}, \quad \frac{\delta^2 F[f]}{\delta f(y) \delta f(z)}$$

(a)

$$F[f] = \int dx f(x)$$

(b)

$$F[f] = \int dx f(x) g(x)$$

(c)

$$F[f] = \int dx e^{f(x)}$$

(d)

$$F[f] = \int dx e^{f(x)+g(x)}$$

(e)

$$F[f] = \int\int dx_1 dx_2 g(x_1, x_2) f(x_1) f(x_2)$$

(f)

$$F[f] = \int\int\int dx_1 dx_2 dx_3 g(x_1, x_2) f(x_1) f(x_2) f(x_3)$$

6. Consider the following functional defined by the series:

$$F[f] = 1 + \int dx_1 f(x_1) g_1(x_1) + \frac{1}{2!}\int\int dx_1 dx_2 g_2(x_1, x_2) f(x_1) f(x_2)$$

$$+\frac{1}{3!}\int\int\int dx_1 dx_2 dx_3 g_3(x_1, x_2, x_3) f(x_1) f(x_2) f(x_3) + \cdots$$

Verify that to order $n = 3$,

$$g_n(x_1, \cdots, x_n) = \left[\frac{\delta^n F[f]}{\delta f(x_1) \cdots \delta f(x_n)}\right]_{f=0}.$$

[This result shows how a generating functional can be Taylor expanded as a power series.]

# Chapter 6

# Path Integral Quantisation of the Klein-Gordon Field

## 6.1 Introduction

In the previous chapter we were concerned with the path integral formulation of non-relativistic quantum mechanics of a single particle. We summarise the three key results proved there:

The ground state to ground state transition amplitude:

$$Z[J] =< 0, \infty|0, -\infty >^J \tag{6.1.1}$$

The path integral expression for $Z[J]$:

$$Z[J] \propto N \int \mathcal{D}q(t) e^{i\int_{-\infty}^{\infty} dt L(q,\dot{q})+Jq} \tag{6.1.2}$$

The time ordered vacuum expectation value of Heisenberg operators:

$$i^n < 0|T(\hat{q}(t_1)\hat{q}(t_2)\cdots\hat{q}(t_n))|0 > = \left[\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right]_{J=0} \tag{6.1.3}$$

All these results are only valid if we regard time $t$ as having been analytically continued off the real axis by a small $\epsilon$ rotation. This proceedure allowed the ground state to be isolated, and having used this property we cannot ever return to the real time axis. Thus in the evaluation of $Z[J]$ we should replace $t$ by $t(1 - i\epsilon)$ before doing the path integrations. The time ordered product of Heisenberg operators has also been analyticically continued.

In chapter 2 we discussed the free Klein-Gordon field and showed how it could be regarded as a continuum of harmonic oscillators, one at each spacetime point, and each coupled to their nearest neighbours in a Lorentz invariant way. The path integral results in chapter 4 may be taken to apply to each of these little oscillators. The first step in going to field theory is clearly to generalise

the path integral results from a single coordinate $q(t)$ to many coordinates $q_i(t)$, one for each oscillator. We then need to take the continuum limit,

$$q_i(t) \rightarrow \phi(x, y, z, t) \tag{6.1.4}$$

just as we did to arrive at canonical field theory. Thus the Heisenberg operators $\hat{q}(t)$ in Eq.6.1.3 become Heisenberg fields $\hat{\phi}(x, y, z, t)$ whose eigenstates are given by

$$\hat{\phi}(x, y, z, t)|\phi(x, y, z, t) >= \phi(x, y, z, t)|\phi(x, y, z, t) > \tag{6.1.5}$$

Recall the point to point amplitude

$$< q_f, t_f|q_i, t_i >= \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q}-H(p,q))\right)} \tag{6.1.6}$$

In field theory this applies separately to each of the little field oscillators

$$< \phi_f(x, y, z, t_f)|\phi_i(x, y, z, t_i) >= \int \frac{\mathcal{D}\phi\mathcal{D}\pi}{2\pi} e^{i\left(\int_{t_i}^{t_f} dt \int d^3\mathbf{x}(\pi\dot{\phi}-\mathcal{H}(\pi,\phi))\right)} \tag{6.1.7}$$

The RHS involves no operators, and is a functional integral over all classical field functions $\phi(x, y, z, t)$ and $\pi(x, y, z, t)$ satisfying some field boundary conditions at $t_i$ and $t_f$.

In field theory the generalisation of the driving force $J(t)$ is to add a driving force to each of the individual field oscillators, so that

$$J(t) \rightarrow J(x, y, z, t) \tag{6.1.8}$$

where $J(x, y, z, t)$ is called a *source* rather than driving term; the effect is the same however since it is an external "hand" which pushes each of the little oscillators comprising the field. Thus the field transition amplitude in the presence of a source is,

$$< \phi_f(x, y, z, t_f)|\phi_i(x, y, z, t_i) >^J= \int \frac{\mathcal{D}\phi\mathcal{D}\pi}{2\pi} e^{i\left(\int_{t_i}^{t_f} dt \int d^3\mathbf{x}(\pi\dot{\phi}-\mathcal{H}(\pi,\phi)+J\phi)\right)} \tag{6.1.9}$$

The field theory generalisation of the transition amplitude from the ground state at time $-\infty$ to the ground state at time $+\infty$ in the presence of driving term, is then the corresponding *vacuum to vacuum* transition amplitude in the presence of a *source*. The *vacuum* is simply the ground state of the field. The three key results reproduced at the start of this section then become generalised to field theory an the obvious way.

The vacuum to vacuum transition amplitude in field theory is defined to be the *generating functional*:

$$Z[J] \equiv< 0, \infty|0, -\infty >^J \tag{6.1.10}$$

The path integral expression for the generating functional $Z[J]$:

$$Z[J] \propto \int \mathcal{D}\phi e^{i\left(\int_{-\infty}^{\infty} dt \int d^3\mathbf{x}(\mathcal{L}+J\phi)\right)} \tag{6.1.11}$$

The time ordered vacuum expectation value of Heisenberg field operators:

$$i^n < 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n))|0> = \left[\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J(x)=0} \tag{6.1.12}$$

where $|0>$ is the time independent vacuum, and we have have allowed the Heisenberg field operators to correspond to the coordinates of different field oscillators at different times, i.e. $x_i = (t_i, \mathbf{x}_i)$. We emphasise again that the time ordered product of Heisenberg operators is not equal to the time ordered product of operators considered previously in the interaction picture. However there will be a relationship between the quantity in Eq.6.1.12 and S-matrix elements as we shall see.

The vacuum expectation value of a time ordered product of $n$ Heisenberg field operators is of course the *n particle Green's function* defined in Eq.3.6.2,

$$\mathcal{G}(x_1, x_2, \cdots, x_n) \equiv < 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n))|0> \tag{6.1.13}$$

Thus

$$i^n\mathcal{G}(x_1, x_2, \cdots, x_n) = \left[\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J(x)=0} \tag{6.1.14}$$

According to the LSZ reduction formula in Eq.3.6.12 the $n$ particle Green's function is related to the S-matrix element involving a total of $n$ particles in initial plus final state. However we shall first consider free field theory and later consider the interacting case.

## 6.2 The Feynman Propagator (again)

Consider the free field Lagrangian density of the real KG field, from Eq.2.3.2

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 \tag{6.2.1}$$

The path integral expression for the generating functional is:

$$Z_0[J] \propto \int \mathcal{D}\phi \exp\left(i\int d^4x \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 + J\phi\right]\right) \tag{6.2.2}$$

We use the identity

$$\int(\partial_\mu\phi)(\partial^\mu\phi)d^4x = \int \partial_\mu(\phi\partial^\mu\phi)d^4x - \int \phi\partial^2\phi d^4x$$

and convert the first term on the rhs to a surface integral using the four dimensional version of Gauss's divergence theorem. The surface term vanishes if $\phi \to 0$ at infinity, leaving

$$\int (\partial_\mu \phi)(\partial^\mu \phi) d^4x = -\int \phi \partial^2 \phi d^4x$$

and hence

$$Z_0[J] \propto \int \mathcal{D}\phi \exp\left(-i \int d^4x \left[\frac{1}{2}\phi(\partial^2 + m^2)\phi - J\phi\right]\right) \tag{6.2.3}$$

Now we wish to evaluate the functional integral in Eq.6.2.3 explicitly, remembering that we must do so off the real time axis because it is only there that the results involving $Z[J]$ are valid. To begin with we recall the most useful integral in the world from the Appendix of the last chapter, in Eqs.5.6.10 and 5.6.11,

$$Z_0' = \int_{-\infty}^{\infty} \prod_{i=1}^{n} dq_i e^{-\frac{1}{2}\sum_{i,j=1}^{n} q_i K_{ij} q_j + \sum_{k=1}^{n} J_k q_k}$$

$$Z_0' = \frac{(2\pi)^{n/2}}{\sqrt{det(K)}} e^{+\frac{1}{2}\sum_{i,j=1}^{n} J_i (K^{-1})_{ij} J_j}$$

which is an $n$ dimensional integral and $K_{ij}$ is an $n \times n$ symmetric matrix, while $J_k$ is an $n$ component column vector. The field theory limit of this integral is

$$Z_0'[J] = \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int d^4x d^4y \left[\phi(x) K(x,y) \phi(y)\right] + \int d^4x J(x)\phi(x)\right) \tag{6.2.4}$$

$$Z_0'[J] = \frac{1}{\sqrt{det(K)}} \exp\left(+\frac{1}{2}\int d^4x d^4y \left[J(x) K^{-1}(x,y) J(y)\right]\right) \tag{6.2.5}$$

The reason why the factors of $2\pi$ have disappeared in the result is related to the definition of $det(K)$. Since this factor multiplying the exponential is a constant as far as $J$ is concerned, and Eq.6.2.2 is only a proportionality we shall not worry about it. If you wish to worry about it see Bailin and Love p.1.

Now we shall write the exponent of Eq.6.2.3 in a way so that it more closely resembles Eq.6.2.4,

$$\left(-\frac{1}{2}\int d^4x d^4y \left[\phi(x) i\delta^4(x-y)(\partial^2 + m^2)\phi(y)\right] + \int d^4x (iJ(x))\phi(x)\right)$$

We can now use the result in Eq.6.2.5 with the transcriptions:

$$K(x,y) \to i\delta^4(x-y)(\partial^2 + m^2)$$

$$J(x) \to iJ(x)$$

Thus we find,

$$Z_0[J] \propto \exp\left(\frac{i}{2}\int d^4x d^4y J(x) \left[\delta^4(x-y)(\partial^2 + m^2)\right]^{-1} J(y)\right) \tag{6.2.6}$$

If we define as

$$\Delta_F(x,y) \equiv -\left[\delta^4(x-y)(\partial^2+m^2)\right]^{-1} \tag{6.2.7}$$

then the final result is:

$$Z_0[J] \propto \exp\left(\frac{-i}{2}\int d^4x d^4y J(x)\Delta_F(x,y)J(y)\right) \tag{6.2.8}$$

What is this inverse operator? Clearly we have

$$\int d^4y [\Delta_F(x,y)]^{-1}\Delta_F(y,z) = \delta(x-z)$$

Hence from its definition we have

$$\int d^4y \left[\delta^4(x-y)(\partial^2+m^2)\right]\Delta_F(y,z) = -\delta(x-z)$$

Integrating over $y$, we find the equation which must be satisfied by $\Delta_F(x,z)$,

$$\left[(\partial^2+m^2)\right]\Delta_F(x,z) = -\delta(x-z) \tag{6.2.9}$$

The solution to this equation is formally

$$\Delta_F(x,z) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}e^{-ik.(x-z)}$$

as may be verified by substitution. However this result is ambiguous: there are poles at $k_0^2 = \mathbf{k}^2 + m^2$.

In order to resolve this ambiguity we only need recall that in all of the results above time must be analytically continued off the real time axis, corresponding to a rotation of $\epsilon$ in the complex time plane. Once we remember this fact then we must replace $t$ by $(1-i\epsilon)t$ everywhere and in particular Eq.6.2.6 and Eq.6.2.7 are strictly only valid if analytically continued off the real time axis. This implies that the defining equation for the inverse operator should have been strictly

$$\left[((1+2i\epsilon)\partial_0^2 - \partial_i^2 + m^2)\right]\Delta_F(x,z) = -\delta(x-z) \tag{6.2.10}$$

and the solution should have been

$$\Delta_F(x,z) = \int \frac{dk_0 d^3\mathbf{k}}{(2\pi)^4}\frac{1}{k_0^2-\mathbf{k}^2-m^2}e^{-i(1-i\epsilon)k_0.(x_0-z_0)+i\mathbf{k}.(\mathbf{x}-\mathbf{z})}$$

After a change of variables to $k_0' = (1-i\epsilon)k_0$, this looks like, dropping the primes,

$$\Delta_F(x,z) = \int \frac{dk_0 d^3\mathbf{k}}{(2\pi)^4}\frac{1}{(1+2i\epsilon)k_0^2-\mathbf{k}^2-m^2}e^{-ik_0.(x_0-z_0)+i\mathbf{k}.(\mathbf{x}-\mathbf{z})}$$

which shows that the previous poles are now avoided. What has happened is that the rotation of the time axis in the complex time plane has led to a rotation

of the energy axis in the complex energy plane corresponding to the change of variables above. From the point of view of the rotated energy variable the poles are shifted off the energy axis and are now at:

$$k_0 = \pm E(1 - i\epsilon) = \pm(E - i\epsilon)$$

where $E = |\sqrt{\mathbf{k}^2 + m^2}|$ and we have exploited the fact that we are going to take $\epsilon$ to zero at the end of the calculation. Thus we could equally well write,

$$\Delta_F(x, z) = \int \frac{dk_0 d^3\mathbf{k}}{(2\pi)^4} \frac{1}{k_0^2 - \mathbf{k}^2 - m^2 + i\epsilon} e^{-ik_0.(x_0 - z_0) + i\mathbf{k}.(\mathbf{x}-\mathbf{z})}$$

or simply

$$\Delta_F(x - z) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik.(x-z)} \tag{6.2.11}$$

where in the last line we have written $\Delta_F(x-z)$ to emphasise that the Feynman propagator only depends on the difference between the two points. The factor $1/(k^2 - m^2 + i\epsilon)$ is just the Fourier transform of $\Delta_F(x, z)$ and is written

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon} \tag{6.2.12}$$

Now we see that $\Delta_F(x, z)$ in Eq.6.2.11 is identical to the definition of the Feynman propagator in Eq.3.5.16. The rotation of the time axis in the complex plane through a small angle $\epsilon$ which was always implicit in the definition of $Z[J]$ (otherwise none of the results involving $Z[J]$ would have been true) has led to Eq.6.2.8 in which $Z[J]$ is expressed in terms of the Feynman propagator $\Delta_F(x, z)$. The Feynman propagator $\Delta_F(x, z)$ is thus seen to be the inverse operator shown in Eq.6.2.7 where this operator appears in the original expression for $Z[J]$ in Eq.6.2.3. The strategy to find the propagator is thus to express the exponent of the functional integral in the form: $\phi(x)$.OPERATOR.$\phi(y)$ then take the inverse of the OPERATOR.

The rotation of the real time axis in the complex plane is seen to be equivalent to a rotation of the real energy axis in the complex energy plane, and provides a way of avoiding the poles which is equivalent to adding a small quantity $\epsilon$ to the denominator of the Feynman propagator. The effect of this is that after the energy integral is performed two $\theta$ functions in time are generated, in such a way that propagation is allowed forwards in time both for $x_0 > y_0$ and $y_0 > x_0$. The practical effect of this is that when we draw our Feynman diagrams in position space we do not have to worry about the time ordering of the propagator since both time ordering possibilities are covered.

## 6.3 Green's Functions in Free Field Theory

In the previous section we managed to perform the functional integrations in the generating functional for the free field theory case, and ended up with the

result in Eq.6.2.8,

$$Z_0[J] \propto \exp\left(\frac{-i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\right) \tag{6.3.1}$$

where $\Delta_F(x-y)$ was shown to be our old friend the Feynman propagator. If it weren't for the awkward constant of proportionality we would now be in a position to calculate the $n$ particle Green's functions for free field theory from Eq.6.1.14

$$i^n\mathcal{G}(x_1,x_2,\cdots,x_n) = \left[\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J(x)=0} \tag{6.3.2}$$

where we set $Z[J] = Z_0[J]$ for free field theory. Recall the definition of the Green's functions from Eq.6.1.13

$$\mathcal{G}(x_1,x_2,\cdots,x_n) \equiv <0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n))|0> \tag{6.3.3}$$

where $\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots$ are Heisenberg field operators.

In order to find the constant of proportionality above, we recall that our original defintion of $Z[J]$ in Eq.6.1.10 shows that it is normalised to

$$Z_0[J=0] = 1 \tag{6.3.4}$$

since in the absence of a source the vacuum will remain the vacuum with unit amplitude. Since the constant of proportionality is independent of $J$ (otherwise we would not call it a constant) it is clear that it must always arrange itself to be equal to unity in order to satisfy Eq.6.3.4. Thus Eq.6.3.1 become simply

$$Z_0[J] = \exp\left(\frac{-i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\right) \tag{6.3.5}$$

Now that $Z_0[J]$ is known exactly there is now no obstacle to us using it to crank out the Green's functions by functional differentiation. This of course is why we call it the generating functional: it may be used to generate the Green's functions. Thus we find the two, three and four point Green's functions to be:

$$\mathcal{G}_0(x_1,x_2) = i\Delta_F(x_1-x_2) \tag{6.3.6}$$

$$\mathcal{G}_0(x_1,x_2,x_3) = 0 \tag{6.3.7}$$

$$\begin{aligned}\mathcal{G}_0(x_1,x_2,x_3,x_4) &= i^2[\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)\\ &+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)]\end{aligned} \tag{6.3.8}$$

In general the odd number of point Green's functions are zero and the even number of point Green's functions involve pairs of Feynman propagators. The astute will realise that the result for the four point Green's function in Eq.**??** is the same as the result of using Wick's theorem for the time ordered product of four field operators in Eq.3.5.9. This is to be expected since the Green's functions are *defined* as the VEV of a time ordered product of field operators,

and there is no distiction between the interaction picture and the Heisenberg picture for free field theory. Note that we have obtained the *same* Green's functions using the path integral formalism as using the canonical formalism. From the point of view of physics only the Green's functions are really important, and it does not matter whether we have calculated the Green's functions using canonical methods or using functional methods since we have arrived at the same results.

We have already introduced Feynman diagrams as a way of representing the Feynman propagator:

$$i\Delta_F(x_1 - x_2) = \quad ----------$$

where the line is drawn between the two spactime points $x_1$ and $x_2$. This notation also serves for the two point Green's function since it is just equal to $i\Delta_F(x_1 - x_2)$. Thus the Feynman propagator is also a Green's function. Incidentally this justifies the use of the name "Green's function" to objects of this kind since the Feynman propagator is the solution to the differential equation 6.2.10 which is what classical applied mathematicians would recognise as a Green's function (apart from the appearance of $\epsilon$ which would probably cause them some confusion!)

We can similarly write the four point Green's functions using Feynman diagram between four spacetime points $x_1, x_2, x_3, x_4$:

It will be noticed that the four spacetime points are not all simultaneously connected together by lines. In fact only pairs of points are connected by lines in any particular diagram. Diagrams such as this are called *disconnected* diagrams and the corresponding Green's functions are called *disconnected* Green's functions. The LSZ reduction formula relates the Green's functions to S-matrix elements and we know that the latter always involve connected Feynman diagrams. Thus only the connected Green's functions are the ones of physical interest. It turns out that there is a different generating functional which generates only connected Green's functions which is denoted by $W[J]$. We shall take the definition of $W[J]$ to be:

$$Z[J] \equiv e^{iW[J]} \tag{6.3.9}$$

and then demonstrate that this definition leads to *connected* Green's functions $G(x_1, x_2, \cdots, x_n)$, given by

$$i^n G(x_1, x_2, \cdots, x_n) = \left[ i \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right]_{J(x)=0} \tag{6.3.10}$$

Note the extra factor of $i$ in the above result as compared to Eq.6.3.2.

In the case of free field theory it is immediately clear that the definition of $W_0[J]$ means that it can be identified with the exponent of $Z_0[J]$ in Eq.6.3.5:

$$iW_0[J] = \left( \frac{-i}{2} \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \right) \tag{6.3.11}$$

The only non-zero connected Green's function is therefore

$$G_0(x_1, x_2) = \mathcal{G}_0(x_1, x_2) = i\Delta_F(x_1 - x_2) \tag{6.3.12}$$

Thus $W[J]$ has generated the only connected Green's function in the free theory: the two point function. This vindicates the definition of $W_0[J]$ in the case of the free theory at least.

## 6.4 Green's Functions for $\lambda\phi^4$ Theory

Consider the interacting field Lagrangian density of the real KG field,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{6.4.1}$$

The path integral expression for the generating functional is:

$$Z[J] \propto \int \mathcal{D}\phi \exp\left( i \int d^4x \left[ \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi \right] \right) \tag{6.4.2}$$

This may be written as

$$Z[J] \propto \int \mathcal{D}\phi \exp\left( -i \int d^4x \left[ \frac{1}{2}\phi(\partial^2 + m^2)\phi + \frac{\lambda}{4!}\phi^4 - J\phi \right] \right) \tag{6.4.3}$$

but, unlike the free field case, the functional integral cannot be performed. This comes as no surprise, since we were unable to solve the field equations in the interacting theory either. Our strategy now as then is to develop a perturbation expansion in the parameter $\lambda$.

In the case $\lambda = 0$ the functional integrals can be performed and the result was given earlier:

$$Z_0[J] = \exp\left( \frac{-i}{2} \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \right) \tag{6.4.4}$$

where the Feynman propagator is:

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik.(x-y)} \tag{6.4.5}$$

For non-zero $\lambda$ the generating functional is given by (see Appendix for proof):

$$Z[J] = exp\left[ i \int d^4x \mathcal{L}_I \left( -i \frac{\delta}{\delta J(x)} \right) \right] Z_0[J] \tag{6.4.6}$$

where we have written

$$\mathcal{L}_I(\phi) = -\frac{\lambda}{4!}\phi^4 \tag{6.4.7}$$

and therefore

$$\mathcal{L}_I \left( -i \frac{\delta}{\delta J(x)} \right) = -\frac{\lambda}{4!} \left( -i \frac{\delta}{\delta J(x)} \right)^4 \tag{6.4.8}$$

The perturbation series comes from expanding the exponential as a power series,

$$exp\left[ i \int d^4x \mathcal{L}_I \left( -i \frac{\delta}{\delta J(x)} \right) \right] = 1 + i \int d^4x \mathcal{L}_I \left( -i \frac{\delta}{\delta J(x)} \right)$$
$$+ \frac{i^2}{2!} \int d^4x d^4y \mathcal{L}_I \left( -i \frac{\delta}{\delta J(x)} \right) \mathcal{L}_I \left( -i \frac{\delta}{\delta J(y)} \right) + \cdots$$

Thus the lowest non-trivial approximation for the interaction in Eq.6.4.7 is

$$exp\left[ i \int d^4x \mathcal{L}_I \left( -i \frac{\delta}{\delta J(x)} \right) \right] \approx 1 + i \int d^4x \frac{-\lambda}{4!} \left( -i \frac{\delta}{\delta J(x)} \right)^4$$

To this approximation we have:

$$Z[J] \approx Z_0[J] - \frac{i\lambda}{4!} \int d^4x \frac{\delta^4 Z_0}{\delta J(x)^4} \tag{6.4.9}$$

We can easily calculate

$$\frac{\delta Z_0}{\delta J(x)} = -\int d^4y \Delta_F(x-y) J(y) Z_0[J]$$

and repeated differentiation gives

$$Z[J] \approx [1 - \frac{i\lambda}{4!} \int d^4x 3(i\Delta_F(0))^2$$

$$+ 6i\Delta_F(0) \frac{i\lambda}{4!} \int d^4x \int d^4y_1 d^4y_2 i\Delta_F(x-y_1) i\Delta_F(x-y_2) J(y_1) J(y_2)$$

$$- \frac{i\lambda}{4!} \int d^4x \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 i\Delta_F(x-y_1) i\Delta_F(x-y_2) i\Delta_F(x-y_3) i\Delta_F(x-y_4)$$
$$\times J(y_1) J(y_2) J(y_3) J(y_4)] Z_0[J] \tag{6.4.10}$$

The Green's functions are as usual given by

$$i^n\mathcal{G}(x_1,x_2,\cdots,x_n)=\left[\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J(x)=0} \tag{6.4.11}$$

In the free theory we found

$$\mathcal{G}_0(x_1,x_2)=i\Delta_F(x_1-x_2) \tag{6.4.12}$$

$$\mathcal{G}_0(x_1,x_2,x_3)=0 \tag{6.4.13}$$

$$\mathcal{G}_0(x_1,x_2,x_3,x_4)=i^2[\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)$$
$$+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)] \tag{6.4.14}$$

In the interacting theory the Green's functions will differ from these. For example the zero point Green's function which in the free theory is just

$$Z_0[J=0]=1 \tag{6.4.15}$$

in the interscting theory to lowest order in $\lambda$ is

$$Z[J=0]=1-\frac{i}{8}\lambda\int d^4x i\Delta_F(x-x)i\Delta_F(x-x) \tag{6.4.16}$$

Note the normalisation of the full generating functional is still based on the free normalisation at this stage. Of course the "correct" normalisation is $Z[J=0]=1$ corresponding to a properly normalised vacuum. But we shall for the moment stick to free field normalisation.

The two point Green's function to lowest order in $\lambda$ is

$$\mathcal{G}(x_1,x_2)=\mathcal{G}_0(x_1,x_2)-\frac{i}{2}\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_2) \tag{6.4.17}$$

The four point Green's function to lowest order in $\lambda$ is

$$\mathcal{G}(x_1,x_2,x_3,x_4)=\mathcal{G}_0(x_1,x_2,x_3,x_4)$$
$$-\frac{i}{2}\mathcal{G}_0(x_1,x_2)\lambda\int d^4x i\Delta_F(x_3-x)i\Delta_F(x-x)i\Delta_F(x-x_4)$$
$$-\frac{i}{2}\mathcal{G}_0(x_3,x_4)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_2)$$
$$-\frac{i}{2}\mathcal{G}_0(x_1,x_3)\lambda\int d^4x i\Delta_F(x_2-x)i\Delta_F(x-x)i\Delta_F(x-x_4)$$
$$-\frac{i}{2}\mathcal{G}_0(x_2,x_4)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_3)$$
$$-\frac{i}{2}\mathcal{G}_0(x_1,x_4)\lambda\int d^4x i\Delta_F(x_2-x)i\Delta_F(x-x)i\Delta_F(x-x_3)$$
$$-\frac{i}{2}\mathcal{G}_0(x_2,x_3)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_4)$$

$$-i\lambda \int d^4x i\Delta_F(x_1 - x) i\Delta_F(x_2 - x) i\Delta_F(x_3 - x) i\Delta_F(x_4 - x) \qquad (6.4.18)$$

Recall the Feynman rule for the propagator:

$$i\Delta_F(x_1 - x_2) = \quad ----------$$

We now supplement this with a new Feynman rule for the vertex of four lines which is equal to $-i\lambda$:

We can then draw Feynman diagrams to represent the Green's functions of the interacting theory to lowest order in perturbation theory. The two point Green's function is thus:

where the first term is the free-propagator, and the second term uses the new Feynman rule with an extra factor of a half, and the prescription that we must integrate over the coordinates $x$ of any internal vertex.

The four point Green's function is thus:

(see Bailin and Love p.63).

As in the free theory it is easier to work with the connected Green's functions which are generated by $W[J]$ defined in Eq.6.3.9 as

$$Z[J] \equiv e^{iW[J]} \tag{6.4.19}$$

The claim is that the *connected* Green's functions $G(x_1, x_2, \cdots, x_n)$, are given by

$$i^n G(x_1, x_2, \cdots, x_n) = \left[ i \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right]_{J(x)=0} \tag{6.4.20}$$

where we substantiated this claim in the free field theory. Let us now substantiate this claim in the interacting theory, at least to lowest order in $\lambda$.

From the above definition we have

$$iW[J] = \ln Z[J] \tag{6.4.21}$$

Using the approximation $\ln(1+x) \approx x$ we have from Eq.6.4.10, to lowest order in $\lambda$,

$$\begin{aligned} iW[J] \approx \ln(Z_0[J]) - \frac{i\lambda}{4!} \int d^4x 3(i\Delta_F(0))^2 \\ +6\frac{i\lambda}{4!} \int d^4x \int d^4y_1 d^4y_2 i\Delta_F(y_1 - x) i\Delta_F(x - x) i\Delta_F(x - y_2) J(y_1) J(y_2) \\ -\frac{i\lambda}{4!} \int d^4x \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 i\Delta_F(y_1 - x) i\Delta_F(y_2 - x) i\Delta_F(y_3 - x) i\Delta_F(y_4 - x) \\ \times J(y_1) J(y_2) J(y_3) J(y_4) \end{aligned} \tag{6.4.22}$$

Now the first term on the rhs is just the free result obtained in Eq.6.3.11,

$$\ln(Z_0[J]) = iW_0[J] = \left( \frac{-i}{2} \int d^4x d^4y J(x) \Delta_F(x - y) J(y) \right) \tag{6.4.23}$$

The additional terms on the rhs are the order $\lambda$ corrections. The result in Eq.6.4.20 then leads to connected Green's functions which just correspond to the only the subset of diagrams in the Green's functions which are connected. Explicitly we find the connected two point Green's function to be just equal to the two point Green's function in Eq.6.4.17

$$G(x_1, x_2) = \mathcal{G}(x_1, x_2) = \mathcal{G}_0(x_1, x_2) - \frac{i}{2}\lambda \int d^4x i\Delta_F(x_1 - x) i\Delta_F(x - x) i\Delta_F(x - x_2) \tag{6.4.24}$$

The connected four point Green's function, on the other hand, is just equal to the single connected diagram of the four point Green's function, corresponding to the last term in Eq.6.4.18,

$$G(x_1, x_2, x_3, x_4) = -i\lambda \int d^4x i\Delta_F(x_1 - x) i\Delta_F(x_2 - x) i\Delta_F(x_3 - x) i\Delta_F(x_4 - x) \tag{6.4.25}$$

corresponding to just the last of the diagrams listed previously.

Here we note that the factor of 4! we put into the definition of the coupling in the interaction has been cancelled by the 4! ways in which the four external points $x_1, x_2, x_3, x_4$ are identified with the four sources at points $J(y_1), J(y_2), J(y_3), J(y_4)$ in the expansion of the generating functional in Eq.6.4.10. If we had not had the foresight to do this then the Feynman rule would have had a factor of 4! instead which is more inconvenient.

Note that the connected four point Green's function is zero in the free field theory.

## 6.5 The $2 \to 2$ Scattering Amplitude from LSZ

We first recall the LSZ reduction formula for the case of $2 \to 2$ scattering in Eq.3.6.11,

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= (\frac{i}{\sqrt{Z}})^2(\frac{-i}{\sqrt{Z}})^2 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{-i(p_1.x_1+p_2.x_2-q_1.y_1-q_2.y_2)}$$

$$\times(\partial^2_{x_1}+m^2)(\partial^2_{x_2}+m^2)(\partial^2_{y_1}+m^2)(\partial^2_{y_2}+m^2)\mathcal{G}(x_1,x_2,y_1,y_2) \qquad (6.5.1)$$

The four point Green's function to lowest order in $\lambda$ is given in Eq.6.4.18, repeated below,

$$\begin{aligned}
\mathcal{G}(x_1,x_2,y_1,y_2) &= i\Delta_F(x_1-x_2)i\Delta_F(y_1-y_2) \\
&+i\Delta_F(x_1-y_1)i\Delta_F(x_2-y_2)+i\Delta_F(x_1-y_2)i\Delta_F(x_2-y_1) \\
&-\frac{i}{2}i\Delta_F(x_1-x_2)\lambda\int d^4x i\Delta_F(y_1-x)i\Delta_F(x-x)i\Delta_F(x-y_2) \\
&-\frac{i}{2}i\Delta_F(y_1-y_2)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_2) \\
&-\frac{i}{2}i\Delta_F(x_1-y_1)\lambda\int d^4x i\Delta_F(x_2-x)i\Delta_F(x-x)i\Delta_F(x-y_2) \\
&-\frac{i}{2}i\Delta_F(x_2-y_2)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-y_1) \\
&-\frac{i}{2}i\Delta_F(x_1-y_2)\lambda\int d^4x i\Delta_F(x_2-x)i\Delta_F(x-x)i\Delta_F(x-y_1) \\
&-\frac{i}{2}i\Delta_F(x_2-y_1)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-y_2) \\
&-i\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x_2-x)i\Delta_F(y_1-x)i\Delta_F(y_2-x) \qquad (6.5.2)
\end{aligned}$$

We now insert these Green's functions into the LSZ formula and use the result in Eq.6.2.9

$$\left[(\partial^2+m^2)\right]\Delta_F(x,z) = -\delta(x-z) \qquad (6.5.3)$$

to evaluate the S-matrix element.

The first three terms in the Green's function arise from the free Green's function. Two applications of Eq.6.5.3 leave terms like,

$$(\partial_{x_2}^2 + m^2)(\partial_{y_2}^2 + m^2)\delta^4(x_1 - x_2)\delta^4(y_1 - y_2)$$

and then we use results like

$$e^{-ip_2.x_2}\partial_{x_2}^2\delta^4(x_1 - x_2) = \delta^4(x_1 - x_2)\partial_{x_2}^2 e^{-ip_2.x_2}$$

which leads to a terms like

$$(p_2^2 - m^2)(q_2^2 - m^2)$$

which vanish on the mass shell. The next six terms also vanish by a similar argument. The only non-zero contribution comes from the last term, which corresponds to the connected diagram. Let us evaluate this explicitly. We have:

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= (\frac{i}{\sqrt{Z}})^2(\frac{-i}{\sqrt{Z}})^2 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{-i(p_1.x_1 + p_2.x_2 - q_1.y_1 - q_2.y_2)}$$

$$\times(\partial_{x_1}^2 + m^2)(\partial_{x_2}^2 + m^2)(\partial_{y_1}^2 + m^2)(\partial_{y_2}^2 + m^2)$$

$$-i\lambda \int d^4x i\Delta_F(x_1 - x) i\Delta_F(x_2 - x) i\Delta_F(y_1 - x) i\Delta_F(y_2 - x)$$

We easily find

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= (-i\lambda)(2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2) \tag{6.5.4}$$

The S-matrix element in Eq.6.5.4 conforms to the general structure in Eq.3.4.24. Thus we can identify the "scattering amplitude" or "matrix element" in this case as

$$i\mathcal{M}_f i = -i\lambda$$

which is just equal to the Feynman rule for the vertex introduced earlier.

Note that so far Feynman diagrams have been introduced to represent the position space $n$ point Green's functions. Similar diagrams can clearly also be used to represent the momentum space scattering amplitude $i\mathcal{M}_f i$, with the following changes:

0. Only connected diagrams contribute.

1. The external legs are now labelled by their momentum which is on the mass shell.

2. The rule for the vertex is $-i\lambda$.

3. Four-momenta are conserved at each vertex.

4. Internal propagator lines are given by Eq.6.2.12 for $\Delta_F(k)$.

5. Integrate over each internal loop momentum $k$ with weight $d^4k/(2\pi)^4$.

In the present example the Feynman diagram consists of the single diagram involving the vertex, but with the legs labelled by momentum.

## 6.6 Appendix: Proof of Eq.6.4.6

Here we prove

$$Z[J] = exp\left[i\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right]Z_0[J]$$

Our starting point is

$$Z[J] = \int \mathcal{D}\phi \exp\left(i\int d^4x\left[\mathcal{L}_0 + \mathcal{L}_I(\phi) + J\phi\right]\right)$$

$$= \int \mathcal{D}\phi\left(\exp i\int d^4x\mathcal{L}_I(\phi)\right)\left(\exp i\int d^4y(\mathcal{L}_0 + J\phi)\right)$$

We expand the first exponential

$$\left(\exp i\int d^4x\mathcal{L}_I(\phi)\right) = 1 + i\int d^4x\mathcal{L}_I(\phi) + \frac{i^2}{2!}\int d^4xd^4y\mathcal{L}_I(\phi(x))\mathcal{L}_I(\phi(y)) + \cdots$$

Now since

$$i\phi(x)\exp i\int d^4z(\mathcal{L}_0 + J\phi) = \frac{\delta}{\delta J(x)}\exp i\int d^4z(\mathcal{L}_0 + J\phi)$$

it follows that

$$\left(\exp i\int d^4x\mathcal{L}_I(\phi)\right)\left(\exp i\int d^4z(\mathcal{L}_0 + J\phi)\right)$$

$$= \int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\left(\exp i\int d^4z(\mathcal{L}_0 + J\phi)\right)$$

Since the operator

$$\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)$$

is independent of $\phi(x)$ it can be taken outside the functional integral and we have

$$Z[J] = \left(\exp\left[i\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right]\right)Z_0[J]$$

where

$$Z_0[J] = \int \mathcal{D}\phi \exp\left(i\int d^4x\left[\mathcal{L}_0 + J\phi\right]\right)$$

which completes the proof.

## 6.7 Problems Set 6

1. In the case of the free theory calculate the four point Green's function in Eq.6.3.8

$$\mathcal{G}_0(x_1, x_2, x_3, x_4) = i^2[\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4)$$
$$+\Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)]$$

by applying functional differentiation four times to the generating functional.

2. Consider an interacting theory based on the KG Lagrangian for a real scalar field but with an interaction

$$\mathcal{L}_{int} = -\frac{\lambda}{3!}\phi^3$$

Calculate the three point Green's functions to order $\lambda$. Justify the use of the combinatorial factor 3! by identifying the Feynman rule for the vertex.

3. Draw the Feynman diagrams for the connected four particle Green's functions for the $\phi^3$ theory in Qu.2 above, to order $\lambda^2$. Using the Feynman rules, write down expressions for the connected four particle Green's functions to order $\lambda^2$. Using the LSZ formula calculate the $2 \to 2$ S-matrix element for $\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4)$ from the connected four particle Green's functions to order $\lambda^2$. Draw the Feynman diagrams which represent the scattering amplitude to this order, labelling all four-momenta.

# Chapter 7

# The Dirac Equation

## 7.1 Relativistic Wave Equations: Reprise

Recall the Schrodinger equation

$$H\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{7.1.1}$$

where $H$ is the Hamiltonian (i.e. the energy operator). In relativity the square of the Hamiltonian is given by

$$H^2 = \mathbf{P}^2 + m^2 \tag{7.1.2}$$

Naively the relativistic Schrodinger equation looks like

$$\sqrt{\mathbf{P}^2 + m^2}\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{7.1.3}$$

but this is difficult to interpret because of the square root. There are two ways forward:

(1) Work with $H^2$. By iterating the Schrodinger equation we have

$$H^2\phi(t) = -\frac{\partial^2\phi(t)}{\partial t^2} \tag{7.1.4}$$

which is known as the Klein-Gordon (KG) equation. In this case the wavefunction describes spinless bosons.

(2) Invent a new Hamiltonian $H_D$ which is linear in momentum, and whose square is equal to $H^2$ given above, $H_D^2 = \mathbf{P}^2 + m^2$. In this case we have

$$H_D\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{7.1.5}$$

which is known as the Dirac equation, with $H_D$ being the Dirac Hamiltonian. In this case the wavefunction describes spin one half fermions, as we shall see.

So far we have been concerned with spin zero particles described by the KG equation. Now it is time to discuss the spin 1/2 particles from which all the atoms in the universe are made.

## 7.2 The Dirac Equation

Dirac wanted an equation first order in time derivatives and Lorentz covariant, so it had to be first order in spatial derivatives too. His starting point was to assume a Hamiltonian of the form,

$$H_D = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \beta m \tag{7.2.1}$$

where $P_i$ are the three components of the momentum operator $\mathbf{P}$, and $\alpha_i$ and $\beta$ are some "unknown quantities", which as will be seen below cannot simply be commuting numbers. When the requirement that the $H_D^2 = \mathbf{P}^2 + m^2$ is imposed, this implies that $\alpha_i$ and $\beta$ must be interpreted as $4 \times 4$ matrices, as we shall discuss. The first step is to write the momentum operators explicitly in terms of their differential operators, using Eq.**??**, then the Dirac equation 7.1.5 becomes, using the Dirac Hamiltonian in Eq.7.2.1,

$$i\,\frac{\partial \psi}{\partial t} = (-i\,\alpha{\cdot}\nabla + \beta\mathbf{m})\psi \tag{7.2.2}$$

which is the position space Dirac equation. Remember that in field theory, the Dirac equation is the equation of motion for the field operator describing spin 1/2 fermions. In order for this equation to be Lorentz covariant, it will turn out that $\psi$ cannot be a scalar under Lorentz transformations. In fact this will be precisely how the equation turns out to describe spin 1/2 particles. We will return to this below.

If $\psi$ is to describe a free particle it is natural that it should satisfy the Klein-Gordon equation so that it has the correct energy-momentum relation. This requirement imposes relationships among the $\alpha$ and $\beta$. To see these, apply the operator on each side of equation (7.2.2) twice, i.e. iterate the equation,

$$-\frac{\partial^2 \psi}{\partial t^2} = [-\alpha^i \alpha^j \nabla^i \nabla^j - i\,(\beta\alpha^i + \alpha^i \beta) m \nabla^i + \beta^2 m^2]\psi$$

with an implicit sum over i and j from 1 to 3. The Klein-Gordon equation by comparison is

$$-\frac{\partial^2 \psi}{\partial t^2} = [-\nabla^i \nabla^i + m^2]\psi \tag{7.2.3}$$

If we do not assume that the $\alpha^i$ and $\beta$ commute then the KG will clearly be satisfied if

$$\begin{aligned} \alpha_i \alpha_j + \alpha_j \alpha_i &= 2\delta_{ij} \\ \beta\alpha_i + \alpha_i \beta &= 0 \\ \beta^2 &= 1 \end{aligned} \tag{7.2.4}$$

for $i, j = 1, 2, 3$. It is clear that the $\alpha_i$ and $\beta$ cannot be ordinary numbers, but it is natural to give them a realisation as matrices. In this case, $\psi$ must be a multi-component *spinor* on which these matrices act.

▷ **Exercise 7.2.1**
Prove that any matrices $\alpha$ and $\beta$ satisfying equation (7.2.4) are traceless with eigenvalues $\pm 1$. Hence argue that they must be even dimensional.

In two dimensions a natural set of matrices for the $\alpha$ would be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2.5}$$

However, there is no other independent $2 \times 2$ matrix with the right properties for $\beta$, so the smallest dimension for which the Dirac matrices can be realised is four. One choice is the *Dirac representation*

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2.6}$$

Note that each entry above denotes a two-by-two block and that the 1 denotes the $2 \times 2$ identity matrix.

There is a theorem due to Pauli which states that all sets of matrices obeying the relations in (7.2.4) are equivalent. Since the Hermitian conjugates $\alpha^\dagger$ and $\beta^\dagger$ clearly obey the relations, you can, by a change of basis if necessary, assume that $\alpha$ and $\beta$ are Hermitian. All the common choices of basis have this property. Furthermore, we would like $\alpha_i$ and $\beta$ to be Hermitian so that the Dirac Hamiltonian (7.4.1) is Hermitian.

▷ **Exercise 7.2.2**
Derive the continuity equation $\partial_\mu J^\mu = 0$ for the Dirac equation with

$$\rho = J^0 = \psi^\dagger(x)\psi(x), \qquad \mathbf{J} = \psi^\dagger(x)\alpha\psi(\mathbf{x}). \tag{7.2.7}$$

We will see in section 7.8 that $(\rho, \mathbf{J})$ does indeed transform as a four-vector.

## 7.3 Free Particle Solutions I: Interpretation

We look for plane wave solutions of the form

$$\psi = \begin{pmatrix} \chi(\mathbf{p}) \\ \phi(\mathbf{p}) \end{pmatrix} e^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \tag{7.3.1}$$

where $\phi(\mathbf{p})$ and $\chi(\mathbf{p})$ are two-component spinors which depend on momentum $\mathbf{p}$ but are independent of $\mathbf{x}$. Using the Dirac representation of the matrices, and inserting the trial solution into the Dirac equation gives the pair of simultaneous equations

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}, \tag{7.3.2}$$

There are two simple cases for which Eq.7.3.2 can readily be solved, namely

(1) $\mathbf{p} = 0$, $m \neq 0$ corresponding physically to an electron in its rest frame.

(2) $m = 0$, $\mathbf{p} \neq 0$ corresponding physically to a massless neutrino.

For case (1), an electron in its rest frame, the equations 7.3.2 decouple and become simply,

$$E\chi = m\chi, \qquad E\phi = -m\phi \tag{7.3.3}$$

so that in this case we see that $\chi$ corresponds to solutions with $E = m$, while $\phi$ corresponds to solutions with $E = -m$ : negative energy solutions!

These negative energy solutions persist for an electron with $\mathbf{p} \neq 0$ for which the solutions to Eq.7.3.2 are readily seen to be

$$\phi = \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}+\mathbf{m}}\,\chi, \qquad \chi = \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}-\mathbf{m}}\,\phi. \tag{7.3.4}$$

Thus the general positive energy solutions with $E = +|\sqrt{\mathbf{p}^2 + m^2}|$ are:

$$\psi(x) = \begin{pmatrix} \chi \\ \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}+\mathbf{m}}\chi \end{pmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}, \tag{7.3.5}$$

while the general negative energy solutions with $E = -|\sqrt{\mathbf{p}^2 + m^2}|$ are:

$$\psi(x) = \begin{pmatrix} \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}-\mathbf{m}}\phi \\ \phi \end{pmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}, \tag{7.3.6}$$

for arbitrary constant $\phi$ and $\chi$. Clearly when $\mathbf{p} = 0$ these solutions reduce to the positive and negative energy solutions discussed previously. Now, since $E^2 = \mathbf{p}^2 + m^2$ by construction, we find, just as we did for the Klein-Gordon equation (**??**), that there exist positive and negative energy solutions given by equations (7.3.5) and (7.3.6) respectively. Once again, the existence of negative energy solutions vitiates the interpretation of $\psi$ as a wavefunction.

Dirac interpreted the negative energy solutions by postulating the existence of a "sea" of negative energy states. The vacuum or ground state has all the negative energy states full. An additional electron must now occupy a positive energy state since the Pauli exclusion principle forbids it from falling into one of the filled negative energy states. By promoting one of these negative energy states to a positive energy one, by supplying energy, you create a pair: a positive energy electron and a hole in the negative energy sea corresponding to a positive energy positron. This was a radical new idea, and brought pair creation and antiparticles into physics. Positrons were discovered in cosmic rays by Carl Anderson in 1932.

The problem with Dirac's hole theory is that it doesn't work for bosons, such as particles governed by the Klein Gordon equation, for example. Such particles have no exclusion principle to stop them falling into the negative energy states, releasing their energy. We need a new interpretation and turn to Feynman for our answer.

According to Feynman and quantum field theory, we should interpret the emission (absorption) of a negative energy particle with momentum $p^\mu$ as the absorption (emission) of a positive energy antiparticle with momentum $-p^\mu$. So, in Figure 7.3.1, for example, an electron–positron pair is created at point $A$.

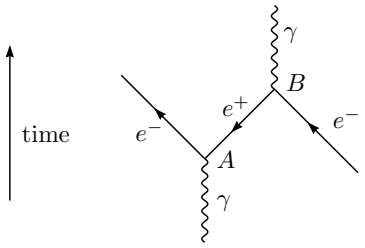



Figure 7.3.1 Feynman interpretation of a process in which a negative energy electron is absorbed. Time increases moving upwards.

The positron propagates to point $B$ where it is annihilated by another electron.

Thus Feynman tells us to keep both types of free particle solution. One is to be used for particles and the other for the accompanying antiparticles. Let's return to our spinor solutions and write them in a conventional form. Take the positive energy solution of equation (7.3.5) and write,

$$\sqrt{E+m}\begin{pmatrix}\chi_r\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\mathbf{E}+\mathbf{m}}\chi_r\end{pmatrix}e^{-ip\cdot x}\equiv u_p^r e^{-ip\cdot x}. \tag{7.3.7}$$

For the former negative energy solution of equation (7.3.6), change the sign of the energy, $E\to -E$, and the three-momentum, $\mathbf{p}\to -\mathbf{p}$, to obtain,

$$\sqrt{E+m}\begin{pmatrix}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\mathbf{E}+\mathbf{m}}\chi_r\\ \chi_r\end{pmatrix}e^{ip\cdot x}\equiv v_p^r e^{ip\cdot x}. \tag{7.3.8}$$

In these two solutions $E$ is now (and for the rest of the course) always positive and given by $E=(\mathbf{p}^2+m^2)^{1/2}$. The subscript $r$ takes the values $1,2$, with

$$\chi_1=\begin{pmatrix}1\\0\end{pmatrix},\qquad \chi_2=\begin{pmatrix}0\\1\end{pmatrix}. \tag{7.3.9}$$

For the simple case $\mathbf{p}=0$ we may interpret $\chi_1$ as the spin-up state and $\chi_2$ as the spin-down state. Thus for $\mathbf{p}=0$ the 4-component wavefunction has a very simple interpretation: the first two components describe electrons with spin-up and spin-down, while the second two components describe positrons with spin-up and spin-down. Thus we understand on physical grounds why the wavefunction had to have four components. The general case $\mathbf{p}\neq 0$ is slightly more involved and is considered in the next section.

At this point I would like to introduce another notation, and define

$$\omega_p\equiv\sqrt{\mathbf{p}^2+m^2}, \tag{7.3.10}$$

so that, $\omega_p$ is the energy (positive) of a particle or anti-particle with three-momentum $\mathbf{p}$ (I write the subscript $p$ instead of $\mathbf{p}$, but you should remember it really means the three-momentum). I will tend to use $E$ or $\omega_p$ interchangeably.

The $u$-spinor solutions will correspond to particles and the $v$-spinor solutions to antiparticles. The role of the two $\chi$'s will become clear in the following section, where it will be shown that the two choices of $r$ are spin labels. Note that each spinor solution depends on the three-momentum $\mathbf{p}$, so it is implicit that $p^0 = \omega_p$. In the expansion of the Dirac quantum field operator in terms of plane waves,

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3\, 2\omega_p} \sum_{r=1,2} \left[\hat{b}(p,r) u_p^r e^{-ip\cdot x} + \hat{d}^\dagger(p,r) v_p^r e^{ip\cdot x}\right] \tag{7.3.11}$$

the operator $\hat{b}$ annihilates a fermion of momentum $(\omega_p, \mathbf{p})$ and spin $r$, whilst $\hat{d}^\dagger$ creates an antifermion of momentum $(\omega_p, \mathbf{p})$ and spin $r$. The Hermitian conjugate Dirac field contains operators which do the opposite. This discussion should be clearer after your quantum field theory lectures.

The vacuum state $|0\rangle$ is defined by,

$$b(p,r)\,|0\rangle = d(p,r)\,|0\rangle = 0, \tag{7.3.12}$$

for every momentum $p = (\omega_p, \mathbf{p})$ and spin label $r$. This ensures the interpretation above: particles are created by the "daggered" operators and destroyed by the undaggered ones.

## 7.4 Free Particle Solutions II: Spin

Now it's time to justify the statements we have been making that the Dirac equation describes spin-1/2 particles. The Dirac Hamiltonian in momentum space is given in Eq.7.2.1 as

$$H_D = \alpha\cdot\mathbf{P} + \beta\mathbf{m} \tag{7.4.1}$$

and the orbital angular momentum operator is

$$\mathbf{L} = \mathbf{R} \times \mathbf{P}.$$

Normally you have to worry about operator ordering ambiguities when going from classical objects to quantum mechanical ones. For the components of $\mathbf{L}$ the problem does not arise — why not?

Evaluating the commutator of $\mathbf{L}$ with $H_D$,

$$\begin{aligned} [\mathbf{L}, H_D] &= [\mathbf{R}\times\mathbf{P}, \alpha\cdot\mathbf{P}] \\ &= [\mathbf{R}, \alpha\cdot\mathbf{P}]\times\mathbf{P} \\ &= i\alpha\times\mathbf{P}, \end{aligned} \tag{7.4.2}$$

we see that the orbital angular momentum is not conserved (otherwise the commutator would be zero). We'd like to find a *total* angular momentum $\mathbf{J}$ which *is* conserved, by adding an additional operator $\mathbf{S}$ to $\mathbf{L}$,

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \quad [\mathbf{J}, H_D] = 0 \tag{7.4.3}$$

To this end, consider the three matrices,

$$\mathbf{\Sigma} \equiv \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\alpha. \tag{7.4.4}$$

where the first equivalence is merely a definition of $\mathbf{\Sigma}$ and the last equality can readily be verified. The $\mathbf{\Sigma}/2$ have the correct commutation relations to represent angular momentum, since the Pauli matrices do, and their commutators with $\alpha$ and $\beta$ are,

$$[\mathbf{\Sigma}, \beta] = 0, \qquad [\Sigma_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k. \tag{7.4.5}$$

▷ **Exercise 7.4.1**
Verify the commutation relations in equation (7.4.5).

From the relations in (7.4.5) we find that

$$[\mathbf{\Sigma}, H_D] = -2i\alpha \times \mathbf{P}.$$

Comparing this with the commutator of $\mathbf{L}$ with $H_D$ in equation (7.4.2), you readily see that

$$\left[\mathbf{L} + \tfrac{1}{2}\mathbf{\Sigma}, H_D\right] = 0,$$

and we can identify

$$\mathbf{S} = \frac{1}{2}\mathbf{\Sigma}.$$

as the additional quantity which when added to $\mathbf{L}$ in Eq.7.4.3 yields a conserved total angular momentum $\mathbf{J}$. We interpret $\mathbf{S}$ as an angular momentum *intrinsic* to the particle. Now

$$\mathbf{S}^2 = \frac{1}{4}\begin{pmatrix} \sigma\cdot\sigma & 0 \\ 0 & \sigma\cdot\sigma \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and recalling that the eigenvalue of $\mathbf{J}^2$ for spin $j$ is $j(j+1)$, we conclude that $\mathbf{S}$ represents spin-1/2 and the solutions of the Dirac equation have spin-1/2 as promised.

We worked in the Dirac representation of the matrices for convenience, but the result is of course independent of the representation.

Now consider the $u$-spinor solutions $u_p^r$ of equation (7.3.7). Choose $\mathbf{p} = (0, 0, p_z)$ and write

$$u_\uparrow = u^1_{p_z} = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix}, \qquad u_\downarrow = u^2_{p_z} = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}. \tag{7.4.6}$$

It is easy to see that,

$$S_z u_\uparrow = \frac{1}{2}u_\uparrow, \qquad S_z u_\downarrow = -\frac{1}{2}u_\downarrow.$$

So, these two spinors represent spin up and spin down along the $z$-axis respectively. For the $v$-spinors, with the same choice for $\mathbf{p}$, write,

$$v_\downarrow = v^1_{p_z} = \begin{pmatrix} \sqrt{E-m} \\ 0 \\ \sqrt{E+m} \\ 0 \end{pmatrix}, \qquad v_\uparrow = v^2_{p_z} = \begin{pmatrix} 0 \\ -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} \end{pmatrix}, \tag{7.4.7}$$

where now,

$$S_z v_\downarrow = \frac{1}{2} v_\downarrow, \qquad S_z v_\uparrow = -\frac{1}{2} v_\uparrow.$$

This apparently perverse choice of up and down for the $v$'s is because, as you see in equation (7.3.11) for the quantum Dirac field, $u_\uparrow$ multiplies an annihilation operator which *destroys* a particle with momentum $p_z$ and spin up, whereas $v_\downarrow$ multiplies an operator which *creates* an antiparticle with momentum $p_z$ and spin up.

## 7.5 Normalisation, Gamma Matrices

We have included a normalisation factor $\sqrt{E+m}$ in our spinors. With this factor,

$$u^{r\dagger}_p u^s_p = v^{r\dagger}_p v^s_p = 2\omega_p \delta^{rs}. \tag{7.5.1}$$

This corresponds to the standard relativistic normalisation of $2\omega_p$ particles per unit volume. It also means that $u^\dagger u$ transforms like the time component of a 4-vector under Lorentz transformations as we will see in section 7.8.

▷ **Exercise 7.5.1**
Check the normalisation condition for the spinors in equation (7.5.1).

I will now introduce (yet) more standard notation. Define the *gamma matrices*,

$$\gamma^0 = \beta, \qquad \gamma = \beta\alpha. \tag{7.5.2}$$

In the Dirac representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}. \tag{7.5.3}$$

In terms of these, the relations between the $\alpha$ and $\beta$ in equation (7.2.4) can be written compactly as,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{7.5.4}$$

Combinations like $a_\mu \gamma^\mu$ occur frequently and are conventionally written as,

$$\not{a} = a_\mu \gamma^\mu = a^\mu \gamma_\mu,$$

pronounced "a slash." Note that $\gamma^\mu$ is not, despite appearances, a 4-vector — it just denotes a set of four matrices. However, the notation is deliberately

suggestive, for when combined with Dirac fields you can construct quantities which transform like vectors and other Lorentz tensors (see the next section).

Let's close this section by observing that using the gamma matrices the Dirac equation (7.2.2) becomes

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{7.5.5}$$

or in momentum space,

$$(p\!\!\!/ - m)\psi = 0. \tag{7.5.6}$$

The spinors $u$ and $v$ satisfy

$$\begin{array}{rcl}(p\!\!\!/ - m)u_p^r &=& 0\\ (p\!\!\!/ + m)v_p^r &=& 0\end{array} \tag{7.5.7}$$

▷ **Exercise 7.5.2**
Derive the momentum space equations satisfied by $u_p^r$ and $v_p^r$.

## 7.6 Lorentz Covariance

We want the Dirac equation (7.5.5) to preserve its form under Lorentz transformations (LT's). Let $\Lambda^\mu{}_\nu$ represent an LT,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{7.6.1}$$

A familiar example of a LT is a boost along the z-axis, for which

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix},$$

with as usual $\beta = v$ (in units of c) and $\gamma = (1-\beta^2)^{-1/2}$. LT's can be thought of as generalised rotations.

The requirement is,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \quad \longrightarrow \quad (i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0,$$

where $\partial_\mu = \Lambda^\sigma{}_\mu\partial'_\sigma$. This last equality follows because

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\sigma}{\partial x^\mu}\frac{\partial}{\partial x'^\sigma} = \Lambda^\sigma{}_\mu\frac{\partial}{\partial x'^\sigma}$$

where Eq.7.6.1 has been used in the last step. We know that 4-vectors get their components mixed up by LT's, so we expect that the components of $\psi$ might get mixed up also,

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x') \tag{7.6.2}$$

where $S(\Lambda)$ is a $4\times 4$ matrix acting on the spinor index of $\psi$. Note that the argument $\Lambda^{-1}x'$ is just a fancy way of writing $x$, so each component of $\psi(x)$ is transformed into a linear combination of components of $\psi(x)$.

It is helpful to recall that for a vector field, the corresponding transformation is

$$A^\mu(x) \to A'^\mu(x')$$

where $x' = \Lambda x$. This makes sense physically if one thinks of space rotations of a vector field. For example the wind arrows on a weather map of England are an example of a vector field: at each point on the map there is associated an arrow. Consider the wind direction at a particular point on the map, say Abingdon. If the map of England is rotated, then one would expect on physical grounds that the wind vector at Abingdon always point in the same physical direction and have the same length. In order to achieve this, both the vector itself must rotate, and the point to which it is attached (Abingdon) must be correctly identified after the rotation. Thus the vector at the point $x'$ (corresponding to Abingdon in the rotated frame) is equal to the vector at the point $x$ (corresponding to Abingdon in the unrotated frame), but rotated so as to keep the physical sense of the vector the same in the rotated frame (so that the wind always blows towards Oxford, say, in the two frames). Thus having correctly identified the same point in the two frames all we need to do is rotate the vector:

$$A'^\mu(x') = \Lambda^\mu_\nu A^\mu(x).$$

A similar thing also happens in the case of the 4-component spinor field above, except that we do not (yet) know how the components of the wavefunction themselves must transform, i.e. we do not know $S$.

To determine $S$ we rewrite the Dirac equation in terms of the primed variables (just a mathematical substitution),

$$(i\gamma^\mu \Lambda^\sigma{}_\mu \partial'_\sigma - m)\psi(\Lambda^{-1}x') = 0. \tag{7.6.3}$$

Some new matrices can be defined, $\gamma'^\sigma \equiv \gamma^\mu \Lambda^\sigma{}_\mu$ which satisfy the same anti-commutation relations as the $\gamma^\mu$'s in equation (7.5.4),

$$\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}. \tag{7.6.4}$$

▷ **Exercise 7.6.1**
Check relation (7.6.4).

Now we invoke the theorem (Pauli's theorem) which states that any two representations of the gamma matrices are equivalent. This means that there is a matrix $S(\Lambda)$ such that

$$\gamma'^\mu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda). \tag{7.6.5}$$

This allows us to rewrite equation (7.6.3) as

$$(i\gamma^\mu \partial'_\mu - m)S(\Lambda)\psi(\Lambda^{-1}x') = 0,$$

or using Eq.7.6.2,

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0, \tag{7.6.6}$$

so that the Dirac equation does indeed preserve its form in the primed frame.

To construct $S$ explicitly we must solve Eq.7.6.5, which may be written as,

$$\gamma^\mu \Lambda^\sigma{}_\mu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda). \tag{7.6.7}$$

For an infinitesimal LT, it can be verified that,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \epsilon(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu) \tag{7.6.8}$$

where $\epsilon$ is an infinitesimal parameter and $\rho$ and $\sigma$ are fixed. Since this expression is antisymmetric in $\rho$ and $\sigma$ there are six choices for the pair $(\rho, \sigma)$ corresponding to three rotations and three boosts.

For example a boost along the z-axis corresponds to $\rho = 0$, $\sigma = 3$, since in this case,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \epsilon(g^{0\mu}\delta^3{}_\nu - g^{3\mu}\delta^0{}_\nu)$$
$$= \begin{pmatrix} 1 & 0 & 0 & -\epsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon & 0 & 0 & 1 \end{pmatrix},$$

which can be identified with the previous example with $\beta = -\epsilon$ and $\gamma = 1$ in the low velocity limit.

Writing,

$$S(\Lambda) = 1 + i\epsilon s^{\rho\sigma} \tag{7.6.9}$$

where $s^{\rho\sigma}$ is a matrix to be determined for each choice of $\rho$ and $\sigma$, we find that equation (7.6.5) for $\gamma'$ is satisfied by,

$$s^{\rho\sigma} = \frac{i}{4}\left[\gamma^\rho, \gamma^\sigma\right] \equiv \frac{1}{2}\,\sigma^{\rho\sigma}. \tag{7.6.10}$$

Here, I have taken the opportunity to define the matrix $\sigma^{\rho\sigma}$. Thus $S$ is given explicitly in terms of gamma matrices, for any LT specified by $\rho$, $\sigma$ and $\epsilon$.

▷ **Exercise 7.6.2**
Verify that equation (7.6.5) relating $\gamma'$ and $\gamma$ is satisfied by $s^{\rho\sigma}$ defined through equations (7.6.9) and (7.6.10).

We have thus determined how $\psi$ transforms under LT's. To find quantities which are Lorentz invariant, or transform as vectors or tensors, we need to introduce the Pauli and Dirac adjoints. The Pauli adjoint $\overline{\psi}$ of a *spinor* $\psi$ is defined by

$$\overline{\psi} \equiv \psi^\dagger\gamma^0 = \psi^\dagger\beta. \tag{7.6.11}$$

The Dirac adjoint of a *matrix* $A$ is defined by

$$(\overline{\psi}A\phi)^* = \overline{\phi}\,\overline{A}\psi. \tag{7.6.12}$$

For Hermitian $\gamma^0$ it is easy to show that

$$\overline{A} = \gamma^0 A^\dagger \gamma^0. \tag{7.6.13}$$

Some properties of the Pauli and Dirac adjoints are:

$$\begin{aligned} \overline{(\lambda A + \mu B)} &= \lambda^* \overline{A} + \mu^* \overline{B}, \\ \overline{AB} &= \overline{B}\,\overline{A}, \\ \overline{A\psi} &= \overline{\psi}\,\overline{A}. \end{aligned}$$

With these definitions, $\overline{\psi}$ transforms as follows under LT's:

$$\overline{\psi} \to \overline{\psi}' = \overline{\psi} S^{-1}(\Lambda) \tag{7.6.14}$$

▷ **Exercise 7.6.3**

(1) Verify that $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$. This says that $\overline{\gamma}^\mu = \gamma^\mu$.

(2) Using (7.6.9) and (7.6.10) verify that $\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)$, i.e. $\overline{S} = S^{-1}$. So $S$ is not unitary in general, although it *is* unitary for rotations (when $\rho$ and $\sigma$ are spatial indices). This is because the rotations are in the unitary $O(3)$ subgroup of the nonunitary Lorentz group. Here you show the result for an infinitesimal LT, but it is true for finite LT's.

(3) Show that $\overline{\psi}$ satisfies the equation

$$\overline{\psi}\,(-i\overleftarrow{\partial\!\!\!/} - m) = 0$$

where the arrow over $\partial\!\!\!/$ implies the derivative acts on $\overline{\psi}$.

(4) Hence prove that $\overline{\psi}$ transforms as in equation (7.6.14).

Note that result (2) of the problem above can be rewritten as $\overline{S}(\Lambda) = S^{-1}(\Lambda)$, and equation (7.6.5) for the similarity transformation of $\gamma^\mu$ to $\gamma'^\mu$ takes the form,

$$\overline{S}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu. \tag{7.6.15}$$

Combining the transformation properties of $\psi$ and $\overline{\psi}$ in equations (7.6.2) and (7.6.14) we see that the bilinear $\overline{\psi}\psi$ is Lorentz invariant. In section 7.8 we'll consider the transformation properties of general bilinears.

Let me close this section by recasting the spinor normalisation equations (7.5.1) in terms of "Dirac inner products." The conditions become,

$$\begin{aligned} \overline{u}^r_p u^s_p &= 2m\delta^{rs} \\ \overline{u}^r_p v^s_p &= 0 \qquad = \quad \overline{v}^r_p u^s_p \\ \overline{v}^r_p v^s_p &= -2m\delta^{rs} \end{aligned} \tag{7.6.16}$$

▷ **Exercise 7.6.4**
Verify the normalisation properties in the above equations (7.6.16).

## 7.7 Parity

In the next section we are going to construct quantities bilinear in $\psi$ and $\overline{\psi}$, and classify them according to their transformation properties under LT's. We normally use LT's which are in the connected Lorentz Group, $SO(3,1)$, meaning they can be obtained by a continuous deformation of the identity transformation. Indeed in the last section we considered LT's very close to the identity in equation (7.6.8). The full Lorentz group has four components generated by combining the $SO(3,1)$ transformations with the discrete operations of parity or space inversion, $P$, and time reversal, $T$,

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

LT's satisfy $\Lambda^T g \Lambda = g$ (see the preschool problems), so taking determinants shows that $\det \Lambda = \pm 1$. LT's in $SO(3,1)$ have determinant 1, since the identity does, but the $P$ and $T$ operations have determinant $-1$.

Let's now find the action of parity on the Dirac wavefunction and determine the wavefunction $\psi_P$ in the parity-reversed system. According to the discussion of the previous section, and using the result of equation (7.6.15), we need to find a matrix $S$ satisfying

$$\overline{S}\gamma^0 S = \gamma^0, \qquad \overline{S}\gamma^i S = -\gamma^i.$$

It's not hard to see that $S = \overline{S} = \gamma^0$ is an acceptable solution, from which it follows that the wavefunction $\psi_P$ is

$$\psi_P(t, \mathbf{x}) = \gamma^0 \psi(t, -\mathbf{x}). \tag{7.7.1}$$

In fact you could multiply $\gamma^0$ by a phase and still have an acceptable definition for the parity transformation.

In the nonrelativistic limit, the wavefunction $\psi$ approaches an eigenstate of parity. Since

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the $u$-spinors and $v$-spinors at rest have opposite eigenvalues, corresponding to particle and antiparticle having opposite *intrinsic* parities.

## 7.8 Bilinear Covariants

Now, as promised, we will construct and classify the bilinears. To begin, observe that by forming products of the gamma matrices it is possible to construct 16 linearly independent quantities. In equation (7.6.10) we have defined

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu],$$

and now it is convenient to define

$$\gamma_5 \equiv \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{7.8.1}$$

with the properties,

$$\gamma_5^\dagger = \gamma_5, \qquad \{\gamma_5, \gamma^\mu\} = 0.$$

Then the set of 16 matrices

$$\Gamma : \{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$$

form a basis for gamma matrix products.

Using the transformations of $\psi$ and $\overline{\psi}$ from equations (7.6.2) and (7.6.14), together with the similarity transformation of $\gamma^\mu$ in equation (7.6.15), construct the 16 fermion bilinears and their transformation properties as follows:

$$\begin{array}{rcll} \overline{\psi}\psi & \to & \overline{\psi}\psi & \text{S scalar} \\ \overline{\psi}\gamma_5\psi & \to & \det(\Lambda)\,\overline{\psi}\gamma_5\psi & \text{P pseudoscalar} \\ \overline{\psi}\gamma^\mu\psi & \to & \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi & \text{V vector} \\ \overline{\psi}\gamma^\mu\gamma_5\psi & \to & \det(\Lambda)\,\Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\gamma_5\psi & \text{A axial vector} \\ \overline{\psi}\sigma^{\mu\nu}\psi & \to & \Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma\overline{\psi}\sigma^{\lambda\sigma}\psi & \text{T tensor} \end{array} \tag{7.8.2}$$

▷ **Exercise 7.8.1**
Verify the transformation properties of the bilinears in equation (7.8.2).

Observe that $\overline{\psi}\gamma^\mu\psi = (\rho, \mathbf{J})$ is just the current we found earlier in equation (7.2.7). Classically $\rho$ is positive definite, but for the quantum Dirac field you find that the space integral of $\rho$ is the charge operator, which counts the number of electrons minus the number of positrons,

$$Q \sim \int d^3x\, \psi^\dagger\psi \sim \int d^3p\, [b^\dagger b - d^\dagger d\,].$$

The continuity equation $\partial_\mu J^\mu = 0$ expresses conservation of electric charge.

## 7.9 Charge Conjugation

There is one more discrete invariance of the Dirac equation in addition to parity. It is charge conjugation, which takes you from particle to antiparticle and vice versa. For scalar fields the symmetry is just complex conjugation, but in order for the charge conjugate Dirac field to remain a solution of the Dirac equation, you have to mix its components as well:

$$\psi \to \psi_C = C\overline{\psi}^{\,T}.$$

Here $\overline{\psi}^{\,T} = \gamma^{0T}\psi^*$ and $C$ is a matrix satisfying the condition

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu.$$

In the Dirac representation,

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}.$$

I refer you to textbooks such as [1] for details.

When Dirac wrote down his equation everybody thought parity and charge conjugation were exact symmetries of nature, so invariance under these transformations was essential. Now we know that neither of them, nor the combination *CP*, are respected by the standard electroweak model.

## 7.10 Neutrinos

In the particle data book [2] you will find only upper limits for the masses of the three neutrinos, and in the standard model they are massless. Let's look therefore at solutions of the Dirac equation with $m = 0$. From Eq.7.3.2 we have in this case

$$E\phi = \sigma\cdot\mathbf{p}\,\chi, \qquad E\chi = \sigma\cdot\mathbf{p}\,\phi. \tag{7.10.1}$$

These equations can easily be decoupled by taking the linear combinations and defining in a suggestive way the two component spinors $\nu_L$ and $\nu_R$,

$$\nu_R \equiv \chi + \phi, \quad \nu_L \equiv \chi - \phi \tag{7.10.2}$$

which leads to

$$E\nu_R = \sigma\cdot\mathbf{p}\,\nu_R, \qquad E\nu_L = -\sigma\cdot\mathbf{p}\,\nu_L. \tag{7.10.3}$$

Since $E = |\mathbf{p}|$ for massless particles, these equations may be written,

$$\frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}\nu_L = -\nu_L, \qquad \frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}\nu_R = \nu_R \tag{7.10.4}$$

Since $\frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}$ is identified as the helicity operator (i.e. the spin operator projected in the direction of motion of the momentum of the particle) we see that the $\nu_L$ corresponds to solutions with negative helicity, while $\nu_R$ corresponds to solutions with positive helicity. In other words $\nu_L$ descibes a left-handed neutrino while $\nu_R$ describes a right-handed neutrino – and each type of neutrino is descibed by a two-component spinor.

The two-component spinors describing neutrinos transform very simply under LT's,

$$\nu_L \to e^{\frac{i}{2}\sigma.(\theta - \mathbf{i}\phi)}\nu_L \tag{7.10.5}$$

$$\nu_R \to e^{\frac{i}{2}\sigma.(\theta + \mathbf{i}\phi)}\nu_R \tag{7.10.6}$$

where $\theta = \mathbf{n}\theta$ corresponding to space rotations through an angle $\theta$ about the unit $\mathbf{n}$ axis, and $\phi = \mathbf{v}\phi$ corresponding to Lorentz boosts along the unit vector $\mathbf{v}$ with a speed $v = \tanh\phi$. Under parity transformations they become transformed into each other,

$$\nu_L \leftrightarrow \nu_R \tag{7.10.7}$$

so a theory which involves only $\nu_L$ without $\nu_R$ (such as the standard model) manifestly violates parity.

Although massless neutrinos can be described very simply using two component spinors as above, they may also be incorporated into the four-component formalism as follows. From equation (7.2.2) we have, in momentum space,

$$|\mathbf{p}|\psi = \alpha{\cdot}\mathbf{p}\,\psi.$$

For such a solution,

$$\gamma_5\psi = \gamma_5\frac{\alpha{\cdot}\mathbf{p}}{|\mathbf{p}|}\psi = 2\frac{\mathbf{S}{\cdot}\mathbf{p}}{|\mathbf{p}|}\psi,$$

using the spin operator $\mathbf{S} = \frac{1}{2}\mathbf{\Sigma} = \frac{1}{2}\gamma_5\alpha$, with $\mathbf{\Sigma}$ defined in equation (7.4.4). But $\mathbf{S}{\cdot}\mathbf{p}/|\mathbf{p}|$ is the projection of spin onto the direction of motion, known as the *helicity*, and is equal to $\pm 1/2$. Thus $(1{+}\gamma_5)/2$ projects out the neutrino with helicity $1/2$ (right handed) and $(1{-}\gamma_5)/2$ projects out the neutrino with helicity $-1/2$ (left handed),

$$\frac{(1{+}\gamma_5)}{2}\psi \equiv \psi_R, \quad \frac{(1{-}\gamma_5)}{2}\psi \equiv \psi_L, \tag{7.10.8}$$

which defines the four-component spinors $\psi_R$ and $\psi_L$.

To date, only left handed neutrinos have been observed, and only left handed neutrinos appear in the standard model. Since

$$\gamma^0\frac{1}{2}(1{-}\gamma_5)\psi = \frac{1}{2}(1{+}\gamma_5)\gamma^0\psi,$$

any theory involving only left handed neutrinos necessarily violates parity - as we saw before in the two-component formalism.

Finally note that in the Dirac representation which we have been using,

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{7.10.9}$$

and the relation between the two-component and four-component formalisms is via the change of variables in Eq.7.10.2. However there exists a representation in which this change of variables is done automatically and the (massless) Dirac equation falls apart into the two two-component equations discussed above. In this chiral representation,

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.10.10}$$

and hence,

$$\frac{(1{+}\gamma_5)}{2}\psi = \begin{pmatrix} 0 \\ \nu_R \end{pmatrix}, \quad \frac{(1{-}\gamma_5)}{2}\psi = \begin{pmatrix} \nu_L \\ 0 \end{pmatrix}. \tag{7.10.11}$$

where we have identified $\nu_R$ and $\nu_L$ as the two-component spinors discussed previously. These results are also applicable to the electron in the approximation that its mass is neglected, by the simple transcription $\nu_R \to e_R$, $\nu_L \to e_L$.

The standard model (and the minimal supersymmetric standard model) contains only left handed massless neutrinos, and neutrino mass terms are forbidden by gauge symmetry, at least given the limited number of fields present in the standard model. If extra fields (e.g. right handed neutrinos) are added then neutrino masses become possible. If neutrino oscillations are confirmed as the solution to the solar neutrino problem, or are discovered in laboratory experiments, then such a modification would become a necessity.

# Chapter 8

# The Free Dirac Field

## 8.1 Canonical Quantisation

Dirac Field Theory is defined to be the theory whose field equations correspond to the Dirac equation. We regard the two Dirac fields $\psi(x)$ and $\overline{\psi}(x)$ as being dynamically independent fields and postulate the Dirac Lagrangian density:

$$\mathcal{L} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{8.1.1}$$

The Euler-Lagrange equation analagous to Eq.2.2.10

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} - \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0 \tag{8.1.2}$$

leads to the Dirac equation.

The canonical momentum from Eq.2.2.5 is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}(x)} = i\psi^\dagger(x) \tag{8.1.3}$$

The Hamiltonian density is analagous to Eq.2.2.7

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger i\frac{\partial\psi}{\partial t} \tag{8.1.4}$$

which is not positive definite. The general solution to the Dirac equation may be expanded in terms of plane waves analagously to Eq.2.3.19

$$\psi(\mathbf{x},t) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}\sum_{\alpha=1,2}[b_\alpha(\mathbf{k})u^\alpha(\mathbf{k})e^{-ik.x} + d^\dagger_\alpha(\mathbf{k})v^\alpha(\mathbf{k})e^{ik.x}] \tag{8.1.5}$$

$$\overline{\psi}(\mathbf{x},t) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}\sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})\overline{u}^\alpha(\mathbf{k})e^{ik.x} + d_\alpha(\mathbf{k})\overline{v}^\alpha(\mathbf{k})e^{-ik.x}] \tag{8.1.6}$$

The total Hamiltonian is

$$H = \int d^3x\mathcal{H} \tag{8.1.7}$$

After some algebra we find

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}k_0 \sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})b_\alpha(\mathbf{k}) - d_\alpha(\mathbf{k})d^\dagger_\alpha(\mathbf{k})] \tag{8.1.8}$$

So far no commutation relations have been assumed, and $H$ could quite easily be negative, unlike the Hamiltonian in the case of the charged scalars for example which was positive definite as seen in Eq.2.3.37. In order to give a positive definite Hamiltonian we require the creation and annihilation operators to satisfy *anticommutation* relations, first proposed by Wigner:

$$\left\{b_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}')\right\} = (2\pi)^3\frac{k_0}{m}\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\alpha\alpha'} \tag{8.1.9}$$

$$\left\{d_\alpha(\mathbf{k}), d^\dagger_{\alpha'}(\mathbf{k}')\right\} = (2\pi)^3\frac{k_0}{m}\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\alpha\alpha'} \tag{8.1.10}$$

$$\{b_\alpha(\mathbf{k}), b_{\alpha'}(\mathbf{k}')\} = 0 \tag{8.1.11}$$

$$\left\{b^\dagger_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0 \tag{8.1.12}$$

$$\{d_\alpha(\mathbf{k}), d_{\alpha'}(\mathbf{k}')\} = 0 \tag{8.1.13}$$

$$\left\{d^\dagger_\alpha(\mathbf{k}), d^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0 \tag{8.1.14}$$

The Hamiltonian is then defined as the normal ordered version of Eq.8.1.8 *but with a change of sign for each interchange of operator*

$$H = \int d^3x : \psi^\dagger i\frac{\partial\psi}{\partial t} : \tag{8.1.15}$$

which results in

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}k_0 \sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})b_\alpha(\mathbf{k}) + d_\alpha(\mathbf{k})d^\dagger_\alpha(\mathbf{k})] \tag{8.1.16}$$

which is now postive definite.

Anticommutation implies Fermi statistics for example:

$$\left\{b^\dagger_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0$$

$$\Rightarrow b^\dagger_\alpha(\mathbf{k})b^\dagger_\alpha(\mathbf{k}) = 0$$

$$\Rightarrow b^\dagger_\alpha(\mathbf{k})b^\dagger_\alpha(\mathbf{k})|0>= 0$$

so that two quanta in the same state are not allowed (Pauli exclusion principle).

The charge operator is the analogue of Eq.2.3.36

$$Q = \int d^3\mathbf{x} : j_0(x) := \int d^3\mathbf{x} : \psi^\dagger i\partial\psi :$$

$$Q = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}k_0 \sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})b_\alpha(\mathbf{k}) - d^\dagger_\alpha(\mathbf{k})d_\alpha(\mathbf{k})] \tag{8.1.17}$$

which shows that $b^\dagger$ creates fermions while $d^\dagger$ creates antifermions of opposite charge.

Finally the equal time commutation relations are (after some algebra):

$$\left\{\psi_i(\mathbf{x},t),\psi_j^\dagger(\mathbf{x}',t)\right\}=\delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij} \tag{8.1.18}$$

$$\{\psi_i(\mathbf{x},t),\psi_j(\mathbf{x}',t)\}=0 \tag{8.1.19}$$

$$\left\{\psi_i^\dagger(\mathbf{x},t),\psi_j^\dagger(\mathbf{x}',t)\right\}=0 \tag{8.1.20}$$

In fact at all times we have:

$$\{\psi(x),\psi(y)\}=0 \tag{8.1.21}$$

## 8.2 Path Integral Quantisation

We have seen that Green's functions in quantum field theory may be generated from generating functionals, which are functional integrals over classical field functions. In the case of scalar field theory the classical field functions are commuting c-numbers. However in the case of spinor field theory the classical field functions cannot be regarded as commuting c-numbers. The problem is that the classical Hamiltonian is also given by Eq.8.1.8 and is similarly non-positive definite. Thus it would appear that classical spinor field theory does not make physical sense. Thus spinor fields are essentially non-classical, and this poses a problem for the path integral approach where functional integrals over classical fields will occur.

In chapter 2 we discussed the free Klein-Gordon field and showed how it could be regarded as a continuum of harmonic oscillators, one at each spacetime point, and each coupled to their nearest neighbours in a Lorentz invariant way. The path integral results in chapter 4 may be taken to apply to each of these little oscillators. In chapter 6 we generalised the path integral results from a single coordinate $q(t)$ to many coordinates $q_i(t)$, one for each oscillator. We then considered the continuum limit,

$$q_i(t)\rightarrow\phi(x,y,z,t) \tag{8.2.1}$$

just as we did to arrive at canonical field theory. Thus the Heisenberg operators $\hat{q}(t)$ in Eq.6.1.3 become Heisenberg fields $\hat{\phi}(x,y,z,t)$ whose eigenstates are given by

$$\hat{\phi}(x,y,z,t)|\phi(x,y,z,t)>=\phi(x,y,z,t)|\phi(x,y,z,t)> \tag{8.2.2}$$

and the generating functional involves the eigenvalue of the field $\phi(x,y,z,t)$ rather than the field operator.

Now we want to do the same thing for the spinor field theory, and the generating functional will involve the eigenvalue field on the right hand side of the equation:

$$\hat{\psi}(x,y,z,t)|\psi(x,y,z,t)>=\psi(x,y,z,t)|\psi(x,y,z,t)> \tag{8.2.3}$$

However if we apply the same field operator twice we encounter a problem:

$$\hat{\psi}^2(x,y,z,t)|\psi(x,y,z,t)>=\psi^2(x,y,z,t)|\psi(x,y,z,t)> \tag{8.2.4}$$

The problem is that the lhs is zero by the anticommutatation property assumed for the fermion field operator, but the rhs is apparently non-zero. In order to overcome this problem we must no longer regard the field eigenvalue $\psi(x,y,z,t)$ as being a commuting c-number, but instead regard it as an *anticommuting Grassmann variable* which obeys the anticommutation relations:

$$\{\psi(x),\psi(y)\}=0 \tag{8.2.5}$$

Then the product of two identical field eigenvalues is also zero and the previous problem is overcome.

What exactly are Grassmann variables? Well they are well defined mathematical objects which multiply operators and vectors just as if they were c-numbers, but have the property that they anticommute amongst themselves. They are discussed further in the Appendix. Since anticommuting field operators implies Grassmann field eigenvalues, the classical spinor field must be regarded as Grassmannian. Thus the classical Hamiltonian (i.e. the energy) becomes neither positive nor negative but rather as function of Grassmann variables, and this solves its positivity problem.

Clearly we must quantise the Dirac fields using functional methods based on spinor fields corresponding to Grassmann variables. By analogy with Eq.6.1.11 the generating functional for the free Dirac field is:

$$Z_0[\eta,\overline{\eta}]\propto\int\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{i(\int d^4x(\mathcal{L}+\overline{\eta}\psi+\overline{\psi}\eta))} \tag{8.2.6}$$

where we have introduced a source term $\overline{\eta}(x)$ for $\psi(x)$ and another one $\eta(x)$ for $\overline{\psi}(x)$, and both the sources and the fields are Grassmann variables. The Lagrangian is just the free Dirac Lagrangian introduced earlier.

By analogy with Eq.6.1.13 the Green's functions may be defined by the VEV of the Heisenberg field operators

$$\mathcal{G}(x_1,\cdots,x_n;y_1,\cdots,y_n)\equiv<0|T(\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)\overline{\hat{\psi}}(y_1)\cdots\overline{\hat{\psi}}(y_n))|0> \tag{8.2.7}$$

where the time ordering puts earlier times to the right as usual but in doing so we must introduce a minus sign every time two fermion fields are anticommuted. Again by analogy with Eq.6.1.14 the Green's functions are given from the generating functional as,

$$i^n\mathcal{G}(x_1,\cdots,x_n;y_1,\cdots,y_n)=$$

$$\left[\frac{\delta^{2n}Z_0[\eta,\overline{\eta}]}{\delta\overline{\eta}(x_1)\cdots\delta\overline{\eta}(x_n)\delta\eta(y_1)\cdots\delta\eta(y_n)}\right]_{\eta=0,\overline{\eta}=0} \tag{8.2.8}$$

## 8.3 The Feynman Propagator for the Dirac Field

We are familiar with the Feynman propagator of the scalar field as being the VEV of the time ordered product of two Heisenberg fields in the free field theory (or of two interaction picture fields in the interacting field theory which amounts to the same thing). In the language of Green's functions, the Feynman propagator is just the two point Green's function in the free field theory. Previously we calculated the Feynman propagator in two ways: by using canonical methods in section 3.5, and using functional methods in section 6.2. Here we shall use the functional approach, following the steps analagous to section 6.2.

We first consider the free Dirac field Lagrangian density

$$\mathcal{L} = \overline{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \tag{8.3.1}$$

The path integral expression for the generating functional is explictly:

$$Z_0[\eta, \overline{\eta}] \propto \int \mathcal{D}\psi \mathcal{D}\overline{\psi}$$

$$\exp\left(i \int d^4x \left[\overline{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) + \overline{\eta}\psi + \overline{\psi}\eta\right]\right)$$

As in the scalar case, in order to identify the Feynman propagator, we wish to perform the functional integral over the fields, and as in the previous case it is possible to do so for the free field theory. We first recall the Grassmannian version of the most important integral in the world from the Appendix:

$$Z_0' = \int \prod_{i=1}^{n} da_i e^{-\sum_{i,j=1}^{n} a_i K_{ij} a_j + \sum_{k=1}^{n} \eta_k a_k} \tag{8.3.2}$$

$$Z_0' = \sqrt{det(K)} e^{-\frac{1}{4}\sum_{i,j=1}^{n} \eta_i (K^{-1})_{ij} \eta_j} \tag{8.3.3}$$

The continuum version of this is:

$$Z_0[\eta] = \int \mathcal{D}\psi \exp(-\frac{1}{2}\int d^4x d^4x' \left[\psi(x')A(x',x)\psi(x)\right] + \int d^4x \eta(x)\psi(x))$$

$$Z_0[\eta] = \sqrt{det(A)} \exp(-\frac{1}{2}\int d^4x d^4x' \left[\eta(x')A^{-1}(x',x)\eta(x)\right]$$

This applies to real Grassmannian functions. For complex Grassmannian functions we need the following generalisation:

$$Z_0[\eta, \eta^*] = \int \mathcal{D}\psi \mathcal{D}\psi^*$$

$$\exp\left(i \int d^4x d^4x' \left[\psi^*(x')A(x',x)\psi(x)\right] + i\int d^4x \eta^*(x)\psi(x) + \psi^*(x)\eta(x)\right)$$

$$Z_0[\eta, \eta^*] = \sqrt{det(iA)} \exp(-i \int d^4x d^4x' \left[\eta^*(x')A^{-1}(x',x)\eta(x)\right] \tag{8.3.4}$$

In order to perform the functional integrations we first write the generating functional as:

$$Z_0[\eta,\overline{\eta}] \propto \int \mathcal{D}\psi\mathcal{D}\overline{\psi}$$
$$\exp(i\int d^4xd^4x'\{\overline{\psi}(x')\left[\delta^4(x-x')(i\gamma^\mu\partial^x_\mu - m)\right]\psi(x)$$
$$+i\int d^4x\overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x)\})$$

We identify the operator in square brackets as $A(x',x)$,

$$A(x',x) \equiv \left[\delta^4(x-x')(i\gamma^\mu\partial^x_\mu - m)\right] \tag{8.3.5}$$

and using Eq.8.3.4, replacing $\psi^*$ by $\overline{\psi} = \psi^\dagger\gamma^0$, and similarly for $\eta$ we find

$$Z_0[\eta,\overline{\eta}] \propto \sqrt{det(iA)}\exp(-i\int d^4xd^4x'\,[\overline{\eta}(x')S_F(x'-x)\eta(x)] \tag{8.3.6}$$

where

$$S_F(x'-x) = A^{-1}(x',x) \tag{8.3.7}$$

is the Feynman propagator for the Dirac field, as is readily verified by constructing the two point Green's function from Eq.8.3.6.

It only remains to explicitly find an expression for the Feynman propagator. From above we see that

$$S_F(x'-x) = \left[\delta^4(x-x')(i\gamma^\mu\partial^x_\mu - m)\right]^{-1} \tag{8.3.8}$$

Clearly we have

$$\int d^4y[S_F(x,y)]^{-1}S_F(y,z) = \delta(x-z)$$

Hence from its definition we have

$$\int d^4y\left[\delta^4(x-y)(i\gamma^\mu\partial^x_\mu - m)\right]S_F(y,z) = \delta(x-z)$$

Integrating over $y$, we find the equation which must be satisfied by $S_F(x,z)$,

$$(i\gamma^\mu\partial^x_\mu - m)S_F(x,z) = \delta(x-z) \tag{8.3.9}$$

The solution to this equation is formally

$$S_F(x,z) = \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}+m}{k^2-m^2}e^{-ik.(x-z)}$$

as may be verified by substitution. As in the scalar propagator case in section 6.2 the result is ambiguous due to the poles, and as in that case the resolution to the problem is to remember that the generating functional is strictly only defined off the real axis. The analysis is identical to that in section 6.2 and the result is:

$$S_F(x-z) = \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}+m}{k^2-m^2+i\epsilon}e^{-ik.(x-z)} \tag{8.3.10}$$

The Fourier transform of the Feynman propagator is thus

$$S_F(k) = \frac{\not{k}+m}{k^2-m^2+i\epsilon} \tag{8.3.11}$$

## 8.4 Appendix: Grassmann Variables

To begin with consider two Grassmann variables $a, b$ which satisfy the anticommutation relations:

$$\{a, a\} = 0, \quad \{a, b\} = 0, \quad \{b, b\} = 0 \tag{8.4.1}$$

They are not operators; they do not act on any vectors in a vector space. However they do obey an algebra: the Grassmann algebra indicated above. We can think of them as "operators which do not act on anything" if we like (although please don't say this in the presence of a mathematician unless you want to see him or her turn blue.)

Without loss of generality any function of the two Grassmann variables may be written as:

$$f(a, b) = f_0 + f_1 a + \tilde{f}_1 b + f_2 ab \tag{8.4.2}$$

where the $f$'s are non-Grassmannian c-numbers. The *left* derivatives of the function are:

$$\frac{\partial f}{\partial a} = f_1 + f_2 b \tag{8.4.3}$$

$$\frac{\partial f}{\partial b} = \tilde{f}_1 - f_2 a \tag{8.4.4}$$

so called because the infinitesimal denominator is really an inverse Grassmann variable which acts on the *left* in this case. Unless otherwise stated all our derivatives will be *left* derivatives. We find:

$$\frac{\partial^2 f}{\partial a \partial b} = -\frac{\partial^2 f}{\partial b \partial a} = -f_2 \tag{8.4.5}$$

We may also define integration with respect to Grassmann variables. The simplest Grassmann integral is zero:

$$\int da = 0 \tag{8.4.6}$$

(to prove this first show that $(\int da)^2 = -(\int da)^2$). The next simplest Grassmann integral is equal to unity:

$$\int daa = 1 \tag{8.4.7}$$

(in fact since $a^2 = 0$ the above integral serves as a sort of normalisation of the Grassmann variables.) Eqs.8.4.6 and 8.4.7 show that Grassmann integration has the same effect as differentiation.

Using Eq.8.4.2 we find that

$$\int dadb f(a, b) = \frac{\partial^2 f}{\partial a \partial b} \tag{8.4.8}$$

An interesting result is:

$$\int dadbe^{-ba} = \int dadb(1-ba) = 1 \tag{8.4.9}$$

Finally recall the most important integral in the world from Eq.5.6.10:

$$Z_0 = \int_{-\infty}^{\infty} \prod_{i=1}^{n} dq_i e^{-\sum_{i,j=1}^{n} q_i K_{ij} q_j + \sum_{k=1}^{n} J_k q_k} \tag{8.4.10}$$

This is an $n$ dimensional integral and $K_{ij}$ is an $n \times n$ symmetric matrix, while $J_k$ is an $n$ component column vector. Amazingly the above integral can be done for all invertible $K_{ij}$:

$$Z_0 = \frac{\pi^{n/2}}{\sqrt{det(K)}} e^{+\frac{1}{4}\sum_{i,j=1}^{n} J_i (K^{-1})_{ij} J_j} \tag{8.4.11}$$

Now we consider the analagous integral but involving Grassmann variables $a_1, \cdots a_n$ and Grassmann variables $\eta_1, \cdots \eta_n$,

$$Z_0 = \int \prod_{i=1}^{n} da_i e^{-\sum_{i,j=1}^{n} a_i K_{ij} a_j + \sum_{k=1}^{n} \eta_k a_k} \tag{8.4.12}$$

and we assume that $K_{ij}$ is now *antisymmetric* rather than symmetric otherwise we would get a zero result. For a similar reason we must assume that $n$ is even. In this case the result is:

$$Z_0 = \sqrt{det(K)} e^{-\frac{1}{4}\sum_{i,j=1}^{n} \eta_i (K^{-1})_{ij} \eta_j} \tag{8.4.13}$$

The proof of this is found in Bailin and Love 8.1.

# Chapter 9

# The Free Electromagnetic Field

## 9.1 The Classical Electromagnetic Field

The four Maxwell equations are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

It is straightforward to show that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

In covariant form,

$$\partial_\mu j^\mu = 0$$

where $j^\mu = (c\rho, \mathbf{j})$.

It is convenient (and even essential) to introduce scalar and vector potentials $\phi$ and $\mathbf{A}$ by defining

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t.$$

whence two of the Maxwell equations become automatic.

Recall the gauge invariance of electrodynamics which says that $\mathbf{E}$ and $\mathbf{B}$ are unchanged when

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda \quad \text{and} \quad \phi \to \phi - \frac{\partial \Lambda}{\partial t}$$

for any scalar function $\Lambda$. Gauge invariance corresponds to a lack of uniqueness of the scalar and vector potentials. This lack of uniqueness can be reduced by imposing a further condition on the scalar and vector potentials, for example

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

Assuming that $\phi$ and $\mathbf{A}$ can be combined into a four vector

$$A^\mu = (\phi/c, \mathbf{A})$$

this can be written as

$$\partial_\mu A^\mu = 0 \tag{9.1.1}$$

which is known as the *Lorentz gauge* condition. Gauge invariance in four-vector notation is just:

$$A^\mu \rightarrow A^\mu + \partial_\mu \Lambda \tag{9.1.2}$$

Note that even the imposition of the Lorentz gauge condition does not completely fix the vector potential; it merely restricts the function $\Lambda$ to satisfy

$$\partial^2 \Lambda = 0 \tag{9.1.3}$$

With the Lorentz gauge condition Maxwell's equations are equivalent to

$$\partial^2 A^\mu = \mu_0 j^\mu$$

The tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ clearly has six independent components, and can be written:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

It is straightforward to show that,

$$F_{\mu\nu}F^{\mu\nu} = -2\left(\frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2\right)$$

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = -\frac{8}{c}\mathbf{E}\cdot\mathbf{B}$$

where

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of 0123} \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise} \end{cases}$$

This gives the relativistic invariants which can be constructed from $\mathbf{E}$ and $\mathbf{B}$.

It is easy to see that in any gauge the Maxwell equations can be written,

$$\partial_\mu F^{\mu\nu} = j^\nu$$

The Maxwell equations, in this compact form, can be reproduced by the following Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \tag{9.1.4}$$

via the Euler-Lagrange equations for each of the four $A_\mu$ fields separately.

In Lorentz gauge the Lagrangian density has the more general form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.1.5}$$

where $\xi$ is a free parameter. The EL equations then imply

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi}\partial^\nu(\partial_\mu A^\mu) = j^\nu \tag{9.1.6}$$

which reduce to Maxwell's equations in Lorentz gauge. The extra term in the Lagrangian density $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ thus has no effect on physics in Lorentz gauge. In fact it is possible to turn the argument around and use this term to fix the gauge to be Lorentz gauge by imposing current conservation instead of obtaining it as a consequence of Maxwell's equations. If one adds the extra term to the Lagrangian and imposes current conservation then Eq.9.1.6 implies immediately the Lorentz gauge condition by the antisymmetry of $F^{\mu\nu}$. For this reason the extra term is referred to as a *gauge fixing term* and $\xi$ is a Lagrange multiplier. The choice $\xi = 1$ is known as Feynman gauge although it is within the framework of the Lorentz gauge.

As usual we can expand the field $A_\mu(x)$ in its Fourier components

$$A_\mu(x)(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a_\mu(k)e^{-ik.x} + a^*_\mu(k)e^{ik.x}] \tag{9.1.7}$$

where $\omega = k_0 = |\mathbf{k}|$. The Lorentz gauge condition implies

$$k.a(k) = 0 \tag{9.1.8}$$

This implies that

$$a_0(k) = \hat{\mathbf{k}}.\mathbf{a}(k)$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. Thus the time component of $a_\mu$ equals the longitudinal component $\mathbf{a}.\mathbf{k}$. Of course only the transverse components are physical (since the $\mathbf{E}$ and $\mathbf{B}$ fields are always orthogonal to the three momentum) and it can be shown that the contribution to the Hamiltonian from the time component and longitudinal component cancel against each other. In fact it is possible to completely specify the gauge by requiring that

$$a_0(k) = \hat{\mathbf{k}}.\mathbf{a}(k) = 0$$

which is called Coulomb gauge. In Coulomb gauge we can write

$$a_\mu(k) = \sum_{\lambda=1,2} a^\lambda(k)\epsilon^\lambda_\mu(k)$$

where $\epsilon^\lambda_\mu(k)$ are two orthonormal spacelike vectors in the plane transverse to $\mathbf{k}$.

In a general Lorentz gauge we can write:

$$a_\mu(k) = \sum_{\lambda=0,1,2,3} a^\lambda(k)\epsilon^\lambda_\mu(k)$$

where now $\epsilon^\lambda_\mu(k)$ are arbitrary unit four-vectors. Suppose that $\mathbf{k}$ is along the third axis, $k = (\omega, 0, 0, \omega)$ then we can define the basis vectors as:

$$\epsilon^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{9.1.9}$$

so that we call $\lambda = 1, 2$ the physical transverse polarisations, $\lambda = 0$ the unphysical timelike polarisation and $\lambda = 3$ the unphysical longitudinal polarisation. Clearly,

$$k.\epsilon^{1,2} = 0$$

and

$$\epsilon^\lambda.\epsilon^{\lambda'} = g^{\lambda\lambda'}$$

which is in fact a basis independent result, although we shall always work in this basis.

## 9.2 Canonical Quantisation

We shall now quantise the free e.m. theory ($j^\mu = 0$). To quantise the theory canonically we introduce the canonical momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \tag{9.2.1}$$

and impose the equal time covariant canonical commutation relations

$$[A_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{x}', t)] = i g_{\mu\nu}\delta^3(\mathbf{x} - \mathbf{x}') \tag{9.2.2}$$

$$[A_\mu(\mathbf{x}, t), A_\nu(\mathbf{x}', t)] = 0 \tag{9.2.3}$$

$$[\pi_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{x}', t)] = 0 \tag{9.2.4}$$

Now if the Lagrangian were simply

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.2.5}$$

then we would find that

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

which would imply that $\pi^0$ always commutes with $A^0$, which loses us both covariance and quantum mechanics at a stroke!

We clearly need a $\pi^0$ that does not vanish. In order to do this we need to change the Lagrangian without changing the physics. But we have learned how to do this in Lorentz gauge which corresponds to the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.2.6}$$

and the field equations:

$$\partial^2 A_\mu - (1 - \frac{1}{\xi})\partial_\mu(\partial_\nu A^\nu) = 0 \tag{9.2.7}$$

Henceforth for simplicity we shall take $\xi = 1$ which is called Feynman gauge (a sub-class of Lorentz gauge).

At first sight this doesn't help us because we find

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu$$

which apparently vanishes in Lorentz gauge. However we shall only assume that matrix elements of $\partial_\mu A^\mu$ vanish rather than imposing the operator condition that it vanish.

In Feynman gauge we have the field equations:

$$\partial^2 A_\mu = 0 \tag{9.2.8}$$

and we can once again expand the $A_\mu$ field in plane wave solutions similar to the previous section:

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} \sum_{\lambda=0}^{3} [\epsilon_\mu^\lambda(k) a^\lambda(k) e^{-ik.x} + \epsilon_\mu^{\lambda *}(k) a^\lambda(k)^\dagger e^{ik.x}] \tag{9.2.9}$$

Here $\epsilon_\mu^\lambda(k)$ are the set of four linearly independent vectors defined in Eq.9.1.9, but now we regard $a$ and its hermitian conjugate as operators whose commutation relations readily follow from Eq.9.2.2

$$[a^\lambda(k), a^{\lambda'}(k')^\dagger] = -g^{\lambda\lambda'} 2k_0 (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \tag{9.2.10}$$

For longitudinal and transverse photons quantisation proceeds in the usual way. But for timelike photons with $\lambda = \lambda' = 0$ we have a negative quantity on the rhs which gives problems. This leads to timelike photons with negative norm. However it is possible to overcome these problems using the Gupta-Bleuler formalism. However at this point we prefer to abandon the canonical approach and move on to the path integral approach which has its own problems.

## 9.3 Path Integral Quantisation

We have seen that the freedom to make gauge transformations means that the $A^\mu$ fields are not uniquely specified, and this causes problems with the theory in the canonical formalism. It should be no surprise that these problems persist in the path integral approach.

The generating functional in this case is

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x(\mathcal{L}+J^\mu A_\mu\right)} \tag{9.3.1}$$

where $\mathcal{L}$ is the Lagrangian for the free photon field which we might naively take to be

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.3.2}$$

(since we have already found problems with this form in the canonical formalism it really is naive to expect it to work here). The field equations in this case are as in Eq.9.1.6

$$\partial_\mu F^{\mu\nu} = 0 \tag{9.3.3}$$

which can be written as

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\mu = 0 \tag{9.3.4}$$

After partial integration and discarding surface terms we can write the generating functional as

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x \frac{1}{2}A^\mu[g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu]A^\nu + J^\mu A_\mu\right)} \tag{9.3.5}$$

By now we know that the photon propagator $D_{\mu\nu}$ is going to be the inverse of the operator in square brackets, and it will satisfy the equation:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)D^{\nu\lambda}(x-y) = \delta^\lambda_\mu\delta^4(x-y) \tag{9.3.6}$$

If we multiply this equation by $\partial^\mu$ we get zero multiplying $D^{\nu\lambda}(x-y)$ on the lhs and something non-zero on thhe rhs, which would seem to imply that $D^{\nu\lambda}(x-y)$ is infinite. In fact the problem is that the operator in square brackets does not have an inverse! To show this all we need to do is show that it has a zero eigenvalue, and this can easily be done:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\partial^\mu\Omega = 0$$

for any function $\Omega$.

From the point of view of the path integral the problem is that the functional integral is taken over all $A_\mu$ including those related by a gauge transformation, leading to an infinite overcounting in the calculation of the generating functional, and hence an infinite overcounting for the Green's functions which are obtained from it by functional differentiation. To cure this problem we

need to fix a particular gauge, and we do this by imposing the Lorentz gauge condition:

$$\partial_\mu A^\mu = 0 \tag{9.3.7}$$

Recall the Lagrangian with gauge fixing term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.3.8}$$

and the field equations:

$$\partial^2 A_\mu - (1 - \frac{1}{\xi})\partial_\mu(\partial_\nu A^\nu) = 0 \tag{9.3.9}$$

After partial integration and discarding surface terms we can now write the generating functional as

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x \frac{1}{2}A^\mu[g_{\mu\nu}\partial^2 + (\frac{1}{\xi}-1)\partial_\mu\partial_\nu]A^\nu + J^\mu A_\mu\right)} \tag{9.3.10}$$

and the operator in square brackets now has an inverse given by

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} - \frac{\left[g_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\epsilon}e^{-ik.(x-y)} \tag{9.3.11}$$

The Fourier transform of the Feynman propagator is thus

$$D_{\mu\nu}(k) = -\frac{\left[g_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\epsilon} \tag{9.3.12}$$

Amongst this class of gauge choices two common choices are Feynman gauge ($\xi = 1$) and Landau gauge ($\xi = 0$).

# Chapter 10

# Quantum Electrodynamics

## 10.1 Feynman Rules of QED

QED involves the interaction of electrons and photons where the interaction corresponds to the Lagrangian

$$\mathcal{L}_{int} = -e\overline{\psi}\gamma^{\mu}A_{\mu}\psi. \tag{10.1.1}$$

Such an interaction may be introduced by the concept of "minimal substitution" familiar from classical electrodynamics. The momentum and energy become:

$$\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$$

$$E \rightarrow E - e\phi$$

or in four vector notation, the four momentum becomes:

$$p^{\mu} \rightarrow p^{\mu} - eA^{\mu}$$

Applying this classical concept of minimal substitution to the Dirac equation gives:

$$(i\not{D} - m)\psi = 0 \tag{10.1.2}$$

where we have introduced the covariant derivative notation

$$D_{\mu} \equiv \partial_{\mu} + ieA_{\mu}$$

The QED Lagrangian describing electrons, photons and their interactions is then given by,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_{\mu}A^{\mu})^2 + \overline{\psi}(i\not{D} - m)\psi. \tag{10.1.3}$$

Here, $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ is the electromagnetic covariant derivative, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and $(\partial{\cdot}A)^2/2$ is the gauge fixing term for Feynman gauge.

The QED Lagrangian is invariant under a symmetry called *gauge symmetry*, which consists of the simultaneous gauge transformations of the photon field:

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{10.1.4}$$

and a phase transformation on the electron field

$$\psi \to e^{-ie\Lambda}\psi \tag{10.1.5}$$

The point is that the value of the phase transformation given by the same gauge function $\Lambda(x)$ as controls the photon gauge transformation. It is important to emphasise that $\Lambda(x)$ is a function of $x$ so that the action of a derivative on $e^{-ie\Lambda}\psi$ will yield two terms by the product rule. However the simultaneous gauge transformation of the photon field means that the covariant derivative of $\psi$ transforms like $\psi$ itself under the combined gauge transformations above:

$$D_\mu\psi \to e^{-ie\Lambda}D_\mu\psi \tag{10.1.6}$$

Thus the QED Lagrangian is invariant under the simultaneous transformations above, referred to collectively as a gauge transformation.

In this chapter we are going to get some practice calculating cross sections and decay rates in QED. The starting point is the set of Feynman rules in Table 10.1.1 derived from the QED Lagrangian above. The fermion propagator is (up to factors of $i$) the inverse of the operator, $\not{p} - m$, which appears in the quadratic term in the fermion fields, as we saw in section 8.3. The derivation of the photon propagator, along with the need for gauge fixing, was discussed in section 9.3. The external line factors are easily derived by considering simple matrix elements in the operator formalism, where they are left behind from the expansions of fields in terms of annihilation and creation operators, after the operators have all been (anti-)commuted until they annihilate the vacuum. One could consider for example the process $\gamma \to e^+e^-$ in complete analogy to the example in section 3.4, working in the interaction picture. In path integral language the natural objects to compute are Green functions, vacuum expectation values of time ordered products of fields: it takes a little more work to convert them to transition amplitudes and see the external line factors appear, but the proceedure is exactly analagous to $\lambda\phi^4$ theory in section 6.4.

The spinor indices in the Feynman rules are such that matrix multiplication is performed in the opposite order to that defining the flow of fermion number. The arrow on the fermion line itself denotes the fermion number flow, *not* the direction of the momentum associated with the line: I will try always to indicate the momentum flow separately as in Table 10.1.1. This will become clear in the examples which follow. We have already met the Dirac spinors $u$ and $v$. I will say more about the photon polarisation vector $\epsilon$ when we need to use it.

## 10.2 Electron–Muon Scattering

To lowest order in the electromagnetic coupling, just one diagram contributes to this process. It is shown in Figure 10.2.1. The amplitude obtained from this diagram is

$$i\mathcal{M}_{fi} = (-ie)\,\overline{u}(p_c)\gamma^\mu u(p_a)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(-ie)\,\overline{u}(p_d)\gamma_\nu u(p_b). \qquad (10.2.1)$$

Note that I have changed my notation for the spinors: now I label their momentum as an argument instead of as a subscript, and I drop the spin label unless I need to use it. In constructing this amplitude we have followed the fermion lines backwards with respect to fermion flow when working out the order of matrix multiplication.

The cross-section involves the squared modulus of the amplitude, which is

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}\,L^{\mu\nu}_{(e)}L_{(\mu)\,\mu\nu},$$

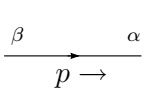
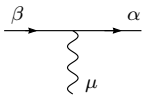
| ‹very ... | draw ... | write ... |
|---|---|---|
| nal photon line |   | $\frac{-ig^{\mu\nu}}{q^2+i\epsilon}$ |
| nal fermion line |   | $\frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i}$ |
| ‹x |   | $-ie\gamma^\mu_{\alpha\beta}$ |
| ‹oing electron | | $\overline{u}^s_p$ |
| ning electron | | $u^s_p$ |
| ‹oing positron | | $v^s_p$ |
| ning positron | | $\overline{v}^s_p$ |
| ‹oing photon | | $\epsilon^{*\mu}$ |
| ning photon | | $\epsilon^\mu$ |
| ‹ach a directed momentum to every internal line | | |
| nserve momentum at every vertex | | |

Table 10.1.1 Feynman rules for QED. $\mu$, $\nu$ are Lorentz indices and $\alpha$, $\beta$ are spinor indices.

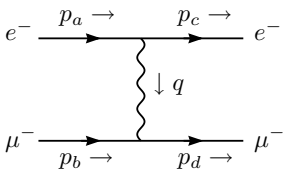



Figure 10.2.1 Lowest order Feynman diagram for electron–muon scattering.

where the subscripts $e$ and $\mu$ refer to the electron and muon respectively and,

$$L^{\mu\nu}_{(e)} = \overline{u}(p_c)\gamma^\mu u(p_a)\overline{u}(p_a)\gamma^\nu u(p_c),$$

with a similar expression for $L^{\mu\nu}_{(\mu)}$.

▷ **Exercise 10.2.1**
Verify the expression for $|\mathcal{M}_{fi}|^2$.

Usually we have an unpolarised beam and target and do not measure the polarisation of the outgoing particles. Thus we calculate the squared amplitudes for each possible spin combination, then average over initial spin states and sum over final spin states. Note that we square and then sum since the different possibilities are in principle distinguishable. In contrast, if several Feynman diagrams contribute to the same process, you have to sum the amplitudes first. We will see examples of this below.

The spin sums are made easy by the following results (I temporarily restore spin labels on spinors):

$$\begin{aligned} \sum_r u^r(p)\,\overline{u}^r(p) &= \not{p} + m \\ \sum_r v^r(p)\,\overline{v}^r(p) &= \not{p} - m \end{aligned} \tag{10.2.2}$$

▷ **Exercise 10.2.2**
Derive the spin sum relations in equation (10.2.2).

Using the spin sums we find,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{4q^4}\ \text{tr}\left(\gamma^\mu(\not{p}_a + m_e)\gamma^\nu(\not{p}_c + m_e)\right)\text{tr}\left(\gamma_\mu(\not{p}_b + m_\mu)\gamma_\nu(\not{p}_d + m_\mu)\right).$$

Since all calculations of cross sections or decay rates in QED require the evaluation of traces of products of gamma matrices, you will generally find a table of "trace theorems" in any quantum field theory textbook [1]. All these theorems can be derived from the fundamental anticommutation relations of the gamma matrices in equation (7.5.4) together with the invariance of the trace under a cyclic change of its arguments. For now it suffices to use,

$$\begin{aligned} \text{tr}(\not{a}\not{b}) &= 4\,a{\cdot}b \\ \text{tr}(\not{a}\not{b}\not{c}\not{d}) &= 4(a{\cdot}b\,c{\cdot}d - a{\cdot}c\,b{\cdot}d + a{\cdot}d\,b{\cdot}c) \\ \text{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 0 \quad \text{for } n \text{ odd} \end{aligned} \tag{10.2.3}$$


Figure 10.3.1 Lowest order Feynman diagrams for electron–electron scattering.

▷ **Exercise 10.2.3**
Derive the trace results in equation (10.2.3)

Using these results, and expressing the answer in terms of the Mandelstam variables of equation (4.5.1), we find,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = \frac{2e^4}{t^2}\Big(s^2+u^2-4(m_e^2+m_\mu^2)(s+u)+6(m_e^2+m_\mu^2)^2\Big).$$

This can now be used in the $2 \rightarrow 2$ cross section formula (4.8.3) to give, in the high energy limit, $s, u \gg m_e^2, m_\mu^2$,

$$\frac{d\sigma}{d\Omega^*} = \frac{e^4}{32\pi^2 s}\,\frac{s^2+u^2}{t^2}. \tag{10.2.4}$$

for the differential cross section in the centre of mass frame.

▷ **Exercise 10.2.4**
Derive the result for the electron–muon scattering cross section in equation (10.2.4).

Other calculations of cross sections or decay rates will follow the same steps we have used above. You draw the diagrams, write down the amplitude, square it and evaluate the traces (if you are using spin sum/averages). There are one or two more wrinkles to be aware of, which we will meet below.

## 10.3 Electron–Electron Scattering

Since the two scattered particles are now identical, you can't just replace $m_\mu$ by $m_e$ in the calculation we did above. If you look at the diagram of Figure 10.2.1 (with the muons replaced by electrons) you will see that the outgoing legs can be labelled in two ways. Hence we get the two diagrams of Figure 10.3.1.

The two diagrams give the amplitudes,

$$\begin{aligned} i\mathcal{M}_1 &= \frac{ie^2}{t}\overline{u}(p_c)\gamma^\mu u(p_a)\overline{u}(p_d)\gamma_\mu u(p_b), \\ i\mathcal{M}_2 &= -\frac{ie^2}{u}\overline{u}(p_d)\gamma^\mu u(p_a)\overline{u}(p_c)\gamma_\mu u(p_b). \end{aligned}$$

Notice the additional minus sign in the second amplitude, which comes from the anticommuting nature of fermion fields. You should accept as part of the


Figure 10.4.1 Lowest order Feynman diagrams for electron–positron scattering in QED.

Feynman rules for QED that when diagrams differ by an interchange of two fermion lines, a relative minus sign must be included. This is important because

$$|\mathcal{M}_{fi}|^2 = |\mathcal{M}_1 + \mathcal{M}_2|^2,$$

so the interference term will have the wrong sign if you don't include the extra sign difference between the two diagrams.

## 10.4 Electron–Positron Annihilation

### 10.4.1 $e^+e^- \to e^+e^-$

For this process the two diagrams are shown in Figure 10.4.1, with the one on the right known as the annihilation diagram. They are just what you get from the diagrams for electron–electron scattering in Figure 10.3.1 if you twist round the fermion lines. The fact that the diagrams are related this way implies a relation between the amplitudes. The interchange of incoming particles/antiparticles with outgoing antiparticles/particles is called *crossing*. This is a case where the general results of crossing symmetry can be applied, and our diagrammatic calculations give an explicit realisation. Theorists spent a great deal of time studying such general properties of amplitudes in the 1960's when quantum field theory was unfashionable.

### 10.4.2 $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to$ hadrons

If electrons and positrons collide and produce muon–antimuon or quark-antiquark pairs, then the annihilation diagram is the only one which contributes. At sufficiently high energies that the quark masses can be neglected, this immediately gives the lowest order QED prediction for the ratio of the annihilation cross section into hadrons to that into $\mu^+\mu^-$,

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_f Q_f^2, \tag{10.4.1}$$

where the sum is over quark flavours $f$ and $Q_f$ is the quark's charge in units of $e$. The 3 comes from the existence of three colours for each flavour of quark. Historically this was important: you could look for a step in the value of $R$ as

your $e^+e^-$ collider's CM energy rose through a threshold for producing a new quark flavour. If you didn't know about colour, the height of the step would seem too large. Incidentally, another place the number of colours enters is in the decay of a $\pi^0$ to two photons. There is a factor of 3 in the amplitude from summing over colours, without which the predicted decay rate would be one ninth of its real size.

At the energies used today at LEP, of course, you have to remember the diagram with a $Z$ replacing the photon. We will say some more about this later.

▷ **Exercise 10.4.1**
Show that the cross-section for $e^+e^- \to \mu^+\mu^-$ is equal to $4\pi\alpha^2/(3s)$, neglecting the lepton masses.

## 10.5 Compton Scattering

The diagrams which need to be evaluated to compute the Compton cross section for $\gamma e \to \gamma e$ are shown in Figure 10.5.1. For unpolarised initial and/or final states, the cross section calculation involves terms of the form

$$\sum_\lambda \epsilon_\lambda^{*\mu}(k)\,\epsilon_\lambda^\nu(k), \tag{10.5.1}$$

where $\lambda$ represents the polarisation of the photon of momentum $k$. Since the photon is massless, the sum is over the two transverse polarisation states, and must vanish when contracted with $k_\mu$ or $k_\nu$. In addition, however, since the photon is coupled to the electromagnetic current $J^\mu = \overline{\psi}\gamma^\mu\psi$ of equation (7.2.7), any term in the polarisation sum (10.5.1) proportional to $k^\mu$ or $k^\nu$ does not contribute to the cross section. This is because the current is conserved, $\partial_\mu J^\mu = 0$, so in momentum space $k_\mu J^\mu = 0$. The upshot is that in calculations you can use,

$$\sum_\lambda \epsilon_\lambda^{*\mu}(k)\,\epsilon_\lambda^\nu(k) = -g^{\mu\nu}, \tag{10.5.2}$$

since the remaining terms on the right hand side do not contribute.

The more precise statement is that

$$\sum_\lambda \epsilon_\lambda^{*\mu}(k) a_\mu\,\epsilon_\lambda^\nu(k) b_\nu = -a.b, \tag{10.5.3}$$

provided that $a_\mu$ and $b_\mu$ are conserved currents, that is $k.a(k) = k.b(k) = 0$. For example if

$$a_\nu = \overline{u}(p_1)\gamma_\nu u(p_3)$$

and $k = p_1 - p_3$ then by using the Dirac equation it follows that $k.a = 0$. Current conservation is also related to gauge invariance of the amplitude. For instance if we worked in a general gauge other than Feynman gauge then any diagram


Figure 10.5.1 Feynman diagrams for Compton scattering.

with an internal photon line would contain an extra term in the numerator of the propagator of the form:

$$(\xi - 1)\frac{k^\mu k^\nu}{k^2}$$

However current conservation guarantees that such a term vanishes leading to an amplitude which does not depend upon the value of $\xi$ and hence is independent of the choice of gauge, or gauge invariant. Gauge invariance is a powerful requirement on the theory, and must be maintained to all orders in perturbation theory. However current conservation by itself is insufficient to guarantee gauge invariance beyond the tree-level. The general conditions which guarantee gauge invariance to all orders are called Ward Identities, but this is a topic for Quantum Field Theory II.

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