

Compton Scattering

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Abstract

In this work tree-level QED diagrams are calculated for Compton scattering. Differential cross sections are plotted in different energy regions.

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1 Introduction

In this work we studied QED prediction for Compton scattering at tree level diagrams. We calculated unpolarized cross section. Firstly squared matrix element was obtained. To obtain unpolarized cross section, we averaged $|M^2|$ over all initial polarization states and summed it over all final polarization states. After having $|M^2|$ we determined Lorentz invariant $\frac{d\sigma}{dt}$. According to general definition of differential cross section phase space integral and flux factor were calculated at the center of mass frame to obtain $\frac{d\sigma}{dt}$. $\frac{d\sigma}{dt}$ was explained in terms of s and u which are Mandelstam variables. Then in the next section $\frac{d\sigma}{d\cos\theta}$ was computed at the lab frame. $\frac{d\sigma}{dt}$ is converted to $\frac{d\sigma}{d\cos\theta}$ in lab frame we took differential of dt in lab frame then $\frac{d\sigma}{dt}$ gives the same result as $\frac{d\sigma}{d\cos\theta}$. In addition, high energy limit ($s \gg m^2$) for Compton scattering was considered. Differential cross section is rewritten for high energy behaviour. Finally angular distribution of $\frac{d\sigma}{d\cos\theta}$ is plotted and $\frac{d\sigma}{dt}$ is plotted against square of the momentum transfer for different energy regions.

2 Compton Scattering

Compton scattering is known as a process whereby photons gain or lose energy from collisions with electrons. Now we will study QED prediction for Compton scattering $e^- \gamma \rightarrow e^- \gamma$. Unpolarized cross section is evaluated for this reaction [1]. At tree level Compton scattering $e^- \gamma \rightarrow e^- \gamma$ is described by two diagrams in figure 1.

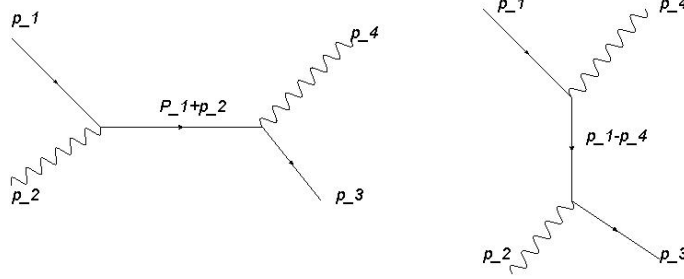


Figure 1: tree level Feynman Diagrams

2.1 Calculation of the Diagrams

$$\begin{aligned}
 iM_s &= \epsilon_\mu(p_2) \epsilon_\nu^* \bar{u}(p_3) (-ie\gamma^\nu) \frac{i(\not{p}_1 + \not{p}_2 + m)}{(p_1 + p_2)^2 - m^2 + i\epsilon} (-ie\gamma^\mu) u_s(p_1) \\
 &= -ie^2 \epsilon_\mu(p_2) \epsilon_\nu^* \bar{u}(p_3) \frac{\bar{u}_s(p_3) \gamma^\nu (\not{p}_1 + \not{p}_2 + m) \gamma_s(p_1)}{(p_1 + p_2)^2 - m^2 + i\epsilon} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 iM_u &= \epsilon_\mu(p_2) \epsilon_\nu^* \bar{u}(p_3) (-ie\gamma^\nu) \frac{i(\not{p}_1 - \not{p}_4 + m)}{(p_1 - p_4)^2 - m^2 + i\epsilon} (-ie\gamma^\mu) u_s(p_1) \\
 &= -ie^2 \epsilon_\mu(p_2) \epsilon_\nu^* \bar{u}(p_3) \frac{\bar{u}_s(p_3) \gamma^\nu (\not{p}_1 - \not{p}_4 + m) \gamma_s(p_1)}{(p_1 - p_4)^2 - m^2 + i\epsilon} \quad (2)
 \end{aligned}$$

$$\begin{aligned}
 iM &= -ie^2 \epsilon_\mu(p_2) * \epsilon_\nu(p_4) \bar{u}_s(p_3) \left[\frac{\gamma^\mu (\not{p}_1 + \not{p}_2 + m) \gamma^\nu}{(p_1 + p_2)^2 - m^2} + \frac{\gamma^\nu (\not{p}_1 - \not{p}_4 + m) \gamma^\mu}{(p_1 - p_4)^2 - m^2} \right] u_s(p_1) \quad (3) \\
 &= -ie^2 \epsilon_\mu(p_2) \epsilon_\nu^*(p_4) M^{\mu\nu}
 \end{aligned}$$

2.2 Ward Identity

We will check ward identity. To do this $p_{2\mu} M^{\mu\nu} = p_{4\nu} M^{\mu\nu} = 0$ has to be checked. The first identity is

$$p_{2\mu} M^{\mu\nu} = \bar{u}_s(p_3) \left[\frac{\gamma^\nu (\not{p}_1 + \not{p}_2 + m) \not{p}_2}{(p_1 + p_2)^2 - m^2} + \frac{\not{p}_2 (\not{p}_1 - \not{p}_4 + m) \gamma^\nu}{(p_1 - p_4)^2 - m^2} \right] u_s(p_1) \quad (4)$$

To make clear this, we use $\not{p}_2 u_s(p_1) = (\not{p}_1 + \not{p}_2 - \not{p}_1) u_s(p_1) = (\not{p}_1 + \not{p}_2 - m) u_s(p_1)$. Similarly we have $\bar{u}_s(p_3) (\not{p}_3 + \not{p}_4 - \not{p}_1) = \bar{u}_s(p_3) (\not{p}_4 - \not{p}_1 + m) = \bar{u}_s(p_3) (\not{p}_1 - \not{p}_4 - m)$. Putting this in, We obtain

$$\begin{aligned} p_{2\mu} M^{\mu\nu} &= \bar{u}_s(p_3) \left[\frac{\gamma^\nu (\not{p}_1 + \not{p}_2 + m) (\not{p}_1 + \not{p}_2 - m)}{(p_1 + p_2)^2 - m^2} - \frac{(\not{p}_1 - \not{p}_4 - m) (\not{p}_1 - \not{p}_4 + m) \gamma^\nu}{(p_1 - p_4)^2 - m^2} \right] u_s(p_1) \\ &= \bar{u}_s(p_3) \left[\frac{[\gamma^\nu (\not{p}_1 + \not{p}_2)^2 - m^2]}{(p_1 + p_2)^2 - m^2} - \frac{[(\not{p}_1 - \not{p}_4)^2 - m^2 \gamma^\nu]}{(p_1 - p_4)^2 - m^2} \right] u_s(p_1) = 0 \quad (5) \end{aligned}$$

Using similiar technique $p_{4\nu} M^{\mu\nu} = 0$. is proved.

2.3 Estimation of Square of Matrix Element

Now, we need to maintain $|M|^2$. To do this, we must first obtain $M^* = ie^2 \epsilon_\mu^*(p_2) \epsilon_\nu(p_4) (M^{\mu\nu})^*$. Since $M^{\mu\nu}$ is a number $(M^{\mu\nu})^* = (M^{\mu\nu})^+$. We will explain it explicitly. We get

$$\begin{aligned} (M^{\mu\nu})^+ &= \left(u_s^+(p_3) \gamma^0 \left[\frac{\gamma^\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu}{(p_1 + p_2)^2 - m^2} + \frac{\gamma^\mu (\not{p}_1 - \not{p}_4 + m) \gamma^\nu}{(p_1 - p_4)^2 - m^2} \right] u_s(p_1) \right) \\ &= u_s^+(p_1) \left[\frac{(\gamma^\nu)^+ (\not{p}_1 - \not{p}_4 + m)^+ (\gamma^\mu)^+}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma^\mu)^+ (\not{p}_1 + \not{p}_2 + m)^+ (\gamma^\nu)^+}{(p_1 + p_2)^2 - m^2} \right] (\gamma^0)^+ u_s(p_3) \\ &= u_s^+(p_1) \left[\frac{(\gamma^\nu)^+ (\not{p}_1 - \not{p}_4 + m)^+ (\gamma^\mu)^+}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma^\mu)^+ (\not{p}_1 + \not{p}_2 + m)^+ (\gamma^\nu)^+}{(p_1 + p_2)^2 - m^2} \right] \gamma^0 u_s(p_3) \\ &= u_s^+(p_1) \gamma^0 \left[\frac{(\gamma^\nu) (\not{p}_1 - \not{p}_4 + m) (\gamma^\mu)}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma^\mu) (\not{p}_1 + \not{p}_2 + m) (\gamma^\nu)}{(p_1 + p_2)^2 - m^2} \right] u_s(p_3) \\ &= \bar{u}_s^+(p_1) \gamma^0 \left[\frac{(\gamma^\nu) (\not{p}_1 - \not{p}_4 + m) (\gamma^\mu)}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma^\mu) (\not{p}_1 + \not{p}_2 + m) (\gamma^\nu)}{(p_1 + p_2)^2 - m^2} \right] u_s(p_3) \quad (6) \end{aligned}$$

The next step is

$$\begin{aligned} |M|^2 &= [-ie^2 \epsilon_\mu^*(p_2) \epsilon_\nu(p_4) M^{\mu\nu}] [ie^2 \epsilon_\alpha^*(p_2) \epsilon_\beta(p_4) (M^{\alpha\beta})^+] \\ &= e^4 \epsilon_\mu(p_2) \epsilon_\alpha^*(p_2) \epsilon_\nu^*(p_4) \epsilon_\beta(p_4) M^{\mu\nu} (M^{\alpha\beta})^+ \quad (7) \end{aligned}$$

We arrange over initial and sum over the final polarizations. This presents

$$\begin{aligned} \frac{1}{2} \sum_{pol} |M|^2 &= \frac{1}{2} e^4 \sum_{pol} \epsilon_\mu(p_2) \epsilon_\alpha^*(p_2) \sum_{pol} \epsilon_\nu^*(p_4) \epsilon_\beta(p_4) M^{\mu\nu} (M^{\alpha\beta})^+ \\ &= \frac{1}{2} e^4 (-g_{\mu\alpha}) (-g_{\nu\beta}) M^{\mu\nu} (M^{\alpha\beta})^+ \\ &= \frac{1}{2} e^4 M^{\mu\nu} M_{\mu\nu}^+ \quad (8) \end{aligned}$$

Let us plug all in. We then have

$$\frac{1}{2} \sum_{pol} |M|^2 = \frac{1}{2} e^4 \bar{u}_s(p_3) \left[\frac{\gamma^\nu (\not{p}_1 + \not{p}_2 + m)}{(p_1 + p_2)^2 - m^2} + \frac{\gamma^\mu (\not{p}_1 - \not{p}_4 + m)}{(p_1 - p_4)^2 - m^2} \right] u_s(p_1)$$

$$\times \bar{u}_s^+(P_1) \left[\frac{\gamma_\nu (\not{p}_1 - \not{p}_4 + m) \gamma_\mu}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma_\mu) (\not{p}_1 + \not{p}_2 + m) (\gamma_\nu)}{(p_1 + p_2)^2 - m^2} \right] u_s(p_3) \quad (9)$$

We average over initial and sum over final spins. This presents

$$\begin{aligned} \frac{1}{2} \sum_{s,\acute{s}} \frac{1}{2} \sum_{pol} |M^2| &= \frac{1}{4} e^4 \sum_{s,\acute{s}} u_s(p_3) \left[\frac{\gamma_\nu (\not{p}_1 + \not{p}_2 + m)}{(p_1 + p_2)^2 - m^2} + \frac{\gamma^\mu (\not{p}_1 - \not{p}_4 + m)}{(p_1 - p_4)^2 - m^2} \right] u_{\acute{s}}(p_1) \\ &\quad \times \bar{u}_{\acute{s}}^+(P_1) \left[\frac{\gamma_\nu (\not{p}_1 - \not{p}_4 + m) \gamma_\mu}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma_\mu) (\not{p}_1 + \not{p}_2 + m) (\gamma_\nu)}{(p_1 + p_2)^2 - m^2} \right] u_s(p_3) \\ &= \frac{1}{4} e^4 \sum_s \bar{u}_s(p_3) \left[\frac{\gamma_\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu}{\gamma^\mu (p_1 + p_2)^2 - m^2} + \frac{\gamma^\mu (\not{p}_1 - \not{p}_4 + m) \gamma^\nu}{(p_1 - p_4)^2 - m^2} \right] (\not{p}_1 + m) \\ &\quad \times \frac{\gamma_\nu (\not{p}_1 - \not{p}_4 + m) \gamma_\mu}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma_\mu) (\not{p}_1 + \not{p}_2 + m) (\gamma_\nu)}{(p_1 + p_2)^2 - m^2} u_s(p_3) \end{aligned} \quad (10)$$

The sum over the \acute{s} spin was simple. To sum over the s let us first write (to neaten notation)

$$\frac{1}{2} \sum_{pol} |M|^2 = \frac{1}{4} e^4 \sum_s \bar{u}_s(p_3) M u_s(p_3) \quad (11)$$

Where we have just determine M as the matrix

$$M = \left[\frac{\gamma_\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu}{\gamma^\mu (p_1 + p_2)^2 - m^2} + \frac{\gamma^\mu (\not{p}_1 - \not{p}_4 + m) \gamma^\nu}{(p_1 - p_4)^2 - m^2} \right] (\not{p}_1 + m) \left[\frac{\gamma_\nu (\not{p}_1 - \not{p}_4 + m) \gamma_\mu}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma_\mu) (\not{p}_1 + \not{p}_2 + m) (\gamma_\nu)}{(p_1 + p_2)^2 - m^2} \right] \quad (12)$$

Now, matrix identity of (11) is written out explicitly. We get

$$\begin{aligned} \frac{1}{2} \sum_{s,\acute{s}} \frac{1}{2} \sum_{pol} |M^2| &= \frac{1}{4} e^4 \sum_s [\bar{u}_s(p_3)]_a M_{ab} [\bar{u}_s(p_3)]_b = \frac{1}{4} e^4 \sum_s M_{ab} [\bar{u}_s(p_3)]_a [\bar{u}_s(p_3)]_b \\ &= \frac{1}{4} e^4 M_{ab} (\not{p}_3 + m)_{ab} \\ &= \frac{1}{4} e^4 Tr [M (\not{p}_3 + m)] \end{aligned} \quad (13)$$

If we put all together, we have

$$\begin{aligned} \frac{1}{2} \sum_{s,\acute{s}} \frac{1}{2} \sum_{pol} |M^2| &= \frac{1}{4} e^4 Tr \left[\frac{\gamma_\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu}{(p_1 + p_2)^2 - m^2} + \frac{\gamma^\mu (\not{p}_1 - \not{p}_4 + m) \gamma^\nu}{(p_1 - p_4)^2 - m^2} \right] (\not{p}_1 + m) \\ &\quad \times \left[\frac{\gamma_\nu (\not{p}_1 - \not{p}_4 + m) \gamma_\mu}{(p_1 - p_4)^2 - m^2} + \frac{(\gamma_\mu) (\not{p}_1 + \not{p}_2 + m) (\gamma_\nu)}{(p_1 + p_2)^2 - m^2} \right] (\not{p}_3 + m) \end{aligned} \quad (14)$$

Now, we will evaluate the trace. We should do this in parts

$$\frac{1}{4} \sum_{spins} |M^2| = \frac{e^4}{4} \left[\frac{I}{(2p_1 \cdot p_2)^2} + \frac{II}{(2p_1 \cdot p_2)(2p_1 \cdot p_4)} + \frac{III}{2(p_1 \cdot p_4)2(p_1 \cdot p_2)} + \frac{IV}{(2p_1 \cdot p_4)^2} \right] \quad (15)$$

$$\begin{aligned} I &= Tr [\gamma^\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu (\not{p}_1 + m) \gamma_\mu (\not{p}_1 + \not{p}_2 + m) \gamma_\nu (\not{p}_3 + m) \\ II &= Tr [\gamma^\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu (\not{p}_1 + m) \gamma_\nu (\not{p}_1 - \not{p}_4 + m) \gamma_\mu (\not{p}_3 + m) \\ III &= Tr [\gamma^\mu (\not{p}_1 - \not{p}_4 + m) \gamma^\nu (\not{p}_1 + m) \gamma_\mu (\not{p}_1 + \not{p}_2 + m) \gamma_\nu (\not{p}_3 + m) \\ IV &= Tr [\gamma^\mu (\not{p}_1 - \not{p}_4 + m) \gamma^\mu (\not{p}_1 + m) \gamma_\nu (\not{p}_1 - \not{p}_4 + m) \gamma_\mu (\not{p}_3 + m) \end{aligned} \quad (16)$$

Half contain an odd number of γ matrices vanish. We use below trace identities

- $\gamma^\mu \gamma^\mu = 4I$
- $\gamma^\mu \not{p} \gamma^\mu = -2 \not{p}$
- $\gamma^\mu \not{p} \not{q} \gamma^\mu = 4pqI$
- $Tr\{\gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_{2n}}\} = \sum_{j=2}^{2n} (-1)^j g^{\mu_1 \mu_j} Tr\{\gamma^{\mu_2} \dots \gamma^{-1} \gamma^{+1} \dots \gamma^{2n}\}$
- $Tr\{\not{p} \not{q} \not{r} \not{s}\} = 4(p \cdot q \cdot r \cdot s - p \cdot r \cdot q \cdot s + p \cdot s \cdot q \cdot r)$ Then using the trace identities for trace number I

$$I = Tr [\gamma^\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu (\not{p}_1 + m) \gamma_\nu (\not{p}_1 + \not{p}_2 + m) \gamma_\mu (\not{p}_3 + m) \quad (17)$$

These trace should be simplified by using trace identities.

$$\begin{aligned} I &= Tr X(\not{p}_3 + m) \\ X &= \gamma^\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu (\not{p}_1 + m) \gamma_\mu (\not{p}_1 + \not{p}_2 + m) \gamma_\nu \\ &= \gamma^\nu (\not{p}_1 + \not{p}_2 + m) (-2p_1 + 4m) (\not{p}_1 + \not{p}_2 + m) \gamma_\nu \\ &= 4(\not{p}_1 + \not{p}_2) \not{p}_1 (\not{p}_1 + \not{p}_2) + m [-16(p_1(p_1 + p_2)16(p_1 + p_2)^2) + \\ &\quad m^2(4 \not{p}_1 - 16(\not{p}_1 + \not{p}_2) + 16m^3) \end{aligned} \quad (18)$$

$$\begin{aligned} I &= Tr X(\not{p}_3 + m) \\ &= 16 \{2 [(p_1 + p_2)p_1] [(p_1 + p_2)p_3] - (p_1 + p_2)^2(p_1 p_3) + [-4(p_1(p_1 + p_2) + 4(p_1 + p_2)^2)]\} \\ &\quad + 16 \{m^2 [(p_1 p_3) - 4(p_1 + p_2)p_3] + 4m^4\} \\ I &= 32 [(p_1 \cdot p_2)(p_1 \cdot p_4) + m^2(p_1 p_2) + m^4] \end{aligned} \quad (19)$$

Here we have applied the fact that momentum conservation means the four-momentum products can be used in terms of invariant variables $p_1 \cdot p_2$; $p_1 \cdot p_4$ and m^2 specially

$$\begin{aligned}
p_2^2 = p_4^2 = 0 \quad p_1^2 = p_3^2 = m^2 \\
p_3 p_2 = p_1 p_4 \\
p_3 p_4 = p_1 p_2 \\
p_1 p_3 = m^2 - p_1 p_2 - p_1 p_4 \\
p_3 p_4 = p_1 p_2 - p_1 p_4
\end{aligned}$$

This expressions can be seen in Mandelstam variables:

$$\begin{aligned}
s &= (p_1 + p_2)^2 = 2p_1 p_2 + m^2 = 2p_3 p_4 + m^2 \\
u &= (p_1 - p_4)^2 = -2p_4 p_1 + m^2 = -2p_2 p_3 + m^2 \\
t &= (p_3 - p_1)^2 = -2p_1 p_3 + 2m^2 = -2p_2 p_4
\end{aligned} \tag{20}$$

IV is calculated by the same expression with $p_2 \leftrightarrow -p_4$ and related calculations shows $II = III$

$$IV = I = 32 [(p_1 \cdot p_2)(p_1 \cdot p_4) + m^2(p_1 p_2) + m^4] \tag{21}$$

$$II = III = 16m^2 [2m^2 + (p_1 p_2) - (p_1 p_4)] \tag{22}$$

If we substitute everything in eq. (15) we reach the squared matrix element:

$$\frac{1}{2} \sum_{s, \acute{s}} \frac{1}{2} \sum_{pol} |M^2| = 2e^4 \left[\frac{p_{12}}{p_{14}} + \frac{p_{14}}{p_{12}} + 2m^2 \left(\frac{1}{p_{12}} - \frac{1}{p_{14}} \right) + m^4 \left(\frac{1}{p_{12}} - \frac{1}{p_{14}} \right)^2 \right] \tag{23}$$

Squared matrix element is written in terms of s and u . p_{12} replace $s - m^2$ and $p_{14} \rightarrow u - m^2$

$$\frac{1}{4} \sum_{spins} |M^2| = 2e^4 \left[\frac{s - m^2}{u - m^2} + \frac{u - m^2}{s - m^2} + 2m^2 \left(\frac{1}{s - m^2} - \frac{1}{u - m^2} \right) + m^4 \left(\frac{1}{s - m^2} - \frac{1}{u - m^2} \right)^2 \right] \tag{24}$$

3 Calculate Differential Cross-section with respect to t for Compton Scattering

Definition of differential cross section is

$$d\sigma = \frac{|M^2|}{4 [(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2}} dLips(s; p_3, p_4) \tag{25}$$

two-particle Lorentz invariant phase space is [2]

$$dLips(s; p_3, p_4) = (2\pi)^4 \delta^4(p_4 + p_3 - p_2 - p_1) \times \frac{1}{(2\pi)^3} \frac{d^3 p_3}{2E_3} \frac{1}{(2\pi)^3} \frac{d^3 p_4}{2E_4} \quad (26)$$

these expressions will be evaluated in the center of momentum frame defined [2]

$$p_1 + p_2 = p_3 + p_4 = 0 \quad (27)$$

$$W = E_1 + E_2 = E_3 + E_4 = (p^2 + m_1^2)^{1/2} + (p^2 + m_2^2)^{1/2} \quad (28)$$

$$dLips(s; p_3, p_4) = \frac{1}{4\pi^2} \delta^4(p_4 + p_3 - p_2 - p_1) \frac{d^3 p_3}{E_3} \frac{d^3 p_4}{E_4} \quad (29)$$

with the 3 momentum δ function, the integral can be eliminated over $d^3 p_4$

$$\int \frac{d^3 p_3}{2E_3} \delta^4(p_4 + p_3 - p_2 - p_1) = \frac{1}{E_4} \delta^4(p_4 + p_3 - p_2 - p_1) \quad (30)$$

p_4 and E_4 are not independent variables. They are evaluated by the conditions [2]

$$p_4 = p_1 + p_2 - p_3 \quad (31)$$

$$E_4 = (|p_4|^2 + m_2^2)^{1/2} \quad (32)$$

Next, convert $d^3 p_3$ to angular variables

$$d^3 p_3 = d\Omega p_3^2 dp_3 \quad (33)$$

The energy and momentum are related by

$$E_3^2 = p_3^2 + m_1^2 \quad (34)$$

$$E_3 dE_3 = p_3 dp_3 \quad (35)$$

with all these changes, we get last step for dLips [2]

$$dLips(s; p_3, p_4) = \frac{1}{(4\pi)^2} d\Omega \frac{p_3 dE_3}{E_4} \delta^4(p_4 + p_3 - p_2 - p_1) \quad (36)$$

$$E_3^2 = p_3^2 + m_2^2 \quad (37)$$

$$E_4^2 = p_3^2 + m_1^2 \quad (38)$$

$$E_3 dE_3 = p dp = E_4 dE_4 \quad (39)$$

$$w' = E_3 + E_4 \quad (40)$$

Then

$$dw' = dE_3 + dE_4 = \frac{w'}{E_3 E_4} p dp = \frac{w'}{E_4} dE_3 \quad (41)$$

equation (33) is used for last two steps this

$$p_3 \frac{dE_3}{E_4} \delta^4(p_4 + p_3 - p_2 - p_1) \quad (42)$$

$$\frac{p}{w'} dw' \delta(w' - w) = \frac{p}{w} \quad (43)$$

We get last step

$dLips(s; p_3, p_4) = \frac{1}{(4\pi)^2} \frac{p}{w} d\Omega$ for two-body phase space in cm frame.

Flux factor f is

$$\begin{aligned} F^2 &= (p_1 \cdot p_2)^2 - m_1^2 m_2^2 \\ p_1 \cdot p_2 &= E_1 \cdot E_2 + p^2 \\ f &= pw = 4p\sqrt{s} \\ d\sigma &= \frac{1}{4f} |F|^2 \frac{1}{(4\pi)^2} \frac{p}{w} d\Omega \end{aligned} \quad (44)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8)^2} |M|^2 \quad (45)$$

The square 4- momentum transfer is defined as the Mandelstam t variable [2]

$$t = q^2 = (p_1 - p_3)^2 = (p_4 - p_2)^2 \quad (46)$$

$$t = -2p^2(1 - \cos \theta) \quad (47)$$

$$dt = 2p^2 d(\cos \theta) \quad (48)$$

$$d\Omega = 2(\cos \theta) \frac{d\sigma}{dt} = \frac{1}{64\pi} \frac{1}{[(p_1 p_2)^2 - m_1^2 m_2^2]} |M|^2 \quad (49)$$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} \frac{1}{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]} |M|^2 \quad (50)$$

or in terms of s [2]

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} \frac{1}{(s - m^2)^2} |M|^2 \quad (51)$$

We are now ready to write expression for the $\frac{d\sigma}{dt}$ in terms of u and s. We substitute equation (23) in eq.(50)

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} \frac{1}{(s - m^2)^2} 2e^4 \left[\frac{s - m^2}{u - m^2} + \frac{u - m^2}{s - m^2} + 2m^2 \left(\frac{1}{s - m^2} - \frac{1}{u - m^2} \right) + m^4 \left(\frac{1}{s - m^2} - \frac{1}{u - m^2} \right)^2 \right] \quad (52)$$

$\frac{d\sigma}{dt}$ was written in terms of s and u. $\frac{d\sigma}{dt}$ is Lorentz invariant.

4 Differential Cross-Section With Respect to $d \cos \theta$ for Compton Scattering

Now we will calculate the differential cross section with respect to scattering angle, $\frac{d\sigma}{d \cos \theta}$ in the rest frame of the incident electron

4.1 Calculation of $\frac{d\sigma}{d \cos \theta}$ In the Lab Frame

If we work in lab frame it makes sense to work in the low energy limit. In this frame momentums are are [3]

$$\begin{aligned} p_1 &= (m, 0, 0, 0,) \\ p_2 &= (w, 0, 0, w) \\ p_3 &= (\acute{E}, \acute{P}) \\ p_4 &= (\acute{w}, \acute{w} \sin \theta, 0, \acute{w} \cos \theta) \end{aligned} \tag{53}$$

These relations suggest [3]

$$\frac{1}{\acute{w}} - \frac{1}{w} = \frac{1}{m}(1 - \cos \theta) \tag{54}$$

and

$$\acute{w} = \frac{w}{1 + \frac{w}{m}(1 - \cos \theta)} \tag{55}$$

Then

$$p_{12} = wm \quad p_{14} = \acute{w}m \tag{56}$$

As a result we have simple formula for M^2

$$\begin{aligned} \frac{1}{4} \sum_{pols} M^2 &= 2e^4 \left[\frac{\acute{w}}{w} + \frac{w}{\acute{w}} - 2(1 - \cos \theta) + (1 - \cos \theta)^2 \right] \\ &= 2e^4 \left[\frac{\acute{w}}{w} + \frac{w}{\acute{w}} - \sin^2 \theta \right] \end{aligned} \tag{57}$$

In the lab frame general formula for differantial cross-section [3]

$$\begin{aligned} d\sigma &= \frac{1}{(2E_1)(2E_2) |\vec{v}_1 - \vec{v}_2|} |M^2| d\Pi_{LIPS} \\ &= \frac{1}{4wm} |M^2| d\Pi_{LIPS} \end{aligned} \tag{58}$$

phase space integral in this frame is

$$\int d\Pi_{LIPS} = \int \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2\dot{E}} \int \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2\dot{w}} [(2\pi)^4 \delta^{(4)}(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu)] \quad (59)$$

The delta function fixes the 3-momenta when we integrate over $d^3 p_4$, leaving the energy constraint

$$\begin{aligned} \int d\Pi_{LIPS} &= \frac{1}{4(2\pi)^2} \int \dot{w}^2 d\Omega d\dot{w} \frac{1}{\dot{w}\dot{E}} \delta(\sum E) \\ &= \frac{1}{8\pi} \int d\cos\theta d\dot{w} \frac{\dot{w}}{\dot{E}} \delta(\sum E) \end{aligned} \quad (60)$$

We integrate over \dot{w} to enforce the energy constraint $\dot{E} + \dot{w} = m + w$. Since \dot{E} and w are already constrained by the electron's on-shell condition we need to be careful [3]

$$\begin{aligned} \dot{E}^2 &= m^2 + (\dot{w} \sin\theta)^2 + (\dot{w} \cos\theta - w)^2 \\ &= m^2 + \dot{w}^2 + w^2 - 2w\dot{w} \cos\theta \end{aligned} \quad (61)$$

$$\dot{E} \frac{d\dot{E}}{d\dot{w}} = \dot{w} - w \cos\theta \quad (62)$$

hence

$$\begin{aligned} \int d\Pi_{LIPS} &= \frac{1}{8\pi} \int d\cos\theta d\dot{w} \frac{\dot{w}}{\dot{E}} \delta(\dot{w} + \dot{E}(\dot{w}) - m - w) \\ &= \frac{1}{8\pi} \int d\cos\theta \frac{\dot{w}}{\dot{E}} \left(1 + \frac{d\dot{E}}{d\dot{w}}\right)^{-1} \\ &= \frac{1}{8\pi} \int d\cos\theta \frac{\dot{w}}{\dot{E}} \left(1 + \frac{\dot{w} - w \cos\theta}{\dot{E}}\right)^{-1} \\ &= \frac{1}{8\pi} \int d\cos\theta \frac{(\dot{w})^2}{\dot{w}m} \\ \frac{d\sigma}{d\cos\theta} &= \frac{1}{4wm} \frac{1}{8\pi} \frac{(\dot{w})^2}{\dot{w}m} 2e^4 \left[\frac{\dot{w}}{w} + \frac{w}{\dot{w}} - \sin^2\theta \right] \\ \frac{d\sigma}{d\cos\theta} &= \frac{\pi\alpha^2}{m^2} \left(\frac{\dot{w}}{w}\right)^2 \left[\frac{\dot{w}}{w} + \frac{w}{\dot{w}} - \sin^2\theta \right] \end{aligned} \quad (63)$$

This is the Klein-Nishina formula. Klein-Nishina calculated this formula firstly in 1929. The differential cross-section of photons scattered from a single free electron in lowest order of quantum electrodynamics is given by The Klein-Nishina formula.

In other variables,

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2} \left(1 + \cos^2\theta - \frac{2w}{m}(1 + \cos^2\theta)(1 - \cos\theta) + O\left(\frac{1}{m^2}\right) \right) \quad (64)$$

In the limit $m \rightarrow \infty$ [3]

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2} [1 + \cos^2\theta] \quad (65)$$

This is well-known Thomson scattering cross sections for classical electromagnetic radiation by a free electron [3]. The cross section becomes energy independent. $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$ is the fine structure constant in natural units.

4.2 By direct conversion of dt into $d \cos \theta$

Now we convert dt into $d \cos \theta$ directly. We calculated $\frac{d\sigma}{dt}$ in the CM frame in previous section. Now, to transform $\frac{d\sigma}{dt}$ into $\frac{d\sigma}{d \cos \theta}$ we consider Compton scattering in the laboratory frame where initial electron is at rest in this case. we take differential of dt in lab frame and write s and u terms in the lab frame.

$$\begin{aligned} s &= (p_1 + p_2)^2 = m^2 + 2mp_2 \\ p_2 &= w \quad p_4 = w' \\ s &= m^2 + 2mw \end{aligned} \tag{66}$$

$$u = (p_1 - p_4)^2 = m^2 - 2mw' \tag{67}$$

momentum transfer between the photons we have

$$\begin{aligned} t &= (p_2 - p_4)^2 = -2p_2p_4 = -2ww' + 2w' \cos \theta = -2ww' = 2mww'(1 - \cos \theta) \\ t &= (p_1 - p_3)^2 = 2m^2 - 2mp_3 = 2m(m - p_3) = 2m(w' - w) \end{aligned} \tag{68}$$

$$2ww'(1 - \cos \theta) = 2m(w - w') \tag{69}$$

We already got the photon energy change after the scattering

$$\left(\frac{1}{w} - \frac{1}{w'}\right) = \frac{1}{m}(1 - \cos \theta) \tag{70}$$

We consider separate terms on right hand side of (51) in the lab frame

$$\frac{m^2}{s - m^2} = \frac{m}{2w}, \quad \frac{m^2}{u - m^2} = -\frac{m}{2w'} \tag{71}$$

$$\frac{m^2}{s - m^2} + \frac{m^2}{u - m^2} = \frac{m}{2} \left(\frac{1}{w} - \frac{1}{w'}\right) = -\frac{1}{2}(1 - \cos \theta) \tag{72}$$

similarly

$$\frac{u - m^2}{s - m^2} = -\frac{w'}{w} \tag{73}$$

$$\begin{aligned} dt &= 2ww'd(\cos \theta) - 2w(1 - \cos \theta)dw' = -\frac{w}{m}(1 - \cos \theta)dt + 2ww'd(\cos \theta) \\ & \qquad \qquad \qquad dt = 2w'^2d(\cos \theta) \end{aligned} \tag{74}$$

Insert eqs. (70) ,(71), (72) and (73) in eq. (51) we have

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2} \left(\frac{w'}{w}\right)^2 \left[\left(\frac{w'}{w}\right) + \left(\frac{w}{w'}\right) - \sin^2\theta \right] \quad (75)$$

We realize that if we convert $\frac{d\sigma}{dt}$ equation into $\frac{d\sigma}{d\cos\theta}$. It is similar to $\frac{d\sigma}{d\cos\theta}$ in section 4.1.

5 High-Energy Behaviour

Now we consider opposite limit $w \gg m$. Work in COM frame is useful in this case. Kinematics of reaction is [3]

$$\begin{aligned} p_1 &= (E, 0, 0, -w) & p_2 &= (w, 0, 0, w) \\ p_3 &= (E, -w \sin\theta, 0, -w \cos\theta) & p_4 &= (E, w \sin\theta, 0, w \cos\theta) \end{aligned}$$

For $w \gg m_e$

$$\begin{aligned} E &\approx m \\ s &\approx 4w^2, \quad u \approx -2p_{14} = 2w^2(1 + \cos\theta) \end{aligned} \quad (76)$$

Our M^2 is

$$\begin{aligned} \frac{1}{4} \sum_{pols} M^2 &= 2e^4 \left[\frac{p_{14}}{p_{12}} + \frac{p_{12}}{p_{14}} \right] = 2e^4 \left[\frac{u}{s} + \frac{s}{u} \right] \\ &= 2e^4 \left[1 + \cos\theta + \frac{1}{1 + \cos\theta} \right] \end{aligned} \quad (77)$$

This is singular at $\theta = \pi$. In this case photon and electron bounce off each other and go backwards. The mass of the electron cuts off the singularity. However the scattering amplitude continue having very large as $\theta \approx \pi$. The differential cross section is [3]

$$\frac{d\sigma}{d\cos\theta} = \frac{e^4}{32\pi s} \left[1 + \cos\theta + \frac{1}{1 + \cos\theta} \right] \quad (78)$$

Which explodes at $\theta = \pi$

6 Plot the Differential cross-section $\frac{d\sigma}{d\cos\theta}$ against $\cos\theta$

When we plot our differential cross sections for different energy limits, it is easy to work in center-of-mass frame for $s - m^2 \gg m^2$ energy limit. It is also easy to work in lab frame for $s - m^2 \approx m^2$ and $s - m^2 \approx m^2 \ll m^2$ energy limits. Our differential cross sections were calculated in lab frame for $s - m^2 \approx m^2$ and $s - m^2 \approx m^2 \ll m^2$ energy regions. Our differential cross sections were also calculated in cm frame for $s - m^2 \gg m^2$ energy limit to plot. In this section units of differential cross section is $\frac{1}{TeV^2}$ for all energy region.

6.1 $s - m_e^2 \ll m_e^2$

Work in the lab frame makes sense for low energy limit. Differential cross section obtained for low energy limit is $\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta)$. Cross section is equal to Thomson scattering at low photon energy [4] We take $\cos\theta$ values from 0 to 180 in this cross section formula. Then plot $\frac{d\sigma}{d\cos\theta}$ against $\cos\theta$

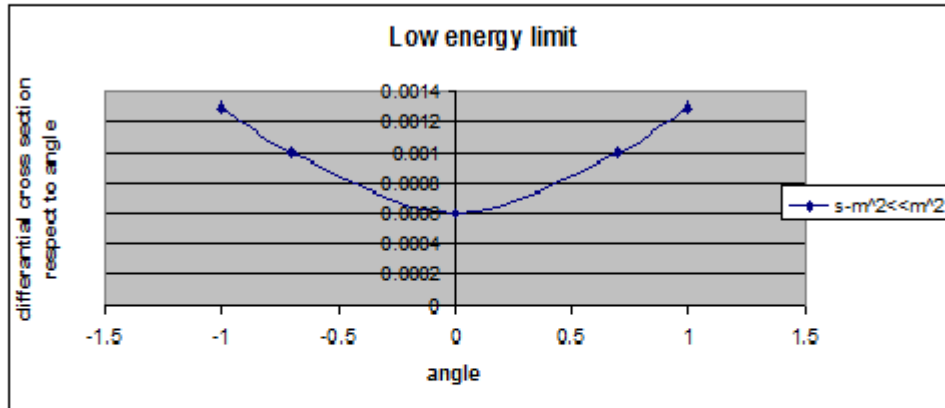


Figure 2: Differential cross section for $s \ll m^2$. $\frac{d\sigma}{d\cos\theta}$ is multiple of $\frac{1}{TeV^2}$ in this graphics.

Y axis represents $\frac{d\sigma}{d\cos\theta}$, X axis represents $\cos\theta$. Differential cross section takes same values between $\cos 0$ and $\cos 180$ also between $\cos 45$ and $\cos 135$. The angular distribution of differential cross section is front-back symmetrical.

6.2 $s - m^2 \approx m^2$ at lab frame

To plot $\frac{d\sigma}{d\cos\theta}$ against $\cos\theta$ in $s - m^2 \approx m^2$ energy limit, $\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2} \left(\frac{w'}{w}\right)^2 \left[\frac{w'}{w} + \frac{w}{w'} - \sin^2\theta\right]$ formula is used. $\cos\theta$ values from 0 to 180 is taken for plotting. To calculate $\frac{d\sigma}{d\cos\theta}$ we take $s = 2m^2$ and $\frac{w'}{w} = \frac{2}{5}$. Figure3 shows our data.

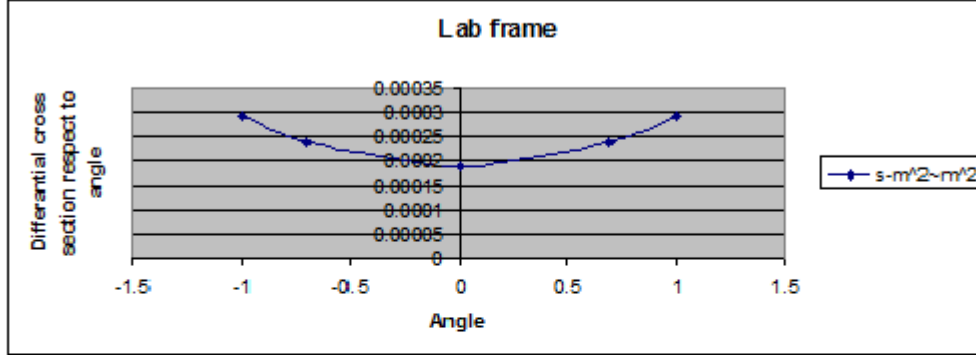


Figure 3: $s - m^2 \approx m^2$ in lab frame. $\frac{d\sigma}{d\cos\theta}$ is multiple of $\frac{1}{TeV^2}$ in this graphics

If we compare this middle energy limit with $s - m^2 \ll m^2$, we see that differential cross section is inversely proportional to energy. Differential cross section is symmetric with respect to forward and backward scattering according to graph. Figure 2 and 3 displays that angular distribution of $\frac{d\sigma}{d\cos\theta}$ is symmetric in low energy cases.

6.3 $s - m_e^2 \gg m_e^2$ at center-of-mass frame

We should work in CM frame for $s - m_e^2 \gg m_e^2$ energy limit. In preceding sections we have obtained differential cross section in the CM frame in high energy limit our formula is

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \left[1 + \cos\theta + \frac{1}{1 + \cos\theta} \right]$$

For $s - m_e^2 \gg m_e^2$ We should approximate for s and take $s = 100m_e^2$ energy value to plot. and $\cos\theta = 0, 45, 90, 135, 160$

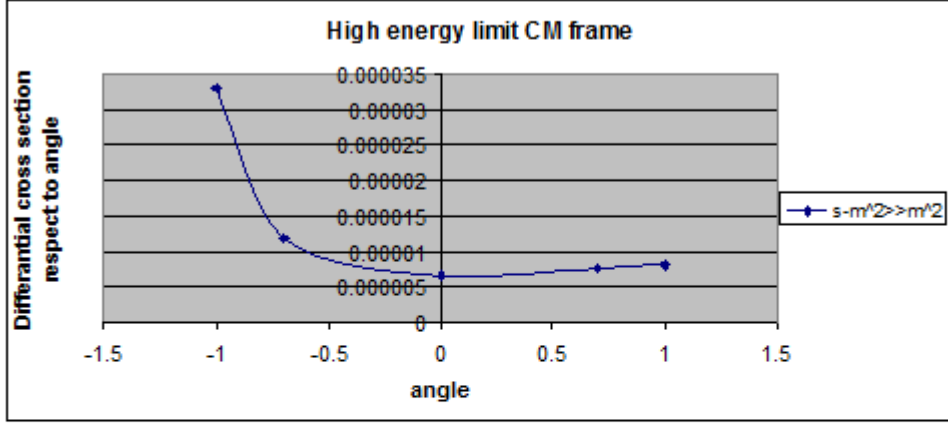


Figure 4: Differential Cross Section for $s - m^2 \gg m^2$. $\frac{d\sigma}{d\cos\theta}$ is multiple of $\frac{1}{TeV^2}$ in this graphics.

Figure 2, 3, 4 shows that differential cross section is inversely proportional to energy. $\frac{d\sigma}{d\cos\theta}$ decreases from $\cos 0$ towards $\cos 90$. However it increases after $\cos 90$ and goes to infinity at $\cos 180$. Scattered photon goes to backward direction at $\theta = 180$.

7 Plot Differential cross Section $\frac{d\sigma}{dt}$ against t

In this section units of $\frac{d\sigma}{dt}$ is $\frac{1}{10^{24}eV^4} = \frac{1}{YeV^4}$ for all energy regions.

7.1 For $s - m_e^2 \ll m_e^2$ case

In previous section $\frac{d\sigma}{dt}$ has been already obtained.

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} \frac{1}{(s - m^2)^2} 2e^4 \left[\frac{s - m^2}{u - m^2} + \frac{u - m^2}{s - m^2} + 2m^2 \left(\frac{1}{s - m^2} - \frac{1}{u - m^2} \right) + m^4 \left(\frac{1}{s - m^2} - \frac{1}{u - m^2} \right)^2 \right]$$

we rearrange this formula in lab frame for low energy case. In this formula $s \rightarrow m^2 + 2mw$ and $u \rightarrow m^2 - 2mw'$. Our $\frac{d\sigma}{dt}$ will be

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{2m^2w^2} \left(\frac{w'}{w} + \frac{w}{w'} - \sin^2\theta \right) \quad (80)$$

For low energies Thomson limit $\frac{w'}{w} \rightarrow 1$ [4]. we left with $\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{2m^2w^2}(1 + \cos^2\theta)$.

Our square momentum-transfer between the initial and final state photons is $t = (p_2 - p_4)^2 = -2ww'(1 - \cos\theta)$ and $dt = 2w^2d\cos\theta$ Now We plot $\frac{d\sigma}{dt}$ against t for different θ values from 0 to 180. We accept $w = 0.1m_e$ to plot figure5. In figure 5 we took t values negative because of its formula $t = -2ww'(1 - \cos\theta)$.

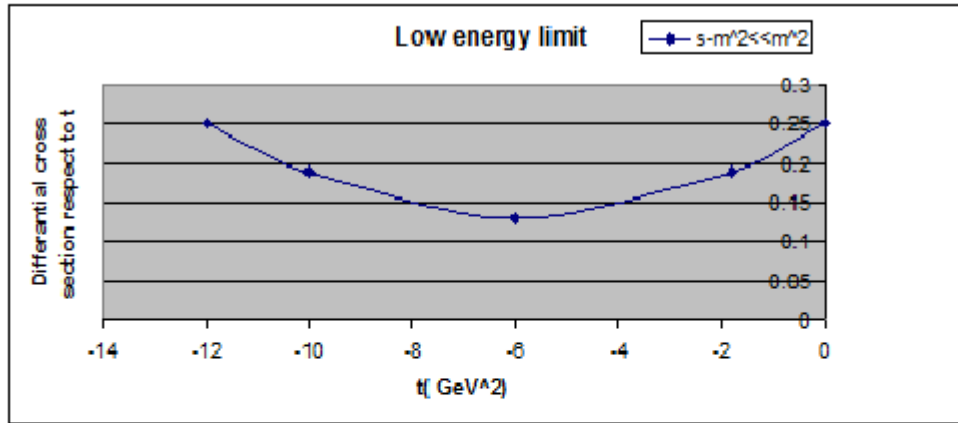


Figure 5: Differential Cross Section for $s - m^2 \ll m^2$. $\frac{d\sigma}{dt}$ is multiple of $\frac{1}{Y_{eV^4}}$ in this figure.

This graph looks like Thomson cross section graph. Differential cross section takes same values between $\cos 0$ and $\cos 180$ also between $\cos 45$ and $\cos 135$.

7.2 $s - m^2 \approx m^2$

We use the equation

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{2m^2w^2} \left(\frac{w'}{w} + \frac{w}{w'} - \sin^2\theta \right) \quad (81)$$

We insert $w = 0.5m_e$, $w' = 0.2m_e$ and $\frac{w'}{w} = (\frac{2}{5})$ values in the equation. We take $\cos\theta$ values from 0 to 180 then plot $\frac{d\sigma}{dt}$ against t .

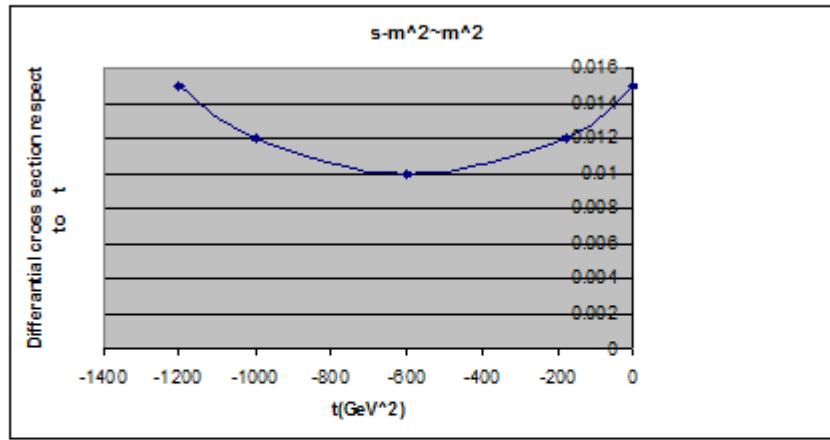


Figure 6: $s - m^2 \approx m^2$ in lab frame. $\frac{d\sigma}{dt}$ is multiple of $\frac{1}{\text{GeV}^4}$ in this figure

Our figure is symmetric.

7.3 For $s \gg m_e^2$ at CM frame

Our general $\frac{d\sigma}{dt}$ equation (51) is

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} \frac{1}{(s - m^2)} |M|^2 \quad (82)$$

At high energy limit mass is negligible $|M|^2$ is

$$\frac{1}{4} \sum M^2 = 2e^4 \left[\frac{p_{14}}{p_{12}} + \frac{p_{12}}{p_{14}} \right] = 2e^4 \left[\frac{u}{s} + \frac{s}{u} \right]$$

we use eq. (76) our $\frac{1}{4} \sum_{pols} |M|^2$ is $\frac{1}{4} \sum M^2 = 2e^4 \left[1 + \cos \theta + \frac{1}{1 + \cos \theta} \right]$. Finally $\frac{d\sigma}{dt}$ is for high energy limit in cm frame

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{1}{16\pi} \frac{1}{s^2} 2e^4 \left[1 + \cos \theta + \frac{1}{1 + \cos \theta} \right] \\ &= \frac{\pi\alpha^2}{2s^2} \left[1 + \cos \theta + \frac{1}{1 + \cos \theta} \right] \end{aligned} \quad (83)$$

Our square momentum-transfer between the initial and final state photons is $t = -2p^2(1 - \cos \theta)$ and $dt = 2p^2 d(\cos \theta)$ We plot $\frac{d\sigma}{dt}$ against t for different θ values from 0 to 160 and we take $w = 5m_e$, $s = 100m_e^2$ to plot. In this case differential cross section blows up at 180. Scattered photon goes backwards through $\cos 180$.

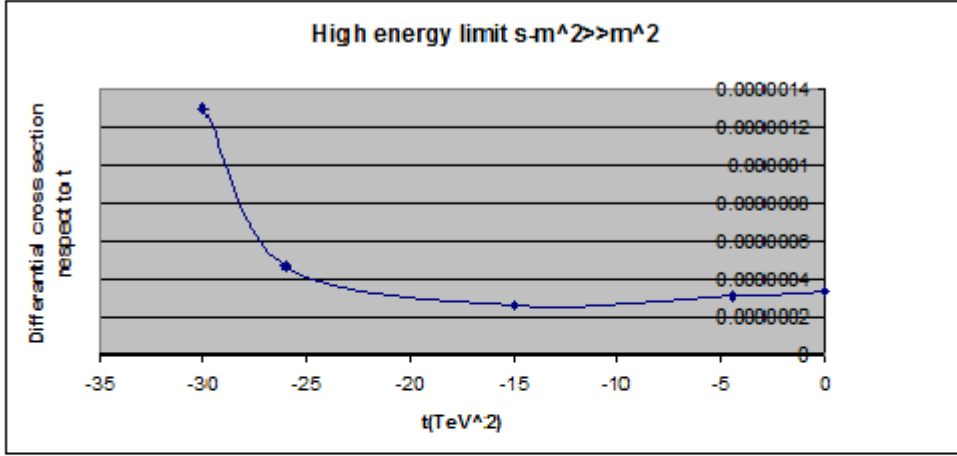


Figure 7: Differential Cross Section for $s - m^2 \gg m^2$. $\frac{d\sigma}{dt}$ is multiple of $\frac{1}{Y_{eV^4}}$ in this figure

Figure7 shows that differantial cross section sharply peaked in the forward $\theta = \cos 160$

at high energy limit and differential cross section is inversely proportional to square of the momentum transfer. This picture is similar to picture of $\frac{d\sigma}{d\cos\theta}$ in high energy case.

8 Conclusion

In this work, QED calculation for Compton scattering was evaluated. $\frac{d\sigma}{d\cos\theta}$ and Lorentz invariant $\frac{d\sigma}{dt}$ were calculated explicitly. when we convert $\frac{d\sigma}{dt}$ to $\frac{d\sigma}{d\cos\theta}$, we see that whereas all frames of references are equivalent, calculations can be of dissimilar magnitudes of complexity depending on the choice of the frame. One of the important section is plotting section. We plotted cross sections for different energy limits. As it is expected differential cross section display different manner between different energy limits. First we plotted $\frac{d\sigma}{d\cos\theta}$ against $d\cos\theta$. In different energy range, one can show that differential cross section is inversely proportional to energy. $\frac{d\sigma}{d\cos\theta}$ in low energy limit (non-relativistic) case is symmetric with respect to forwards and backwards scattering according to figures. Whereas $\frac{d\sigma}{d\cos\theta}$ behaves differently at high energy limit ($s \gg m^2$) peaks forwards in figure4 at $\theta = 180$. In relativistic case photon goes backwards through $\cos 180$. Pictures of $\frac{d\sigma}{dt}$ is similar with $\frac{d\sigma}{d\cos\theta}$. $\frac{d\sigma}{dt}$ in low energy case is also front-back symmetrical. $\frac{d\sigma}{dt}$ in high energy region peaks forwards and photon deflects through 180. $\frac{d\sigma}{dt}$ was calculated in various energy regions. One can see in figure7 square of the momentum transfer is inversly proportional to $\frac{d\sigma}{dt}$. In low energy cases picture of $\frac{d\sigma}{dt}$ is symmetric at $\theta = 0 - 180$ and $\theta = 45 - 135$. In high energy limit square of the momentum transfer is inversley proportional to $\frac{d\sigma}{dt}$. To sum up, there is inverse proportion between energy and differential cross section. $\frac{d\sigma}{d\cos\theta}$ is bigger than $\frac{d\sigma}{dt}$ in the same center-of-mass energy.

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