Calculating cross-sections in Compton scattering processes

Fredrik Björkeroth
School of Physics & Astronomy, University of Southampton

January 16, 2014.

Abstract
We consider the phenomenon of Compton scattering of a photon off an electron. The differential cross-section with respect to the (cosine of) scattering angle $\theta$ and Mandelstam variable $\ell$ is calculated in terms of the scattering probability, given by the square of the matrix element $i\mathcal{M}$. This matrix element is derived directly from the Feynman rules, and the probability evaluated analytically by taking an average over spin states for incoming and outgoing particles. The differential cross-section is found explicitly in both centre-of-mass and lab frames, and is shown to resolve to the Klein-Nishina formula in the lab frame. The differential cross-sections are then plotted against their respective dynamical parameters and the results are remarked upon.
1 Introduction

Central to our understanding of electromagnetic phenomena is the scattering of light off matter. A large amount of information about the physical properties of these processes may be compiled into a cross-section $\sigma$, which can be defined rigorously (and is done in this report). Often a differential cross-section $\frac{d^2 \sigma}{d\Omega}$ is measured in some dynamical variable $x$. The cross-section is directly correlated to the probability of a given interaction, itself the square of the scattering amplitude $iM$; this amplitude is in turn given by quantum field theory. Thus, in establishing scattering processes, the most immediate goal is to derive $iM$. This may be done by considering the Dyson series expansion for the time evolution of the particle states or by an expansion of the path integral [1]. Either method takes you by the Feynman rules for quantum electrodynamics (QED); as these have already been derived [2, 3], one may take a shortcut to the scattering amplitude by simply drawing the relevant Feynman diagrams and writing down the corresponding terms.

2 Deriving the Scattering Probability

There are two diagrams that contribute to Compton scattering at tree-level:

![Feynman diagrams](image)

Fig. 1

where $q_1 = k + p$ and $q_2 = k - p'$, with associated amplitudes $iM_1$ and $iM_2$ respectively. Four-momentum is conserved, i.e.

$$k + p - k' - p' = 0$$

(2.1)

The total amplitude is then $iM = i(M_1 + M_2)$, with an associated scattering probability $\langle |M|^2 \rangle$ such that

$$\langle |M|^2 \rangle \propto |M|^2$$

(2.2)

The proportionality is only an equality if the polarisations of all particles are measured; this is not typically done (or necessarily even practically possible) in experiment. Instead an average over initial states and a sum over final states is taken; we may then interpret $\langle |M|^2 \rangle$ as a polarisation-independent probability. From the Feynman rules, we arrive immediately at the amplitudes

$$iM_1 = -e^2 \varepsilon^{(\lambda)}_\nu(p)(p')(k')(k) \gamma^\rho \frac{i(q_1 + m)}{q_1 - m^2} \gamma^\mu u^{(r)}(k) e^{(\lambda)}_{\mu}(p)$$

(2.3)

$$iM_2 = -e^2 \varepsilon^{(\lambda)}_\mu(p)\bar{\pi}^{(s)}(k') \gamma^\mu \frac{i(q_2 + m)}{q_2 - m^2} \gamma^\nu u^{(r)}(k) e^{*(\lambda)}_{\nu}(p')$$

(2.4)

These describe the interactions of an incoming electron with momentum $k$ and spin $r$, incoming photon with momentum $p$ and polarisation $\lambda$, outgoing electron with momentum $k'$ and spin $s$, and outgoing photon with momentum $p'$ and polarisation $\lambda'$. The spinors obey completeness relations; assuming a normalisation of $2E$ particles per unit volume:

$$\sum_{s=1,2} u^{(s)}(k)\bar{u}^{(s)}(k) = \slashed{k} + m$$

(2.5)

The photon vectors are normalised by the condition

$$\varepsilon^{(\lambda)}_\mu \varepsilon^{*(\lambda')}_{\nu} = \eta^{\lambda\lambda'} \eta_{\mu\nu}$$

(2.6)

To evaluate $\langle |M|^2 \rangle$ we need complex conjugates, and to sum over $r, s, \lambda$ and $\lambda'$. The complex conjugate of $M_1$ is proportional to $[\bar{\pi}^{(s)}(k')\gamma^\rho(q_1 + m)\gamma^\mu u^{(r)}(k)]^\dagger$. Using the properties of $\gamma$-matrices, this expression is shown to
equal \([\Pi^{(r)}(k)\gamma^\mu(q_i + m)\gamma^\nu u^{(s)}(k')]\), with a similar result for \(M_2\). \(\langle |M|^2 \rangle\) is then a sum of terms proportional to quantities like \(u\bar{u}\), with \(u\) and \(\bar{u}\) having the same momentum and spin; \(\Gamma\) is the product of all other factors, and has (two) matrix indices. It is obvious that \(u_i\Gamma_{ij}u_j = u_j\bar{u}_i\Gamma_{ij} = \text{Tr} \{u\bar{u}\Gamma\}\). In essence, to move the final \(u\)-spinor to the front, we need to take the trace of the entire expression. The photon terms (summed over \(\lambda\) and \(\lambda'\)) condense to a product of Minkowski metric tensors as per (2.6). Thus, using (2.5) on the \(u\)-spinor terms,

\[
\langle |M|^2 \rangle = \frac{1}{4} \sum_{r=1}^{2} \sum_{s=1}^{2} |M_1 + M_2|^2
\]

\[
= e^4 \frac{1}{4} \left[ \text{Tr} \left\{ (k' + m)\gamma^\nu (q_i + m)\gamma^\mu (k + m)\gamma_\nu (q_i + m)\gamma_\mu \right\} \right. \\
+ 2 \text{Tr} \left\{ (k' + m)\gamma^\nu (q_i + m)\gamma^\mu (k + m)\gamma_\nu (q_i + m)\gamma_\mu \right\} \\
\left. + \text{Tr} \left\{ (k' + m)\gamma^\nu (q_i + m)\gamma^\mu (k + m)\gamma_\nu (q_i + m)\gamma_\mu \right\} \right]\] (2.7)

Evaluating these traces in terms of momenta is a straightforward but lengthy computation. Ostensibly, in each of the three traces there are 36 terms added; however, as the trace of an odd product of \(\gamma\)-matrices is zero, this leaves 18. We make extensive use of the following identities for \(\gamma\)-matrices:

\[
\begin{align*}
\gamma_\mu \gamma^\mu &= 4 \\
\gamma_\mu \gamma^\nu \gamma^\mu &= -2\gamma^\nu \\
\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\mu &= 4\gamma^\nu \gamma^\rho \\
\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu &= -2\gamma^\nu \gamma^\rho \gamma^\sigma \\
\text{Tr} \{1 \} &= 1 \\
\text{Tr} \{\gamma_\mu \gamma^\nu \} &= 4\eta^{\mu\nu} \\
\text{Tr} \{\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \} &= 4 (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho})
\end{align*}
\] (2.8)

This produces expressions in terms of products \(k \cdot p, k' \cdot p\) and \(p' \cdot p\). To express them slightly differently, we first define the Mandelstam variables \(s, t\) and \(u\) in the usual way

\[
\begin{align*}
s &= (k + p)^2 = m^2 + 2k \cdot p \\
t &= (p' - p)^2 = -2p' \cdot p \\
u &= (k' - p)^2 = m^2 - 2k' \cdot p
\end{align*}
\] (2.9)

and note that \(s\) is interpreted as the square of total momentum of the system – in the centre-of-mass frame, it is equivalent to the square of the total energy. Momentum conservation yields the relationship between them

\[
s + t + u = 2m^2
\] (2.10)

hence we may express any cross-section in terms of only two of these variables. Choosing \(s\) and \(u\) (the momentum cross-transfer between incoming photon and outgoing electron), the expression in (2.7) resolves to

\[
\langle |M|^2 \rangle = 2e^4 \left[ \frac{s - m^2}{u - m^2} - \frac{u - m^2}{s - m^2} + 4m^2 \left( \frac{1}{s - m^2} \frac{1}{u - m^2} \right) + 4m^4 \left( \frac{1}{s - m^2} \frac{1}{u - m^2} \right)^2 \right] (2.11)
\]

Choosing \(s\) and \(t\) instead yields

\[
\langle |M|^2 \rangle = 2e^4 \left[ \frac{(s - m^2)t^3 + (3s^2 - 2m^2s + 3m^4)t^2 + 4s(s - m^2)^2t + 2(s - m^2)^4}{(s - m^2)^2(t + s - m^2)^2} \right] (2.12)
\]

3 Calculating the Differential Cross-section

In order to interpret the probability \(\langle |M|^2 \rangle\) in terms of observable quantities, we turn it into a cross-section. Following the definition of a cross-section given in Ref. [4], if we consider two beams in a collider-like experiment, the cross-section \(\sigma\) will relate the number of scattering events to the physical parameters of the interaction,
specifically the densities of particles in the beams, the beam sizes and shape, and the overlap of their cross-section areas in the beam direction. The differential cross-section \( d\sigma \) describes scattered particles emerging within some momentum range. Equivalently, we may interpret the cross-section as the transition probability per unit time and volume times the number of final (momentum-) states, divided by the incident flux. Generally, for two particles \( A \) and \( B \) scattering into \( n \) particles,

\[
d\sigma = \frac{\langle |M|^2 \rangle}{4E_A E_B |v_A - v_B|} d\Pi_n
\]

where \( d\Pi_n \) is the \( n \)-body Lorentz-invariant phase space measure, defined by

\[
d\Pi_n = \left( \prod_{f=1}^{n} \frac{d^3p_f}{(2\pi)^3 2E_f} \right) (2\pi)^4 \delta^4(p_A + p_B - \Sigma p_f)
\]

In the case of \( 2 \to 2 \) scattering, there are six parameters in (3.2). Four of these may be extinguished by the \( \delta \)-function, which we expand into one- and three-dimensional \( \delta \)-functions over energy and momentum, respectively. We work in the center-of-mass frame. The integral over \( \delta \)-function, which we expand into one- and three-dimensional

\[
\int \delta(E_1 + E_2 - E_A - E_B) = \delta(p_1 - p_0) \left( \frac{p_1}{E_1} + \frac{p_1}{E_2} \right)^{-1}
\]

Gathering up terms, we find that

\[
\int d\Pi_2 = \int d\Omega \frac{1}{16\pi^2} \frac{p_1}{E_1 + E_2}
\]

If the particle beams are colinear, the flux term in the cross-section may be rewritten as

\[
4E_A E_B |v_A - v_B| = 4 \left( (p_A \cdot p_B)^2 - m_A^2 m_B^2 \right)^{\frac{1}{2}}
\]

In Compton scattering in the center-of-mass frame, we have

\[
p_A = (\omega, p) \quad p_1 = (\omega', p')
p_B = (E, -p) \quad p_2 = (E', -p')
\]

where \( \omega \equiv |p| \) and \( \omega' \equiv |p'| \). The flux term condenses to simply \( 4\omega(E + \omega) \). Inserting these terms into (3.4) and (3.5), we arrive at the differential cross-section

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{CM}} = \frac{1}{64\pi^2} \langle |M|^2 \rangle
\]

where we have used the fact that \( \sqrt{s} = E + \omega = E' + \omega' \), from which we also conclude that \( \omega' = \omega \), i.e. the collision is elastic in this frame. In the case where we have an azimuthal symmetry about the beam axis, this simplifies further; expressing the solid angle in terms of polar and azimuthal angles \( \theta \) and \( \phi \), i.e. \( d\Omega = d\cos \theta \, d\phi \), performing the integral over \( d\phi \) simply produces a factor \( 2\pi \).

The scattering angle \( \theta \) is not a Lorentz-invariant quantity; it is valuable to be able to write the cross-section instead in terms of the Mandelstam variable \( t \). Their relationship is given by

\[
t = (p - p')^2 = 2\omega'\omega(\cos \theta - 1)
\]

This in fact serves as a definition of the scattering angle in any frame. In the centre-of-mass frame, differentiation is easy, yielding \( dt = 2\omega^2 \, d\cos \theta \). Using the definition of \( s \) in (2.9), we may express the photon energy \( \omega \) entirely in terms of \( s \) and \( m \), by

\[
\omega^2 = \frac{(s - m^2)^2}{4s}
\]
Everything in the cross-section is now expressible in terms of Lorentz-invariant quantities; the differential cross-section is

\[
\frac{d\sigma}{dt} = \frac{1}{16\pi} \frac{1}{(s - m^2)^2} \langle |M|^2 \rangle 
\]  

(3.9)

In experiment, the scattering angle measured is that in the lab frame, i.e. the rest frame of the incident electron, and we will denote it \(\theta_L\). To evaluate the cross-section in terms of \(\theta_L\), we can make a change of variables \(dt \rightarrow d\cos \theta_L\). Returning to our definition in (3.3), we note that, unlike the centre-of-mass case, \(\omega'\) and \(\theta_L\) are not independent. The kinematic information for the collision is now

\[
\begin{align*}
    p_A &= (\omega, p) \\
    p_B &= (m, 0) \\
    p_1 &= (\omega', p') \\
    p_2 &= (E', p - p')
\end{align*}
\]  

(3.10)

Invoking conservation of momentum with these parameters produces the equation

\[
\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1 - \cos \theta)
\]  

(3.11)

Introducing this result into (3.9) and differentiating with respect to \(\cos \theta_L\), we plug our expression for \(dt\) into (3.9). Making the substitution \(s = (\omega + m)^2 - \omega^2\), we arrive ultimately at

\[
\frac{d\sigma}{d\cos \theta_L} = \frac{1}{32\pi} \frac{(\omega')^2}{m^2 \omega^2} \langle |M|^2 \rangle
\]  

(3.12)

Alternatively, we could have evaluated the differential cross-section from its definition (3.1), working in the lab frame. Using the kinematic parameters in (3.10), the phase space integral is

\[
\int d\Pi_2 = \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{2\omega'} \frac{d^3 k'}{(2\pi)^3} \frac{1}{E'} (2\pi)^4 \delta(4)(k' + p' - k)
\]  

(3.13)

3-momentum conservation extinguishes the integral over \(d^3 k'\), setting \(k' = p - p'\). Assuming an azimuthal symmetry of the collision, we can make the change of variables \(d^3 p' \rightarrow 2\pi (\omega')^2 d\omega' d\cos \theta_L\). To resolve the final integral over \(\omega'\), we make use of the property in (3.3) to rewrite the \(\delta\)-function over energy as

\[
\delta(\omega' + \sqrt{m^2 + (\omega')^2} - \omega - m) = \frac{\delta(\omega' - \omega_0)}{\sqrt{1 + \omega' \omega_0}}
\]  

The integration over \(\omega'\) is now trivial. Collecting terms, the phase space measure resolves to

\[
\int d\Pi_2 = \frac{1}{8\pi} \int d\cos \theta_L \frac{(\omega')^2}{\omega m}
\]

Furthermore, from (3.5) we see that the flux is simply \(4m\omega\). Assembling all components yields a differential cross-section in terms of \(\langle |M|^2 \rangle\) that is identical to that in (3.12), as is expected. Having evaluated \(\langle |M|^2 \rangle\) in terms of \(s\) and \(u\) in (2.11), we use the definitions in (2.9) and (3.10) to write them in terms of \(\omega, \omega'\) and \(\theta\). The final result is

\[
\frac{d\sigma}{d\cos \theta_L} = \frac{e^4}{16\pi m^2} \left\langle \frac{\omega'}{\omega} \right\rangle \omega^2 \left[ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2 \theta_L \right]
\]  

(3.14)

This is the Klein-Nishina formula. It can be made an explicit function of the centre-of-mass energy \(\sqrt{s}\) by noting that

\[\omega = \frac{s - m^2}{2m}\]

Hence we can loosely group interactions into three cases: (1) low-energy collisions where \(s - m^2 \ll m^2\), (2) mid-level collisions where \(s - m^2 \sim m^2\), and (3) high-energy collisions where \(s \gg m^2\). Fig. 2 plots \(\frac{d\sigma}{d\cos \theta_L}\) with respect to \(\cos \theta_L\) for each of these regions. In the low energy region, the cross-section is nearly symmetrical about a scattering angle of 90°, which also serves as a minimum. The function is proportional to \(1 + \cos^2 \theta_L\), which respects the prediction of classical Thomson scattering. As such, low-energy scattering gently favours observation near the axis of the beams, in both the forward and backward directions. In the high-energy region, the cross-section is nearly zero everywhere but in the direction of forward photon momentum. Returning to the differential
cross-sections in the centre-of-mass frame with respect to $t$ (3.9) and $\theta$ (3.7), it is clear that not only the shape of the cross-section but the scale has a strong dependence on the total energy of the interaction; dimensionally, $\frac{d\sigma}{dt}$ has an $s^{-2}$ dependence, and is expressed in units of $[E]^{-4}$. The results are plotted in Fig. 3 and Fig. 4. Fig. 3 shows $\frac{d\sigma}{dt}$ as a function of $t$. $t$ is, like $\theta$, naturally bounded above and below. Inspection of equation (3.8) shows that $-\frac{4\omega^2}{s} < t < 0$, i.e. $t$ is strictly negative and its value is bounded by a function of the total energy (since $\omega$ and $s$ can be transformed into one another). In the high-energy region, there is a preference toward large $|t|$, suggesting interactions with a large momentum transfer are preferred. At low energy, the dependence on centre-of-mass angle $\theta$ is the same as on $\theta_L$, as the two frames of reference are the same in the limit of $\omega \to 0$. At high energy, however, there is a large preference for scattering near $180^\circ$, wherein a scattered photon travels nearly back along its original trajectory. Physically this is not particularly significant, it is merely a reflection on the fact that the frame is moving with respect to both particles (unlike the lab frame). The mid-energy plot indicates a combination of two competing factors, resulting in a maximum at $180^\circ$, with a minimum now at $\theta > 90^\circ$. 
Fig. 3: Differential cross-section $\frac{d\sigma}{dt}$ plotted against $t$. The solid line is the value of the cross-section when $s - m^2 = \xi m^2$ for a given $\xi$. The dotted lines show the curve when $\xi$ has been shifted by a certain percentage from the central value, to demonstrate the volatility of the dependence on $\xi$. Note that the upper dotted line is, for all figures given here, that corresponding to lower $\xi$.

(a) $s - m^2 \ll m^2$. Central $\xi$ is 0.05, with a $\pm 10\%$ shift plotted.

(b) $s - m^2 \sim m^2$. Central $\xi$ is 1, with a $\pm 10\%$ shift plotted.

(c) $s \gg m^2$. Central $\xi$ is 50, with a $\pm 2\%$ shift plotted. Note the particularly strong divergence in the limit of large $|t|$. 
Fig. 4: Differential cross-section \( \frac{d\sigma}{d\cos \theta} \) plotted against \( \cos \theta \). The solid line is the value of the cross-section when \( s - m^2 \ll m^2 \) for a given \( \xi \). The dotted lines show the curve at a \( \pm 20\% \) shift in \( \xi \). Note that the upper dotted line is, for all figures given here, that corresponding to lower \( \xi \).
4 CONCLUSION

Having derived the matrix element for a Compton scattering event, we have gone on to demonstrate how this information relates to physical parameters such as scattering angle and centre-of-mass energy. The differential cross-sections with respect to angle $\theta$ and the Mandelstam variable $t$ were calculated in both the centre-of-mass and lab frames, resulting in the plots Fig. 3 and Fig. 4. In further analysis, we successfully reproduced the Klein-Nishina formula. Our results show that the differential cross-section is strongly dependent on the energy of the system: in the low-energy limit, we reproduce the classical result for Thomson scattering. Consequently at high energy, we note a cross-section that is peaked in the region of parameter space corresponding to a maximal momentum transfer between incoming and outgoing photons, as well as an angular dependence that favours scattering close to the beam axis.
References


