Quantum Field Theory 1 Christmas Problem

Differential Cross Section for Compton Scattering

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Abstract

Differential cross sections for Compton scattering are computed at tree level for two collinear frames - the center of mass frame and the rest frame of the electron. In the center of mass frame, the differential cross section is found with respect to t, the square momentum transfer between initial and final state photons. In the electron's rest frame, the differential cross section is calculated with respect to scattering angle θ and is shown to reproduce the Klein-Nishina formula. The dependency of both differential cross sections on center of mass energy is plotted and the classical cross section of Thomson scattering is reproduced.

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1 Introduction

The cross section is a physically measurable quantity which gives the likelihood of a scattering event to occur [1]. If we consider a beam of particles incident on a static target of particles, then the cross section can be thought of as the effective area of the target which, if intersected by a particle in the beam, will result in an interaction. Crucially, it is a Lorentz invariant quantity; its value will be independent of the frame we choose to work in [2].

For our purposes it is useful to define the *differential* cross section $d\sigma$, for the scattering of two incident particles. With collinear frames, the differential cross section is related to the square matrix element $\mathcal{M}$ (and hence S-matrix element) by [3]

$$d\sigma = \frac{1}{2E_{i_1}2E_{i_2}|\mathbf{v}_{i_1} - \mathbf{v}_{i_2}|}|\mathcal{M}|^2 d\Pi_n, \tag{1.1}$$

where $E_{i_1}(E_{i_2})$ is the energy of the $1^{\text{st}}(2^{\text{nd}})$ initial state, $\mathbf{v}_{i_1}(\mathbf{v}_{i_2})$ is the velocity of the $1^{\text{st}}(2^{\text{nd}})$ initial state related to the momenta by $\mathbf{v} = \mathbf{p}/p^0$ and

$$\int d\Pi_n = \left(\prod_f \int \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^4\left(\sum p_f - \sum p_i\right) \tag{1.2}$$

is known as the phase space integral where p_f and E_f denote the momenta and energies of the final states and p_i represents initial state momenta.

Upon integration, equation (1.1) gives the cross section for scattering into the final-state momentum ranges $d^3p_1...d^3p_n$ [1]. When there are only two products in the final state, we can consider the likelihood for scattering to occur in some element of solid angle $d\Omega = \sin\theta d\theta d\phi$ [2]. Momentum conservation $p_1 + p_2 = p_1' + p_2'$ constrains four components i.e. $|\mathbf{p}_1'|$ and $\mathbf{p}_2'$. This leaves $\int d^3p_1' d^3p_2'$ containing two unconstrained components, which can manifest themselves as the angles θ and ϕ of the momentum of one of the particles. Note that in general the differential cross section is not frame independent as the scattering angles (θ, ϕ) depend on the frame of reference.

In the following we compute the differential cross section for Compton scattering $e^-\gamma \to e^-\gamma$ at tree level. We look at two cases of collinear frames - the center of mass frame and the "lab" frame, in which the electron is initially at rest. We proceed by calculating the summed/averaged matrix element for the process, introducing spin and polarization sums. In the following section we find the phase space integral in the center of mass frame and, together with the matrix element, build the relevant differential cross section $d\sigma/dt$. In section 3 the phase space integral and subsequently, differential cross section $d\sigma/d\cos\theta$, are calculated in the rest frame of the electron. An alternative method of finding $d\sigma/d\cos\theta$ is outlined in this section as well. Finally we explore the dependence of the differential cross sections we find on the center of mass energy of the scattering interaction.

2 Calculation of the square matrix element

2.1 Matrix element for Compton scattering

Two Feynman diagrams contribute to Compton scattering at tree level:

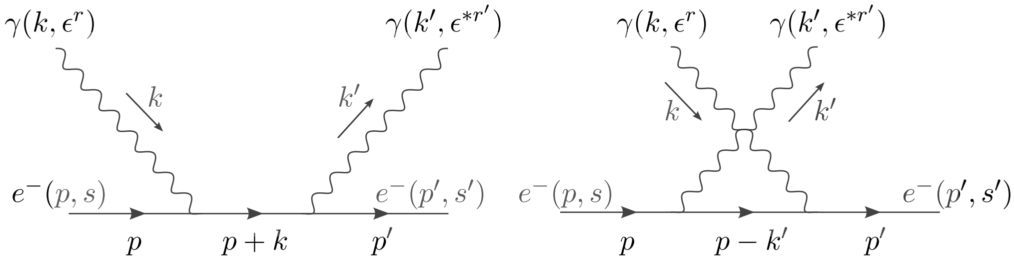

Figure 1: Scattering processes A and B respectively where ϵ^r and $\epsilon^{*r'}$ are polarization vectors of incoming and outgoing photons, s and s' label the spins of incoming and outgoing electrons and p and k label momenta.

The matrix elements for processes A and B can be immediately written down from the Feynman rules. Suppressing spinor indices, we have

$$i\mathcal{M}_A = -ie^2[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu \epsilon_\mu^{*r'} \frac{(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} \gamma^\nu \epsilon_\nu^r u^s(\mathbf{p})] \tag{2.1}$$

and

$$i\mathcal{M}_B = -ie^2[\bar{u}^{s'}(\mathbf{p}')\gamma^\nu \epsilon_\nu^{r} \frac{(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2} \gamma^\mu \epsilon_\mu^{*r'} u^s(\mathbf{p})], \tag{2.2}$$

where the factor of $i\epsilon$ in the denominator is ignored as we have no integrations over momenta. As there is no exchange of fermion legs between the two diagrams, there is no relative minus sign between the two terms in the total matrix element $i\mathcal{M}$, given by

$$\begin{aligned} i\mathcal{M} &= i\mathcal{M}_A + i\mathcal{M}_B \qquad (2.3)\\ &= -ie^2\epsilon_\mu^{*r'}\epsilon_\nu^r[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu \frac{(\not{p} + \not{k} + m)}{(p+k)^2 - m^2}\gamma^\nu u^s(\mathbf{p}) + \bar{u}^{s'}(\mathbf{p}')\gamma^\nu \frac{(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2}\gamma^\mu u^s(\mathbf{p})]. \end{aligned}$$

Conservation of momentum $p + k = p' + k'$, together with the mass shell conditions $p^2 = p'^2 = m^2$ and $k^2 = k'^2 = 0$, give

$$(p+k)^2 - m^2 = 2p.k \qquad \text{and} \qquad (p-k')^2 - m^2 = -2p.k'. \tag{2.4}$$

Using these, we can write the total matrix element more simply as

$$i\mathcal{M} = -ie^2\epsilon_\mu^{*r'}\epsilon_\nu^r[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu\frac{(\not{p}+\not{k}+m)}{2p.k}\gamma^\nu u^s(\mathbf{p}) + \bar{u}^{s'}(\mathbf{p}')\gamma^\nu\frac{(\not{p}-\not{k}'+m)}{-2p.k'}\gamma^\mu u^s(\mathbf{p})]. \quad (2.5)$$

The polarizations and spins of the initial and final states in this process are not measured experimentally and are therefore summed or averaged over [2]. Specifically we average over incoming spins and polarizations and sum over outgoing spins and polarizations. We replace $|\mathcal{M}|^2$ by the summed/averaged matrix element X in the equation for the differential cross section (1.1). Thus the relevant quantity we wish to compute is given by

$$\begin{aligned} X &= \frac{1}{4}\sum_{s,r,s',r'}|\mathcal{M}|^2 \qquad (2.6)\\ &= \frac{1}{4}\sum_{s,r,s',r'}|\mathcal{M}_A|^2 + \frac{1}{4}\sum_{s,r,s',r'}|\mathcal{M}_B|^2 + \frac{1}{4}\sum_{s,r,s',r'}\mathcal{M}_A\mathcal{M}_B^* + \frac{1}{4}\sum_{s,r,s',r'}\mathcal{M}_A^*\mathcal{M}_B. \end{aligned}$$

For ease of computation in what follows, we rewrite the above as

$$X = \frac{e^4}{16}\left[\frac{Z_{AA}}{(p.k)^2} + \frac{Z_{BB}}{(p.k')^2} - \frac{Z_{AB}+Z_{BA}}{(p.k)(p.k')}\right], \quad (2.7)$$

where

$$\begin{aligned} Z_{AA} &= \sum_{s,r,s',r'}|\bar{u}^{s'}(\mathbf{p}')\gamma^\mu\epsilon_\mu^{*r'}(\not{p}+\not{k}+m)\gamma^\nu\epsilon_\nu^r u^s(\mathbf{p})|^2 \qquad (2.8)\\ &= \sum_{s,r,s',r'}\epsilon_\mu^{*r'}\epsilon_\nu^r\epsilon_\sigma^{*r}\epsilon_\rho^{r'}[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu u^s(\mathbf{p})][\bar{u}^{s'}(\mathbf{p}')\gamma^\rho(\not{p}+\not{k}+m)\gamma^\sigma u^s(\mathbf{p})]^*\\ &= \sum_{s,r,s',r'}\epsilon_\mu^{*r'}\epsilon_\nu^r\epsilon_\sigma^{*r}\epsilon_\rho^{r'}[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu u^s(\mathbf{p})\bar{u}^s(\mathbf{p})\gamma^\sigma(\not{p}+\not{k}+m)\gamma^\rho u^{s'}(\mathbf{p}')]\\ Z_{BB} &= \sum_{s,r,s',r'}|\bar{u}^{s'}(\mathbf{p}')\gamma^\nu\epsilon_\nu^r(\not{p}-\not{k}'+m)\gamma^\mu\epsilon_\mu^{*r'}u^s(\mathbf{p})|^2\\ &= \sum_{s,r,s',r'}\epsilon_\nu^r\epsilon_\mu^{*r'}\epsilon_\sigma^{r'}\epsilon_\rho^{*r}[\bar{u}^{s'}(\mathbf{p}')\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu u^s(\mathbf{p})\bar{u}^s(\mathbf{p})\gamma^\sigma(\not{p}-\not{k}'+m)\gamma^\rho u^{s'}(\mathbf{p}')]\\ Z_{AB} &= \sum_{s,r,s',r'}[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu\epsilon_\mu^{*r'}(\not{p}+\not{k}+m)\gamma^\nu\epsilon_\nu^r u^s(\mathbf{p})][\bar{u}^{s'}(\mathbf{p}')\gamma^\rho\epsilon_\rho^r(\not{p}-\not{k}'+m)\gamma^\sigma\epsilon_\sigma^{*r'}u^s(\mathbf{p})]^*\\ &= \sum_{s,r,s',r'}\epsilon_\mu^{*r'}\epsilon_\nu^r\epsilon_\sigma^{r'}\epsilon_\rho^{*r}[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu u^s(\mathbf{p})\bar{u}^s(\mathbf{p})\gamma^\sigma(\not{p}-\not{k}'+m)\gamma^\rho u^{s'}(\mathbf{p}')]\\ Z_{BA} &= \sum_{s,r,s',r'}[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu\epsilon_\mu^{*r'}(\not{p}+\not{k}+m)\gamma^\nu\epsilon_\nu^r u^s(\mathbf{p})]^*[\bar{u}^{s'}(\mathbf{p}')\gamma^\rho\epsilon_\rho^r(\not{p}-\not{k}'+m)\gamma^\sigma\epsilon_\sigma^{*r'}u^s(\mathbf{p})]\\ &= \sum_{s,r,s',r'}\epsilon_\nu^{*r}\epsilon_\mu^{r'}\epsilon_\rho^r\epsilon_\sigma^{*r'}[\bar{u}^s(\mathbf{p})\gamma^\nu(\not{p}+\not{k}+m)\gamma^\mu u^{s'}(\mathbf{p}')\bar{u}^{s'}(\mathbf{p}')\gamma^\rho(\not{p}-\not{k}'+m)\gamma^\sigma u^s(\mathbf{p})]. \end{aligned}$$

Here we have used the identity $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ to perform the complex conjugation. Upon inspection we see that Z_{AA} is equivalent to Z_{BB} with $k \leftrightarrow k'$ and similarly for Z_{AB} and Z_{BA}. Hence only two of the four Z_{XX} terms will have to be computed in (2.7).

2.2 Spin and polarization sums

We wish to simplify our Z_{XX} expressions. We proceed by summing over polarizations; making use of the replacement [2]

$$\sum_r \epsilon_\mu^{*r}\epsilon_\nu^r \rightarrow -\eta_{\mu\nu}, \tag{2.9}$$

with $\eta_{\mu\nu}$ the usual Minkowski metric. Introducing this prescription to Z_{AA}, we obtain

$$\begin{aligned} Z_{AA} &= \sum_{s,s'} \eta_{\mu\rho}\eta_{\nu\sigma}[\bar{u}^{s'}(\mathbf{p}')\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu u^s(\mathbf{p})\bar{u}^s(\mathbf{p})\gamma^\sigma(\not{p}+\not{k}+m)\gamma^\rho u^{s'}(\mathbf{p}')] \\ &= \sum_{s,s'} \bar{u}^{s'}(\mathbf{p}')\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu u^s(\mathbf{p})\bar{u}^s(\mathbf{p})\gamma_\nu(\not{p}+\not{k}+m)\gamma_\mu u^{s'}(\mathbf{p}'). \end{aligned} \tag{2.10}$$

Similarly, for Z_{AB} this substitution yields

$$Z_{AB} = \sum_{s,s'} \bar{u}^{s'}(\mathbf{p}')\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu u^s(\mathbf{p})\bar{u}^s(\mathbf{p})\gamma_\mu(\not{p}-\not{k}'+m)\gamma_\nu u^{s'}(\mathbf{p}'). \tag{2.11}$$

With a view to sum over spins, let us briefly introduce spinor indices (denoted by lower case roman letters) such that for Z_{AA} for example, we write

$$Z_{AA} = \sum_{s,s'} (\bar{u}^{s'}(\mathbf{p}'))_a(\gamma^\mu)^a{}_b(\not{p}+\not{k}+m)^b{}_c(\gamma^\nu)^c{}_d(u^s(\mathbf{p})\bar{u}^s(\mathbf{p}))^d{}_e(\gamma_\nu)^e{}_f(\not{p}+\not{k}+m)^f{}_g(\gamma_\mu)^g{}_h u^{s'}(\mathbf{p}')^h.$$

With all indices explicitly shown, it is clear that we can manipulate the above expression accordingly

$$\begin{aligned} Z_{AA} &= \sum_{s,s'} (\gamma^\mu)^a{}_b(\not{p}+\not{k}+m)^b{}_c(\gamma^\nu)^c{}_d(u^s(\mathbf{p})\bar{u}^s(\mathbf{p}))^d{}_e(\gamma_\nu)^e{}_f(\not{p}+\not{k}+m)^f{}_g(\gamma_\mu)^g{}_h(u^{s'}(\mathbf{p}')\bar{u}^{s'}(\mathbf{p}'))^h{}_a \\ &= \sum_{s,s'} \mathrm{Tr}[\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu u^s(\mathbf{p})\bar{u}^s(\mathbf{p})\gamma_\nu(\not{p}+\not{k}+m)\gamma_\mu u^{s'}(\mathbf{p}')\bar{u}^{s'}(\mathbf{p}')]. \end{aligned}$$

Now, $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$, we can trivially take the sum inside the trace and, using the completeness relation [3]

$$\sum_s u^s(\mathbf{p})\bar{u}^s(\mathbf{p}) = \not{p} + m, \tag{2.12}$$

we can perform the final sum over spins, giving

$$Z_{AA} = \mathrm{Tr}[\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu(\not{p} + m)\gamma_\nu(\not{p} + \not{k} + m)\gamma_\mu(\not{p}' + m)]. \tag{2.13}$$

A similar calculation for Z_{AB} yields

$$Z_{AB} = \mathrm{Tr}[\gamma^\mu(\not{p} + \not{k} + m)\gamma^\mu(\not{p} + m)\gamma_\mu(\not{p} - \not{k}' + m)\gamma_\nu(\not{p}' + m)]. \tag{2.14}$$

As noted before, Z_{BB} and Z_{BA} are simply given by the interchange $k \leftrightarrow k'$ in expressions (2.13) and (2.14) respectively.

2.3 Gamma matrices and trace manipulations

To find more useful forms of the Z_{XX} expressions, we use the following properties of the gamma matrices [1]:

$$\begin{aligned}
\gamma^\nu\gamma_\nu &= 4 && (2.15)\\
\gamma^\nu\gamma^\rho\gamma_\nu &= -2\gamma^\rho && (2.16)\\
\gamma^\nu\gamma^\mu\gamma^\rho\gamma_\nu &= 4\eta^{\mu\rho} && (2.17)\\
\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\nu &= -2\gamma^\sigma\gamma^\rho\gamma^\mu && (2.18)\\
\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= 4\eta^{\mu\nu} && (2.19)\\
\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) && (2.20)\\
\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n+1}}] &= 0. && (2.21)
\end{aligned}$$

First of all we use the cyclicity of the trace to write (2.13) as

$$\begin{aligned}
Z_{AA} &= \mathrm{Tr}[(\not{p} + \not{k} + m)\gamma^\nu(\not{p} + m)\gamma_\nu(\not{p} + \not{k} + m)\gamma_\mu(\not{p}' + m)\gamma^\mu] && (2.22)\\
&= \mathrm{Tr}[(\not{p} + \not{k} + m)(\gamma^\nu\gamma^\rho p_\rho\gamma_\nu + \gamma^\nu m\gamma_\nu)(\not{p} + \not{k} + m)(\gamma_\mu\gamma^\sigma p'_\sigma\gamma^\mu + \gamma_\mu m\gamma^\mu)].
\end{aligned}$$

Now using the first two properties (2.15) and (2.16) above, we obtain

$$Z_{AA} = \mathrm{Tr}[(\not{p} + \not{k} + m)(4m - 2\not{p})(\not{p} + \not{k} + m)(4m - 2\not{p}')]. \tag{2.23}$$

Expanding this expression and by use of property (2.21), Z_{AA} reduces to a trace of only even combinations of gamma matrices (in this case, two or four). We have

$$\begin{aligned}
Z_{AA} = \mathrm{Tr}[&16m^2\not{p}\not{k} - 12m^2\not{p}\not{p}' + 16m^2\not{k}\not{k} - 16m^2\not{k}\not{p}' && (2.24)\\
&+4\not{p}\not{p}\not{p}\not{p}' + 4\not{p}\not{p}\not{k}\not{p}'' + 4\not{k}\not{p}\not{p}\not{p}' + 4\not{k}\not{p}\not{k}\not{p}' + 16m^4].
\end{aligned}$$

To perform the trace over four gamma matrices, we use identity (2.20). For example, taking the 7th term in the above expression, this gives

$$\begin{aligned}
\mathrm{Tr}[4\not{p}\not{p}\not{k}\not{p}'] &= 4\mathrm{Tr}[\gamma^\mu p_\mu \gamma^\nu p_\nu \gamma^\rho k_\rho \gamma^\sigma p'_\sigma] \\
&= 4\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] p_\mu p_\nu k_\rho p'_\sigma \\
&= 16(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) p_\mu p_\nu k_\rho p'_\sigma.
\end{aligned} \tag{2.25}$$

Terms involving two gamma matrices require (2.19). For example

$$\begin{aligned}
\mathrm{Tr}[8m^2 \not{p}\not{k}] &= 8m^2 \mathrm{Tr}[\gamma^\mu \gamma^\nu] p_\mu k_\nu \\
&= 32m^2 \eta^{\mu\nu} p_\mu k_\nu \\
&= 32m^2 p^\nu k_\nu.
\end{aligned} \tag{2.26}$$

After performing the traces over all gamma matrices we find that

$$\begin{aligned}
Z_{AA} &= 64m^2(p.k) - 48m^2(p.p') - 64m^2 k'.p + 64m^2 k.k + 64m^4 \\
&+ 16(p.p')(p.p) + 32(p.p)(k.p') + 32(k.p)(k.p') - 16(k.k)(p.p'),
\end{aligned} \tag{2.27}$$

where we have contracted indices using $\eta^{\mu\nu}$. Note that when carrying out the trace over the m^4 term, we have to take account of the implicit 4 by 4 identity matrix i.e. we in fact compute $\mathrm{Tr}[16m^4 \mathbb{1}] = 16m^4 \mathrm{Tr}[\mathbb{1}]$ where $\mathrm{Tr}[\mathbb{1}] = 4$. This explains the appearance of the cofactor 64.

Again we can use the mass-shell conditions and conservation of momentum to simplify our expression. From these we obtain

$$p'.k = p.k' \qquad \text{and} \qquad p.p' = m^2 + p.k - p.k'. \tag{2.28}$$

Finally, we have

$$Z_{AA} = 32(m^2 p.k + (p.k)(p.k') + m^4). \tag{2.29}$$

The computation of Z_{AB} follows in the same vein. In this case, the cyclicity of the trace offers no immediate simplification and we must expand all terms. We find

$$\begin{aligned}
Z_{AB} &= \mathrm{Tr}[\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu(\not{p} + m)\gamma_\mu(\not{p} - \not{k}' + m)\gamma_\nu(\not{p}' + m)] \qquad (2.30) \\
&= \mathrm{Tr}[8\eta^{\mu\nu} p_\mu k'_\nu \not{p}\not{p}' - 8\eta^{\mu\nu} p_\mu p_\nu \not{p}\not{p}' + 12m^2 \not{p}\not{p} - 8m^2 \not{k}'\not{p} \\
&- 8\eta^{\mu\nu} k_\mu p_\nu \not{p}\not{p}' + 8\eta^{\mu\nu} k_\mu k_\nu \not{p}\not{p}' + 8m^2 \not{p}\not{k} - 4m^2 \not{k}'\not{k} + 12m^2 \not{p}\not{p}' \\
&- 4m^2 \not{k}'\not{p}' + 4m^2 \not{k}\not{p}' - 8m^4],
\end{aligned}$$

where we have employed the additional identities (2.17) and (2.18) in going from the first to the second equality. After performing traces over gamma matrices and using the additional following relations obtained from conservation of momentum,

$$p'.k' = p.k \qquad \text{and} \qquad k.k' = p.k - p.k', \tag{2.31}$$

we find

$$Z_{AB} = 16(2m^4 + (p.k) - (p.k')). \tag{2.32}$$

Altogether we have

$$\begin{aligned} Z_{AA} &= 32(m^2 p.k + (p.k)(p.k') + m^4), && (2.33)\\ Z_{BB} &= 32(-m^2 p.k + (p.k')(p.k) + m^4), && (2.34)\\ Z_{AB} &= Z_{BA} = 16m^2(2m^2 + (p.k) - (p.k')). && (2.35) \end{aligned}$$

Finally, plugging these expressions into our definition (2.36) of the matrix element X, we obtain

$$X = 2e^4\left[\frac{p.k'}{p.k} + \frac{p.k}{p.k'} + 2m^2\left(\frac{1}{p.k} - \frac{1}{p.k'}\right) + m^4\left(\frac{1}{p.k} - \frac{1}{p.k'}\right)^2\right]. \tag{2.36}$$

3 Differential cross section in center of mass frame

3.1 Phase space integral

The last ingredient we need to calculate in our formula for the differential cross section (1.1) is the phase space integral. We specialize to a particular frame of reference to obtain a more explicit expression. Let us choose the center of mass frame for simplicity. We picture the kinematics as

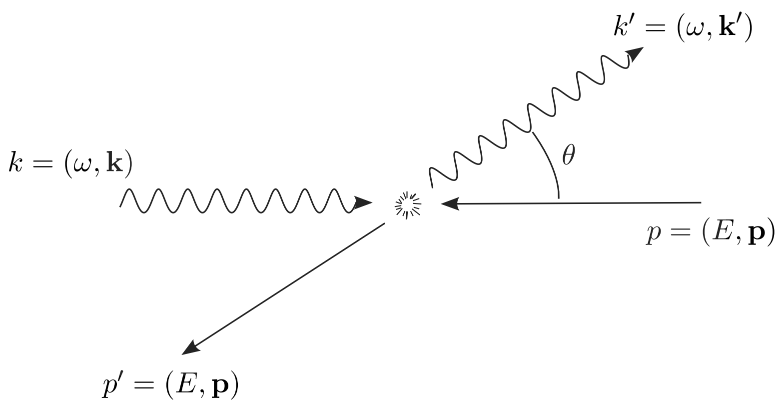

Figure 2: Compton scattering in the center of mass frame.

Note that the initial and final energy of the photon are the same and similarly for the electron. In fact, in the center of mass frame we also have the relation $\sum \mathbf{p}_i = \sum \mathbf{p}_f = 0$ and thus $\mathbf{k} = -\mathbf{p}$ and $\mathbf{k}' = -\mathbf{p}'$.

From the definition (1.2) of the phase space integral, this process gives

$$\begin{aligned} \int d\Pi_2 &= \int \frac{d^3p'}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{2E'2\omega'} (2\pi)^{(4)} \delta^4(p' + k' - p - k), \\ &= \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2E'2\omega'} (2\pi)\delta(E' + \omega' - E - \omega). \end{aligned} \tag{3.1}$$

In going to the second line we have performed the d^3p' integral using the delta function which sets $\mathbf{p}' = -\mathbf{k}'$ and sends $E'(\mathbf{p}') \to E'(\mathbf{k}') = \sqrt{|\mathbf{k}'|^2 + m^2}$.

Transforming to spherical polar coordinates, we have

$$\int d\Pi_2 = \int \frac{dk'}{(2\pi)^3} \frac{d\Omega |\mathbf{k}'|^2}{2E'2\omega'} (2\pi)\delta(E' + \omega' - E - \omega). \tag{3.2}$$

Using the relation

$$\delta(f(x) - f(x_0)) = \frac{\delta(x - x_0)}{|f'(x_0)|} \tag{3.3}$$

and viewing $E' + \omega'$ as a function of $|\mathbf{k}'|$ (since the zero mass of the photon implies $|\mathbf{k}'| = \omega'$), we can perform the dk' integral to find

$$\int d\Pi_2 = \int \frac{d\Omega}{(2\pi)^2} \frac{\omega'}{4E'} \frac{E'}{(\omega' + E')}. \tag{3.4}$$

As the process is symmetric about the collision axis, we can simplify this expression further by integrating the ϕ degree of freedom between 0 and 2π and thus

$$\int d\Pi_2 = \int \frac{d\cos\theta}{8\pi} \frac{\omega'}{\omega' + E'}. \tag{3.5}$$

3.2 Differential cross section

We now have everything we need to construct the differential cross section for Compton scattering. In particular we can compute the cross section with respect to the square momentum transfer between the initial and final state photons, given by the Mandelstam variable t. In the center of mass frame [1]

$$t = (k' - k)^2 = -2k.k' = -2\omega\omega'(1 - \cos\theta), \tag{3.6}$$

where we have used the mass-shell condition $k'^2 = k^2 = 0$ and that $\cos(180 - \theta) = \cos\theta$. Since the energy of the final state photon is independent of θ in the center of mass frame, it is straightforward to express the phase space integral (3.5) in terms of t:

$$dt = 2\omega\omega' d\cos\theta \implies \int d\Pi_2 = \int \frac{dt}{16\pi} \frac{1}{\omega(\omega' + E')}. \tag{3.7}$$

Using the definition (1.1) we are now able to write down an expression for the differential cross section with respect to t:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi} \frac{1}{\omega^2} \frac{1}{|\mathbf{v}_{i_1} - \mathbf{v}_{i_2}|} \frac{1}{E(\omega' + E')} X. \tag{3.8}$$

From conservation of energy $E + \omega = E' + \omega'$ and recalling that $\mathbf{v} = \mathbf{p}/p^0$, we have

$$E(\omega' + E')|\mathbf{v}_{i_1} - \mathbf{v}_{i_2}| = E(\omega + E)|\frac{\mathbf{k}}{\omega} - \frac{\mathbf{p}}{E}|. \tag{3.9}$$

Using the conditions $|\mathbf{k}| = \omega$ and $\sum \mathbf{p}_i = 0$, the above reduces to

$$E(\omega + E)\omega\left(\frac{E + \omega}{E\omega}\right) = (E + \omega)^2. \tag{3.10}$$

Thus our differential cross section simplifies to

$$\frac{d\sigma}{dt} = \frac{1}{64\pi} \frac{1}{\omega^2(E + \omega)^2} X. \tag{3.11}$$

We can rewrite the above equation completely in terms of Lorentz invariant quantities by introducing the Mandelstam variable s. In the center of mass frame, s is related to ω by

$$\begin{aligned} s &= (p + k)^2 = m^2 + 2\omega(E + \omega) \qquad (3.12) \\ \implies &\quad \omega^2 = \frac{(s - m^2)^2}{4s}, \end{aligned}$$

where we have used the fact that the center of mass energy $E_{cm} \equiv \sqrt{s} = (E+\omega)$. Therefore we find

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} \frac{1}{(s - m^2)^2} X. \tag{3.13}$$

Even though we calculated this differential cross section in the center of mass frame, as it only contains Lorentz scalars, it holds in any collinear frame of reference.

4 Differential cross section in rest frame of electron

4.1 Phase space integral

We can also analyse Compton scattering in the rest frame of the electron also referred to as the lab frame. For the kinematics we have

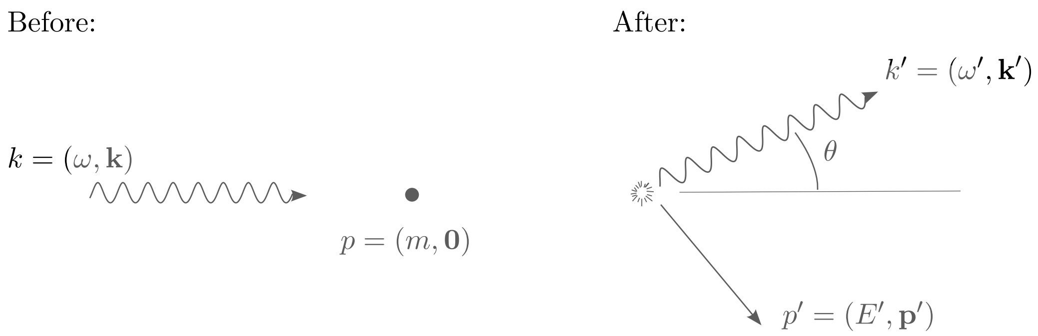

Figure 3: Compton scattering in the lab frame.

(Note that θ, ω and ω' do not necessarily have the same values as of those in Figure 2. We will label the photon's initial and final energy in the lab frame by ω_L and ω'_L when there is a risk of ambiguity.)

To compute the phase space integral in the lab frame, we again use the delta function to perform the d^3p' integration. This sets $\mathbf{p'} = \mathbf{k} - \mathbf{k'}$ due to the electron having zero initial momentum. We find,

$$\int d\Pi_2 = \int \frac{d^3p'}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{2E'2\omega'}(2\pi)^{(4)}\delta^4(p'+k'-p-k), \tag{4.1}$$

$$= \int \frac{dk'}{(2\pi)^3}\frac{d\Omega\omega'}{4E'}(2\pi)\delta(E'+\omega'-E-\omega). \tag{4.2}$$

where $E' = \sqrt{|\mathbf{k}-\mathbf{k}'|^2+m^2} = \sqrt{\omega^2+\omega'^2-2\omega\omega'\cos\theta+m^2}$. As before we perform the dk' integral by using identity (3.3) and viewing $E'+\omega'$ as a function of $|\mathbf{k}'|$. We find

$$\int d\Pi_2 = \int \frac{d\Omega}{(2\pi)^2}\frac{\omega'}{4E'}\frac{1}{|1+\frac{\omega'-\omega\cos\theta}{E'}|}. \tag{4.3}$$

By conservation of energy $E'+\omega' = m+\omega$ and integrating over $d\phi$, we have

$$\int d\Pi_2 = \int \frac{d\cos\theta}{8\pi}\frac{\omega'}{m+\omega(1-\cos\theta)}. \tag{4.4}$$

We can simplify this expression by using Compton's formula for the shift in the photon wavelength [2]. We derive this as follows:

$$\begin{aligned} m^2 &= (p')^2 = (p+k-k')^2 = p^2+2p(k-k')-2k.k' \\ &= m^2+2m(\omega-\omega')-2\omega\omega'(1-\cos\theta) \\ \implies & \frac{1}{\omega'}-\frac{1}{\omega} = \frac{1}{m}(1-\cos\theta). \end{aligned} \tag{4.5}$$

With this equation (4.4) reduces to

$$d\Pi_2 = \int \frac{d\cos\theta}{8\pi}\frac{(\omega')^2}{\omega m}. \tag{4.6}$$

4.2 Differential cross section

We are now in a position to write down the differential cross section in the lab frame with respect to the scattering angle θ. From the definition (1.1) we have

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2m2\omega}\frac{1}{|\mathbf{v}_{i_1} - \mathbf{v}_{i_2}|}\frac{1}{8\pi}\frac{(\omega')^2}{\omega m}X. \tag{4.7}$$

In the lab frame, the electron is initially at rest so $\mathbf{v}_{i_2} = 0$ and, as the photon is massless, it follows that $|\mathbf{v}_{i_1} - \mathbf{v}_{i_2}| = 1$, hence (4.7) becomes

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi}\frac{(\omega')^2}{\omega^2 m^2}X. \tag{4.8}$$

Recall X is given by

$$X = 2e^4\left[\frac{p.k'}{p.k} + \frac{p.k}{p.k'} + 2m^2\left(\frac{1}{p.k} - \frac{1}{p.k'}\right) + m^4\left(\frac{1}{p.k} - \frac{1}{p.k'}\right)^2\right]. \tag{4.9}$$

To evaluate X, we replace $p.k$ and $p.k'$ by $m\omega$ and $m\omega'$ respectively. We find

$$\begin{aligned} X &= 2e^4\left[\frac{m\omega'}{m\omega} + \frac{m\omega}{m\omega'} + 2m^2\left(\frac{1}{m\omega} - \frac{1}{m\omega'}\right) + m^4\left(\frac{1}{\omega m} - \frac{1}{\omega' m}\right)^2\right] \\ &= 2e^4\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2(\cos\theta - 1) + (\cos\theta - 1)^2\right] \\ &= 2e^4\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \end{aligned} \tag{4.10}$$

where we have used (4.5) in going from the first to the second line.

Putting this altogether we have,

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right], \tag{4.11}$$

where $\alpha = e^2/4\pi$ is the fine structure constant. The resulting equation is the Klein-Nishina formula [1].

4.3 Alternative approach

We can also find the result (4.8) by direct conversion of dt to $d\cos\theta$. In the lab frame we have

$$\begin{aligned} t &= (k'-k)^2 = -2\omega\omega'(1-\cos\theta), \qquad (4.12)\\ \implies dt &= -2\omega d\omega' + 2\omega\omega' d\cos\theta + 2\omega\cos\theta d\omega', \end{aligned}$$

where remember ω' is a function of θ. From Compton's formula (4.5) we find

$$d\omega' = \frac{(\omega')^2}{m} d\cos\theta, \qquad (4.13)$$

and so we have

$$\begin{aligned} dt &= \left[-\frac{2\omega(\omega')^2}{m} + 2\omega\omega' + 2\frac{\omega(\omega')^2}{m}\cos\theta \right] d\cos\theta \\ &= \left[2\omega\omega' - 2(\omega')^2 \left(\frac{\omega}{m}(1-\cos\theta) \right) \right] d\cos\theta \qquad (4.14)\\ &= \left[2\omega\omega' - 2(\omega')^2 \left(\frac{\omega}{\omega'} - 1 \right) \right] \\ &= 2(\omega')^2 d\cos\theta. \end{aligned}$$

Hence

$$\left(\frac{d\sigma}{d\cos\theta} \right)_{\text{lab}} = 2(\omega'_L)^2 \left(\frac{d\sigma}{dt} \right)_{\text{com}} = \frac{1}{8\pi} \frac{(\omega_L)^2}{(s-m^2)^2} X, \qquad (4.15)$$

where we have used equation (3.13) in going to the last equality and introduced the subscript L to show that ω' is in the photon's final energy in the lab frame. In the lab frame $s = m^2 + 2m\omega$ and thus upon substitution for $(s-m^2)$, we find

$$\left(\frac{d\sigma}{d\cos\theta} \right)_{\text{lab}} = \frac{1}{32\pi} \frac{(\omega')^2}{m^2\omega^2} X, \qquad (4.16)$$

which agrees with the previous result (4.8)!

5 Cross section dependence on center of mass energy

5.1 $d\sigma/d\cos\theta$ against $\cos\theta$

Recall that in the lab frame, our differential cross section was given by

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2} \left(\frac{\omega'}{\omega} \right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \right]. \qquad (5.1)$$

We want to see how $d\sigma/d\cos\theta$ varies, as a function of $\cos\theta$, with differing values of $\sqrt{s}$. To express the above equation as a function of $\cos\theta$ and s, we first need to substitute for ω' using Compton's formula (4.5). This gives

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(1+\frac{\omega}{m}(1-\cos\theta)\right)^{-2}\left[\left(1+\frac{\omega}{m}(1-\cos\theta)\right)^{-1}+\left(1+\frac{\omega}{m}(1-\cos\theta)\right)+\cos^2\theta-1\right].$$

We can write ω in terms of s in the lab frame by using (5.6):

$$s = m^2 + 2m\omega \implies \omega = \frac{s-m^2}{2m}. \tag{5.2}$$

Thus for the cross section we have

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(1+\frac{s-m^2}{2m^2}(1-\cos\theta)\right)^{-2}\left[\left(1+\frac{s-m^2}{2m^2}(1-\cos\theta)\right)^{-1} + \left(1+\frac{s-m^2}{2m^2}(1-\cos\theta)\right)+\cos^2\theta-1\right]. \tag{5.3}$$

We plot $d\sigma/d\cos\theta$ against $\cos\theta$ for three different values of center of mass energy in the lab frame: $(s-m^2) \ll m^2$ where $\sqrt{s} > m$, $(s-m^2) \sim m^2$ and $s \gg m^2$.

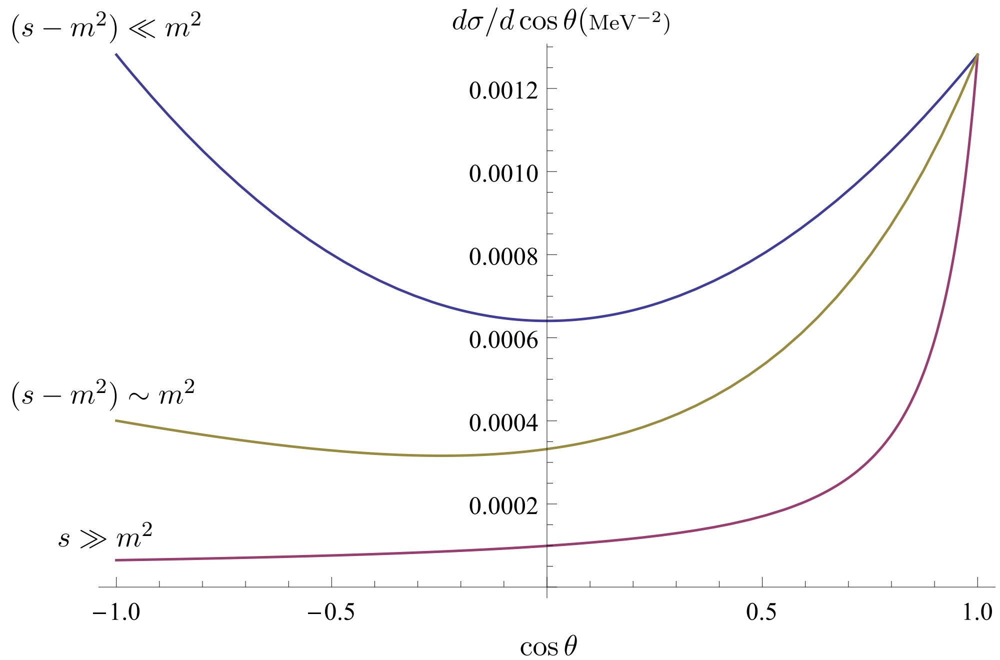

Figure 4: Angular dependence of the differential cross section for three different values of $\sqrt{s}$. $(s-m^2) \ll m^2$ in blue, $(s-m^2) \sim m^2$ in mustard and $s \gg m^2$ in purple.

The low energy limit corresponds to $\omega \to 0$ which, according to (4.5), implies $\omega'/\omega \to 1$ and thus the cross section (4.11) becomes

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1+\cos^2\theta) \implies \sigma = \frac{8\pi\alpha^2}{3m^2}. \tag{5.4}$$

This is the cross section of classical Thomson scattering of electromagnetic radiation by a free electron in which there is no observed shift in the photon's wavelength [1]. The low energy curve takes its highest values for scattering angles close to 0 or π, indicating that forward or backwards scattering of the photon are most likely and also equally probable as the curve is symmetric.

As we move to higher energies, the differential cross section decreases in value as $\theta \to \pi$ evidencing that back scattering becomes increasingly unlikely and forward scattering is favoured.

Comparing all three curves, we see that scattering is more likely to happen at low energy for all values of scattering angle θ apart from at $\theta = 0$ when it is equally likely at all energies considered.

5.2 $d\sigma/dt$ against t

In the center of mass frame we found the differential cross section was given by

$$\frac{d\sigma}{dt} = \frac{1}{16\pi}\frac{1}{(s-m^2)^2}X, \tag{5.5}$$

where X is given by (2.36). We want to see how $d\sigma/dt$ behaves, as a function of t, for differing values of center of mass energy $\sqrt{s}$. For this, we need to express the differential cross section in terms of the Mandelstam variables [1]:

$$s = (p+k)^2 = 2p.k + m^2, \tag{5.6}$$

$$u = (k'-p)^2 = -2k'.p + m^2, \tag{5.7}$$

$$t = (p'-p)^2 = -2p.p' + m^2. \tag{5.8}$$

These three variables are related through the identity [1]

$$s + t + u = \sum_{i=1}^{4} m_i^2, \tag{5.9}$$

where the sum runs over the four external particles.

In the matrix element X, we replace $p.k$ by $\frac{1}{2}(s-m^2)$ and $p.k'$ by $\frac{1}{2}(t+s-m^2)$, using equations (5.6) to (5.9). We find

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha^2}{(s-m^2)^2}\left[\frac{t+s-m^2}{s-m^2} + \frac{s-m^2}{t+s-m^2} + 2m^2\left(\frac{2}{s-m^2} - \frac{2}{t+s-m^2}\right) + m^4\left(\frac{2}{s-m^2} - \frac{2}{t+s-m^2}\right)^2\right]. \tag{5.10}$$

We plot $d\sigma/dt$ against t for three different values of center of mass energy: $(s-m^2) \ll m^2$ where $\sqrt{s} > m$, $(s-m^2) \sim m^2$ and $s \gg m^2$.

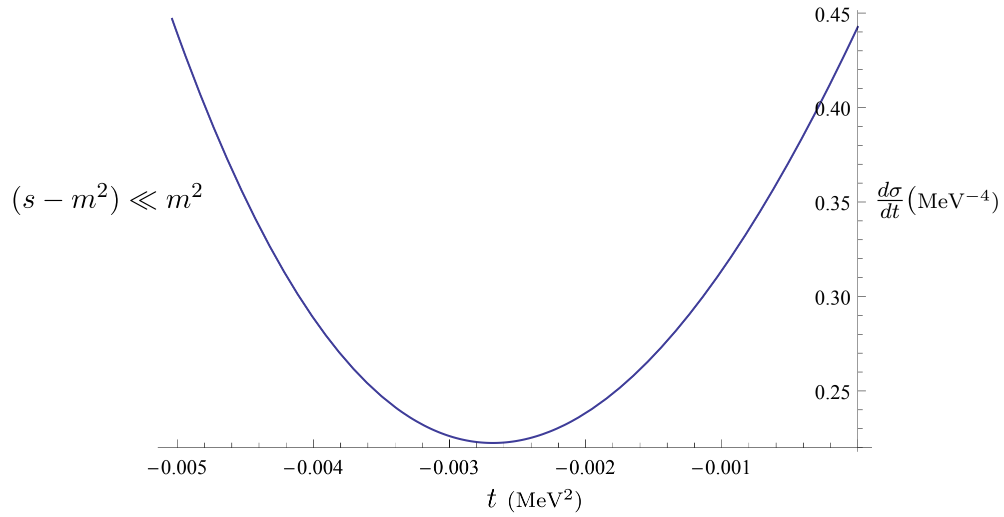

Figure 5: Differential cross section dependence on t for $(s-m^2) \ll m^2$.

At low energy, scattering is most likely to occur for $t = 0$ and $t = -4\omega^2$ corresponding to scattering angle $\theta = 0$ and $\theta = \pi$ respectively. This plot indicates that forward and backward scattering are equally likely to happen.

As energy increases, scattering starts to become less likely at small θ and begins to dominant for values of θ close to π.

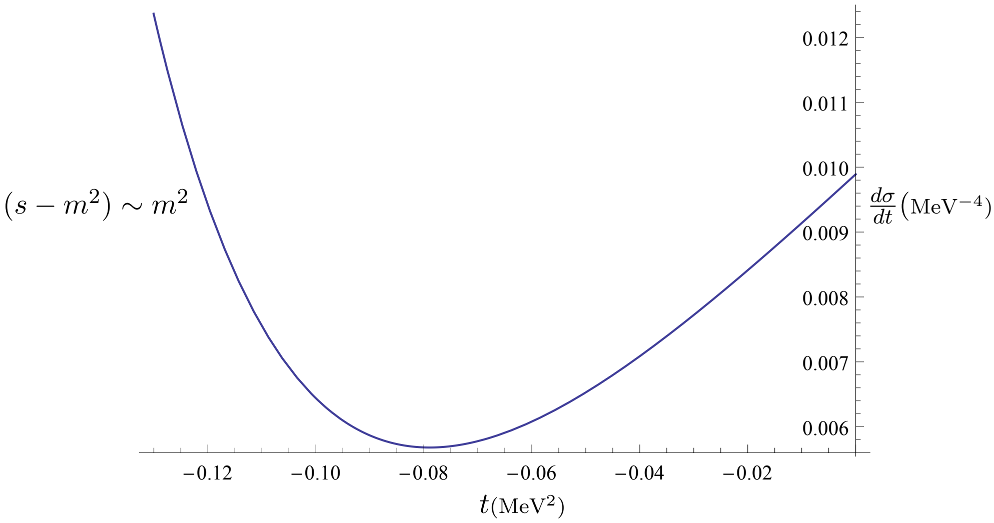

Figure 6: Differential cross section dependence on t for $(s - m^2) \sim m^2$.

The high energy behaviour displayed by the final plot shows that scattering at small θ is greatly suppressed and that the likelihood of scattering only increases as θ tends to π and no longer as $\theta \to 0$ from $\pi/2$. This suggests that the photon is most likely to scatter backwards as viewed from the center of mass frame.

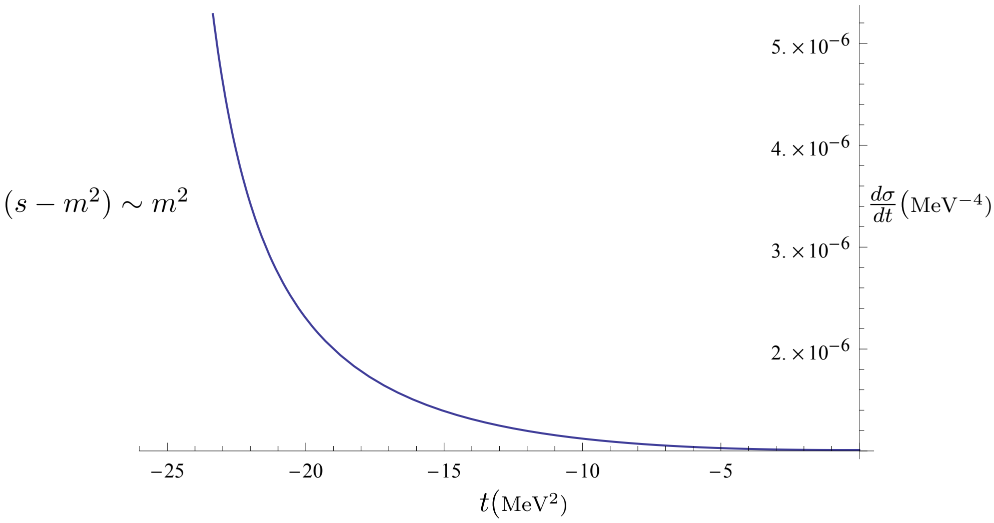

Figure 7: Differential cross section dependence on t for $s \gg m^2$.

Comparing all three curves we see that for all values of t, scattering is again most likely to happen at low energy.

6 Discussions

We computed the differential cross sections $d\sigma/dt$ and $d\sigma/d\cos\theta$ in the center of mass frame and lab frame respectively. We reproduced the Klein-Nishina formula for Compton scattering in the lab frame which reduced to the cross section for classical Thomson scattering at low energy. The corresponding low energy curve was unable to account for the high energy behaviour, displayed by the other curves. Historically, the Compton shift in the wavelength of the photon gave evidence to the notion that light could behave as a particle [3]. At high energy the particle nature of light must be taken into account to explain high energy scattering.

References

[1] Michael E Peskin and Daniel V Schroeder. *An Introduction to Quantum Field Theory.* Westview Press, (1995).

[2] Daniel Waldram. Quantum electrodynamics, (2011). Imperial College Qunatum Fields and Fundamental Forces MSc lecture notes.

[3] David Tong. Quantum field theory, (2006). University of Cambridge Part III Mathematical Tripos lecture notes.