Path Integral in Quantum Field Theory

Alexander Belyaev
(course based on Lectures by Steven King)

Contents

Chapter 1

Preliminaries

1.1 Review of Classical Mechanics of Finite System

Consider a finite classical system of particles whose generalised coordinates are q_i where $i = 1, \ldots, N$. For example $N = 3n$ for n particles in three dimensions. Suppose the Lagrangian given by $L = T - V$, T and V being the kinetic and potential energy, is:

$$L = L(q_i, \dot{q}_i) \tag{1.1.1}$$

where $\dot{q}_i$ denotes the time derivative of q_i. For each set of paths $q_i(t)$ connecting (q_{i1}, t_1) to (q_{i2}, t_2) the action is defined by:

$$S[q_i(t)] = \int_{t_1}^{t_2} L dt \tag{1.1.2}$$

Note that S is a functional, i.e. a number whose value depends on a set of functions $q_i(t)$. The classical paths are those which minimise (or in general extremise) the numerical value of S, subject to the boundary conditions $\delta q_i = 0$ at the initial and final times t_1 and t_2. This is the well-known action principle (which can be derived from Feynman's path integral approach to quantum mechanics – see later) and it leads to the N second order (in time) differential Lagrange equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{1.1.3}$$

In the Lagrange formalism the independent variables are the coordinates q_i and the velocities $\dot{q}_i$. The generalised momenta p_i are derived quantities defined by:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{1.1.4}$$

In the Hamiltonian formalism the variables $\dot{q}_i$ are traded in for p_i, and instead of working with a Lagrangian $L = L(q_i, \dot{q}_i)$ one works with a Hamiltonian

$H = H(q_i, p_i)$ defined by the so-called Legendre transformation:

$$H(q_i, p_i) \equiv \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i) \tag{1.1.5}$$

where it is understood that the $\dot{q}_i$ are to be written as functions of q_i and p_i. From Eqs.1.1.3-1.1.5 one readily obtains the Hamilton equations of motion:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1.1.6}$$

which are $2N$ first order (in time) differential equations. Just as (for a conservative force) $L = T - V$, so $H = T + V$, so the Hamiltonian is just the total energy in this case.

Example 1: The Harmonic Oscillator

First consider a single oscillator in oscillating in one dimension (e.g. mass on a spring)

The Lagrangian is:

$$L = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} \omega^2 q^2 \tag{1.1.7}$$

The generalised momentum is:

$$p = \frac{\partial L}{\partial \dot{q}} = m \dot{q} \tag{1.1.8}$$

The Hamiltonian is:

$$H = p\dot{q} - L = \frac{p^2}{2m} + \frac{\omega^2}{2} q^2 \tag{1.1.9}$$

Example 2: Coupled Harmonic Oscillators

Now consider a linear chain of coupled oscillators, say corresponding to a vibrating line of atoms. Let the coordinate q_i denote the displacement of the $i - th$ atom in the line from its equilibrium position, where we assume that all displacements correspond to longitudinal vibrations along the length of the line of atoms. To make the system finite, let us form a closed loop of N atoms such that $q_{i+N} = q_i$.

The Lagrangian is:

$$L = \sum_{i=1}^{N} \frac{1}{2} {\dot{q}_i}^2 - \frac{\Omega^2}{2} (q_i - q_{i+1})^2 - \frac{\Omega_0^2}{2} (q_i)^2 \tag{1.1.10}$$

where we have taken the atoms to have unit mass, and have included a nearest neighbour coupling plus a second independent frequency Ω_0.

The Lagrange equations of motion are:

$$\ddot{q}_i = -\Omega^2 (2q_i - q_{i-1} - q_{i+1}) - \Omega_0^2 q_i \tag{1.1.11}$$

The generalised momenta are:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i \tag{1.1.12}$$

The Hamiltonian is:

$$H = \sum_{i=1}^{N} \frac{1}{2} {p_i}^2 + \frac{\Omega^2}{2}(q_i - q_{i+1})^2 + \frac{\Omega_0^2}{2}(q_i)^2 \tag{1.1.13}$$

Normal Modes

The oscillators above can be uncoupled by taking a discretised Fourier transform of the form

$$q_j = \sum_{k=-N/2}^{N/2} \frac{\tilde{q}_k}{N^{1/2}} e^{ij2\pi k/N} \tag{1.1.14}$$

$$p_j = \sum_{k=-N/2}^{N/2} \frac{\tilde{p}_k}{N^{1/2}} e^{-ij2\pi k/N} \tag{1.1.15}$$

which satisfies the periodic boundary conditions. Note that $i^2 = -1$ and we have labelled the oscillators by j to avoid confusion.

Since the q_j are real the $\tilde{q}_k$ and $\tilde{p}_k$ are complex and satisfy

$$\tilde{q}_{-k} = \tilde{q}_k^\star, \quad \tilde{p}_{-k} = \tilde{p}_k^\star. \tag{1.1.16}$$

Using the result:

$$\sum_{j=1}^{N} e^{ij2\pi(k-k')/N} = \delta_{kk'} N$$

the Hamiltonian becomes,

$$H = \frac{1}{2} \sum_{k=-N/2}^{N/2} \tilde{p}_k \tilde{p}_k^\star + \omega_k^2 \tilde{q}_k \tilde{q}_k^\star \tag{1.1.17}$$

which corresponds to a sum of uncoupled harmonic oscillators of frequencies given by

$$\omega_k^2 = \Omega^2 4 \sin^2(\pi k/N) + \Omega_0^2 \tag{1.1.18}$$

1.2 Review of Non-Relativistic Quantum Mechanics

In quantum mechanics, [1] physical states are represented by vectors $|\psi>$ in a vector space (Hilbert space). Physical observables are represented by linear

[1] We shall set $h/(2\pi) = c = 1$ throughout.

Hermitian operators $\hat{\omega}$ and the result of a measurement of some observable $\hat{\omega}$ is one of its (real) eigenvalues ω where

$$\hat{\omega}|\omega >= \omega|\omega > \tag{1.2.1}$$

A very useful result is the completeness relation:

$$I = \int_{-\infty}^{\infty} |\omega >< \omega|$$

which follows from the fact that the eigenvectors of an Hermitian operator form a complete orthonormal basis, where I is the unit operator. If the system is in some state $|\psi >$ and some observable $\hat{\omega}$ is measured then the system will be forced into one of the eigenstates $|\psi >\rightarrow |\omega >$. The result of the measurement will be one of the eigenvalues ω which is not classically predictable, but which occurs with a probability $P(\omega)$ given by

$$P(\omega) = \frac{| < \omega|\psi > |^2}{< \psi|\psi >} \tag{1.2.2}$$

The average or expectation value is therefore,

$$< \hat{\omega} >= \frac{< \psi|\hat{\omega}|\psi >}{< \psi|\psi >} \tag{1.2.3}$$

Now let us assume we are given a classical Hamiltonian in some particular basis of coordinates $H = H(q_i, p_i)$. According to quantum mechanics we must replace the number-valued variables q_i and p_i by Hermitian operators which satisfy the commutation relations:

$$[\hat{q}_i, \hat{p}_j] = iI\delta_{ij}, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \tag{1.2.4}$$

where I is the unit operator, 0 is the null operator, $[A, B] = AB - BA$ and δ_{ij} is the Kronecker delta symbol defined in the usual way

$$\delta_{ij} = 1 \quad (i = j)$$

$$\delta_{ij} = 0 \quad (i \neq j)$$

Each coordinate has its own complete set of eigenvectors and real eigenvalues

$$\hat{q}_i|q_i >= q_i|q_i >, \quad \hat{p}_i|p_i >= p_i|p_i >$$

The position and momentum space wavefunctions are thus

$$\psi(q_1, q_2, \ldots) =< q_1| < q_2| \ldots |\psi >$$

and

$$\psi(p_1, p_2, \ldots) =< p_1| < p_2| \ldots |\psi >$$

In one dimension we have simply,

$$\psi(q) =< q|\psi >, \quad \psi(p) =< p|\psi >$$

The way the time dependence of a quantum system is described is a matter of taste and convenience. Schrodinger preferred to regard all the time dependence as residing in the state vector, $|\psi >= |\psi(t) >$, whose time evolution is determined by the equation which bears his name,

$$i\frac{\partial}{\partial t}|\psi(t) >= H|\psi(t) > \tag{1.2.5}$$

If H has no explicit t dependence a number of simplifications follow. To begin with H may be regarded as an energy operator with time independent eigenvectors and eigenvalues,

$$H|E_n >= E_n|E_n > \tag{1.2.6}$$

In addition the formal solution to Schrodinger's equation is,

$$|\psi(t) >= e^{-iHt}|\psi(0) > \tag{1.2.7}$$

where $|\psi(0) >$ is the state vector at time $t = 0$. More generally for two arbitrary times $t_2 > t_1$

$$|\psi(t_2) >= e^{-iH(t_2-t_1)}|\psi(t_1) > \tag{1.2.8}$$

The operator $U(t_2, t_1) = e^{-iH(t_2-t_1)}$ is called the time translation operator.

Restricting ourselves to a single coordinate q for simplicity, pre-multiplying by $< q_2|$ and using the completeness relation, Eq.1.2.8 becomes,

$$< q_2|\psi(t_2) >= \int dq_1 < q_2|U(t_2, t_1)|q_1 >< q_1|\psi(t_1) > \tag{1.2.9}$$

or,

$$\psi(q_2, t_2) = \int dq_1 U(q_2, t_2, q_1, t_1)\psi(q_1, t_1) \tag{1.2.10}$$

where

$$U(q_2, t_2, q_1, t_1) =< q_2|U(t_2, t_1)|q_1 > \tag{1.2.11}$$

is commonly referred to as the propagator. Eq.1.2.10 shows that knowledge of the wavefunction at time t_1, together with the propagator, is sufficient to determine the wavefunction at some later time t_2, providing the integral can be performed. A trivial example is if the initial wavefunction at time $t_1 = 0$ is the Dirac delta function $\psi(q_1, t_1 = 0) = \delta(q_1)$ then the final wavefunction is just the propagator itself, $\psi(q_2, t_2) = U(q_2, t_2, q_1 = 0, t_1 = 0)$. This shows that the propagator is just the amplitude to find the particle at position q_2 at time t_2 given that it was located precisely at position q_1 at time t_1.

Heisenberg preferred an alternative picture of time dependence, in which the time dependence is carried by the operators and the eigenstates of the

operators rather than the state vectors. Again restricting ourselves to a single coordinate q for simplicity, we first summarise the Schrodinger picture:

$$\hat{q}|q>=q|q>, \quad |\psi(t)>=e^{-iHt}|\psi(0)>, \quad \psi(q,t)=<q|\psi(t)> \tag{1.2.12}$$

The Heisenberg picture replaces the static position basis vectors $|q>$ by the time-dependent position basis vectors $|q,t>$ where

$$|q,t>\equiv e^{iHt}|q> \tag{1.2.13}$$

where the sign of the exponent in Eq.1.2.13 should be contrasted to that in Eq.1.2.7. The Heisenberg picture also replaces the static operator coordinate $\hat{q}$ by the time dependent operator $\hat{q}(t)$ where

$$\hat{q}(t) \equiv e^{iHt}\hat{q}e^{-iHt} \tag{1.2.14}$$

With these definitions it is clear that

$$\hat{q}(t)|q,t>=q|q,t> \tag{1.2.15}$$

so that $|q,t>$ is a complete orthonormal basis of the Hermitian operator $\hat{q}(t)$. Physically $|q,t>$ represents a position eigenstate of the particle at the time t. In terms of the Heisenberg basis, the propagator in Eq.1.2.11 is given very simply as

$$U(q_2,t_2,q_1,t_1)=<q_2,t_2|q_1,t_1> \tag{1.2.16}$$

which shows very clearly that the propagator is just the amplitude that the particle is at q_2,t_2 given that it was at q_1,t_1.

The equivalence of the two bases should be clear. The time dependent wavefunction may be expressed either in terms of the "static" Schrodinger position basis or in terms of the "rotating basis" of the Heisenberg picture,

$$\psi(q,t)=<q|\psi(t)>=<q,t|\psi(0)> \tag{1.2.17}$$

In the Schrodinger picture the time dependence of the wavefunction is carried by $|\psi(t)>$ with $|q>$ static, while in the Heisenberg picture the time dependence is carried by $|q,t>$ with the $|\psi(0)>$ static. The two pictures are clearly totally equivalent, it just depends where one wishes to absorb the time translation operator e^{iHt}. In the Heisenberg picture every operator has a time dependence, with the exception of the Hamiltonian itself, and is related to the corresponding static Schrodinger operator by:

$$\hat{\omega}(t) \equiv e^{iHt}\hat{\omega}e^{-iHt} \tag{1.2.18}$$

The Heisenberg equation of motion follows:

$$[H,\hat{\omega}(t)] = -i\dot{\hat{\omega}}(t) \tag{1.2.19}$$

Clearly there are an infinite number of possible "pictures" all related by the same type of unitary transformation. In certain situations the Hamiltonian

naturally splits into two pieces: a "free" Hamiltonian H_0 the solution to which is known, plus a small "interaction" V. In such situations we wish to perform a perturbation expansion in the interaction, and in this situation it is convenient to use the "interaction picture" defined later.

Example 1: The Quantum Harmonic Oscillator

The one-dimensional harmonic oscillator (described by the single coordinate q) we considered previously but with unit mass has the time independent Hamiltonian

$$H(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2) \tag{1.2.20}$$

Classically Hamilton's equations in Eq.1.1.6 yield

$$\dot{q} = p, \quad \dot{p} = -\omega^2 q \tag{1.2.21}$$

In quantum mechanics q and p become operators and acquire hats, which we shall drop out of laziness. [2] Since H is time independent all of the nice results above are applicable, and we can choose to carry the time dependence in either the state vector or the coordinates. If we work in the latter Heisenberg picture then Heisenberg's equation 1.2.19 yields identical results to Eq.1.2.21, except that it now involves operators. In either picture we need to solve the energy eigenvalue problem Eq.1.2.6. To do this one introduces the aptly named raising and lowering operators $a^\dagger$ and a, defined by:

$$a \equiv (\omega/2)^{1/2} q + i(1/2\omega)^{1/2} p, \quad a^\dagger \equiv (\omega/2)^{1/2} q - i(1/2\omega)^{1/2} p \tag{1.2.22}$$

Because $[p,q] = -i$ one finds

$$[a, a^\dagger] = I \tag{1.2.23}$$

Also,

$$H = (a^\dagger a + \frac{1}{2})\omega \tag{1.2.24}$$

Hence,

$$[a, H] = \omega a, \quad [a^\dagger, H] = -\omega a^\dagger \tag{1.2.25}$$

If we are in the Heisenberg picture then all the operators above have a time dependence, and all the relations above apply at each time t. For example it can be shown by solving the equations of motion that

$$a(t) = a(0)e^{-i\omega t}, \quad a^\dagger(t) = a^\dagger(0)e^{i\omega t}$$

In the Schrodinger picture we simply identify $a = a(0)$, $a^\dagger = a^\dagger(0)$. Note that the Hamiltonian is time independent in both pictures.

It can be shown using eq.1.2.25 that the solution to the eigenvalue problem for the lowest energy state $|0>$ is

$$H|0> = \frac{\omega}{2}|0> \tag{1.2.26}$$

[2] Hats off to quantum mechanics!

and that in general

$$H|n>=(n+\frac{1}{2})\omega|n> \tag{1.2.27}$$

where n is any positive integer 0,1,2,.. and the eigenvectors are given by

$$|n>=\frac{1}{\sqrt{n!}}(a^\dagger)^n|0> \tag{1.2.28}$$

The operator $N = a^\dagger a$ is commonly called the occupation number operator. The operators are named raising and lowering operators because

$$a^\dagger|n>=\sqrt{n+1}|n+1>, \quad a|n>=\sqrt{n}|n-1>, \tag{1.2.29}$$

The humble harmonic oscillator, familiar since school days, turns out to sit at the heart of quantum field theory, as we shall see.

Example 2: Quantised Coupled Harmonic Oscillators

Let us now return to our coupled Harmonic oscillator example and quantise it. The Hamiltonian was written in terms of normal mode tilded variables as

$$H=\frac{1}{2}\sum_{k=-N/2}^{N/2}\tilde{p}_k\tilde{p}_k^\star+\omega_k^2\tilde{q}_k\tilde{q}_k^\star \tag{1.2.30}$$

which corresponds to a sum of uncoupled harmonic oscillators of frequencies given by

$$\omega_k^2=\Omega^2 4\sin^2(\pi k/N)+\Omega_0^2 \tag{1.2.31}$$

We now regard $\tilde{p}_k$ and $\tilde{q}_k$ as operators with Eq.1.1.16 generalised to

$$\tilde{q}_{-k}=\tilde{q}_k^\dagger, \quad \tilde{p}_{-k}=\tilde{p}_k^\dagger. \tag{1.2.32}$$

We define creation and annihilation operators as

$$a_k=(\omega_k/2)^{1/2}\tilde{q}_k+i(1/2\omega_k)^{1/2}\tilde{p}_k^\dagger, \quad a_k^\dagger=(\omega_k/2)^{1/2}\tilde{q}_k^\dagger-i(1/2\omega_k)^{1/2}\tilde{p}_k \tag{1.2.33}$$

where in the Heisenberg picture

$$a_k(t)=a_k(0)e^{-i\omega_k t}, a_k^\dagger(t)=a_k^\dagger(0)e^{i\omega_k t}, \tag{1.2.34}$$

The equal time commutation relation is

$$[a_k(t),a_{k'}^\dagger(t)]=I\delta_{kk'} \tag{1.2.35}$$

The Hamiltonian can be expressed as

$$H=\sum_k(a_k^\dagger a_k+1/2)\omega_k \tag{1.2.36}$$

We would like the ground state $|0>$ for which $a_k|0>=0$ to have zero energy rather than a contribution of $1/2\omega_k$ from each oscillator. This can be achieved by defining a new Hamiltonian to be:

$$H'\equiv\sum_k(a_k^\dagger a_k)\omega_k \tag{1.2.37}$$

What is the new Hamiltonian in terms of the original variables? We write this as:

$$H = \sum_{i=1}^{N} : (\frac{1}{2}{p_i}^2 + \frac{\Omega^2}{2}(q_i - q_{i+1})^2 + \frac{\Omega_0^2}{2}(q_i)^2) : \tag{1.2.38}$$

where the : : notation means "put all the annihilation operators to the right". This is called normal ordering. Any normal ordered product of operators acting on the ground state gives zero. It is common convention to write H' as $H^{N.O.}$ or simply drop all the superscripts and write it as H, with the understanding that it is normal ordered. We shall again follow the lazy man's convention of dropping everything.

If $N \gg k$ then the frequencies of the oscillators are given by

$$\omega_k^2 \approx \Omega^2 4(\pi k/N)^2 + \Omega_0^2 \tag{1.2.39}$$

Thus the energy of each excitation is equal to ω_k, where this expression formally resembles the energy of a relativistic particle of mass $m = \Omega_0$ and momentum $p = 2\Omega(\pi k/N)$. Thus, now that we have removed the zero point energy, we may regard the quanta of energy created and destroyed by $a_k^\dagger$ and a_k as formally similar to the energy of free relativistic particles. The ground state of the system $|0>$ corresponds to there being no "particles" and having zero energy, in other words it resembles the relativistic vacuum. The single "particle" state corresponds to $a_k^\dagger|0>$ where the "particle" has momentum $p = 2\Omega(\pi k/N)$. A two "particle" state with momenta p and $p' = 2\Omega(\pi k'/N)$ is given by $a_k^\dagger a_k'^\dagger|0>$, and so on. The "particle" number operator is now given by:

$$N = \sum_k a_k^\dagger a_k \tag{1.2.40}$$

which counts the number of particles with energy ω_k and sums over the allowed energies.

Finally recall the original expansion of the variables in terms of tilded variables,

$$q_j = \sum_{k=-N/2}^{N/2} \frac{\tilde{q}_k}{N^{1/2}} e^{ij2\pi k/N} \tag{1.2.41}$$

$$p_j = \sum_{k=-N/2}^{N/2} \frac{\tilde{p}_k}{N^{1/2}} e^{-ij2\pi k/N} \tag{1.2.42}$$

Now everything is an operator, and inverting Eq.1.2.33 we find

$$\tilde{q}_k = (a_k + a_{-k}^\dagger)/(2\omega_k)^{1/2}, \quad \tilde{p}_k = (a_{-k} - a_k^\dagger)/i(2/\omega_k)^{1/2} \tag{1.2.43}$$

and hence,

$$q_j = \sum_k \frac{a_k}{(2N\omega_k)^{1/2}} e^{ij2\pi k/N} + \frac{a_k^\dagger}{(2N\omega_k)^{1/2}} e^{-ij2\pi k/N} \tag{1.2.44}$$

$$p_j = \sum_k \frac{a_k}{i(2N/\omega_k)^{1/2}} e^{ij2\pi k/N} - \frac{a_k^\dagger}{i(2N/\omega_k)^{1/2}} e^{-ij2\pi k/N} \tag{1.2.45}$$

so that the original operator variables are expressed as a sum of plane waves, with each term multiplied by a creation and annihilation operator appropriate to that particular mode.

1.3 Relativistic Quantum Mechanics

1.3.1 Relativistic Conventions and Notation

In relativity space-time coordinates $(t, x, y, z) \equiv (t, q_i)$ are denoted by the contravariant four-vector

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$$

The covariant four-vector x_μ is obtained by changing the sign of the space components:

$$x_\mu \equiv (x_0, x_1, x_2, x_3) \equiv (t, -x, -y, -z) = g_{\mu\nu} x^\nu$$

where the metric is chosen to be

$$g_{\mu\nu} = g^{\mu\nu} = diag(1, -1, -1, -1)$$

Also note

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

My convention for the totally antisymmetric Levi-Civita tensor is:

$$\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \{\mu, \nu, \lambda, \sigma\} \text{ an even permutation of } \{0, 1, 2, 3\} \\ -1 & \text{if an odd permutation} \\ 0 & \text{otherwise} \end{cases} \tag{1.3.1}$$

Note that $\epsilon^{\mu\nu\lambda\sigma} = -\epsilon_{\mu\nu\lambda\sigma}$.

For example momentum vectors are:

$$p^\mu = (E, p_x, p_y, p_z)$$

with

$$p_1 . p_2 = p_1^\mu p_{2\mu} = E_1 E_2 - p_i p_i$$

The partial derivatives are,

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

In other words,

$$\partial_\mu = (\partial^0, \nabla), \quad \partial^\mu = (\partial^0, -\nabla)$$

The energy-momentum operator in the coordinate basis is written using the substitutions

$$E \to i\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -i\nabla$$

as,

$$p^\mu = i\partial^\mu, \quad p_\mu = i\partial_\mu$$

Thus

$$p^2 = p^\mu p_\mu = -\partial^\mu \partial_\mu = -\partial^2$$

where

$$\partial^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

1.3.2 The Klein-Gordon Equation

Most of what has been said about non-relativistic quantum mechanics applies equally well to the relativistic domain. The only real difference is that the Hamiltonian H must be "souped-up" to become relativistic. For a single spinless boson, this can quite easily be achieved, at least for a free particle. Let us work in the Schrodinger picture and iterate the Schrodinger equation 1.2.5 to yield,

$$-\frac{\partial^2}{\partial t^2}|\psi(t)>= H^2|\psi(t)> \tag{1.3.2}$$

We are now in a position to use the Einstein equation for the square of the energy of a free particle of mass m, which we equate to the square of the Hamiltonian,

$$H^2 = \mathbf{p}^2 + m^2 \tag{1.3.3}$$

where the sum over the three momentum coordinates is implied. In the coordinate basis we have $p_i = -i\frac{\partial}{\partial q_i}$ hence we find

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial q_i^2} + m^2)\phi(q_i, t) = 0 \tag{1.3.4}$$

where it is conventional to denote the wavefunction of a spinless boson by ϕ rather than ψ which is usually reserved for spin-1/2 fermions. Eq.1.3.4 is known as the Klein-Gordon (KG) equation. It applies to a single relativistic spinless boson. However there is a well known problem since the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_i}{\partial q_i} = 0 \tag{1.3.5}$$

(again a sum over i is implied) is satisfied by a non-positive definite probability density. In the case of the non-relativistic Schrodinger equation you all know how to obtain an expression for the probability density and current: take the

complex conjugate of the equation and multiply it by the wavefunction, then subtract this from the original equation multiplied by the complex conjugate of the wavefunction – yielding the results:

$$\rho = \psi^*\psi, \quad j_i = \frac{-i}{2m}(\psi^*\frac{\partial\psi}{\partial q_i} - \psi\frac{\partial\psi^*}{\partial q_i})$$

We are thus entitled to interpret $\psi^*\psi dq_i$ as the probability of finding the particles in some range q_i to $q_i + dq_i$, with the integrated probaility being conserved. In the case of the KG equation a similar procedure yields a non-positive definite probability density:

$$\rho = i(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}) \tag{1.3.6}$$

with a current density:

$$j_i = -i(\phi^*\frac{\partial\phi}{\partial q_i} - \phi\frac{\partial\phi^*}{\partial q_i}) \tag{1.3.7}$$

Physically we cannot tolerate a negative probability density and historically this led Dirac to abandon the KG equation and search for an equation with positive definite probability density. However it turns out that the KG equation describes spinless bosons in the context of quantum field theory.

In relativistic notation the KG equation 1.3.4 becomes

$$(\partial^2 + m^2)\phi(x) = 0 \tag{1.3.8}$$

which looks a bit neater. The continuity equation becomes

$$\partial_\mu j^\mu = 0 \tag{1.3.9}$$

where the four-vector current is

$$j^\mu = (\rho, j_i) = \frac{i}{m}\phi^* \overleftrightarrow{\partial^\mu} \phi \tag{1.3.10}$$

where we have defined

$$A \overleftrightarrow{\partial^\mu} B \equiv \frac{1}{2}(A\partial^\mu B - (\partial^\mu A)B) \tag{1.3.11}$$

The KG equation suffers also from the problem of negative energy solutions which correspond to the two possible solutions of Eq.1.3.4

$$\phi(x) = Ne^{-i(Et - \mathbf{p}.\mathbf{x})} \tag{1.3.12}$$

where N is a normalisation constant and $E = \pm\sqrt{\mathbf{p}^2 + m^2}$. Choosing $N = 1$ corresponds to normalising to $2E$ particles per unit volume. Do not be concerned that the normalisation depends on the energy E, since Lorentz contraction will also play a role in defining the unit volume, and in the end physical quantities will be Lorentz invariant. This is commonly referred to as relativistic normalisation, and we shall use it extensively.

Actually also the Dirac equation has negative energy solutions as well for the same reason, but they are fixed-up by invoking a Dirac sea of filled negative energy levels so that the physical states cannot fall into them due to Pauli exclusion. However since this explanantion relies on the Pauli exclusion principle, it fails for the KG equation which describes spinless bosons. So the KG equation has two killer problems: negative probability and negative energy solutions. In fact there is also a further problem with relativistic equations which describe bosons, and that is that the number of bosons is not physically conserved. For example if two protons are smashed together with enough energy the final state may contain any number of spinless pions. So it is fruitless to try to contruct a conserved probability density for bosons in any case. This situation is also true of photons which certainly are not conserved since they may be absorbed or emitted by atoms. In the case of photons we know that they are described by an electromagnetic field, and in some sense we must identify the photons as the quanta of that field. Historically it was suggested that space was filled with an ether, and that light waves were vibrations in that ether. The modern view is not so dissimilar since it requires space to be inhabited by an electromagnetic field whose quanta are the photons, although the field is more abstract and the physical reality lies in the excitations of the field. In fact the Maxwell field is relativistically invariant, and it can be successfully quantised to yield photons, but the technical complications mean that we must leave this field theory until last. Nevertheless the electromagnetic field is the inspiration we need to solve the problems of the Klein-Gordon equation. It turns out that the solution to both the negative probability, and the negative energy problems, as well as the problem that the number of bosons is not physically conserved anyway, can all be solved by inventing a new field: the Klein-Gordon field. The idea is that, when the KG field is quantised, it will yield quanta which may be identified as particles of mass m. In terms of the field theory, it will turn out that the energy of the quanta is always positive, and the number of bosons is certainly not conserved. The rogue non-positive definite probability density will be reinterpreted as a electric charge density, and the probability currrent as an electric current density, where the idea of the bosons carrying electric charge is connected to the fact that the field is complex. If the field were real the charge and current would be zero.

1.4 Problems Set 1

1. Find the normal modes of a coupled pendulum system where each pendulum has a length l and has a mass m on the end. Use Lagrangian methods with the initial angular variables θ_1 and θ_2. Then diagonalise the coupled matrix equations to obtain normal mode angular variables and hence normal mode frequencies.

2. Find the normal modes of two coupled equal masses m on springs sliding on a frictionless table with a one dimensional motion along the length of the springs which are attached to two fixed points at either end. Take the spring constant for the outer springs to be k and the spring constant for the spring which connects the two masses to be g. Find the Lagrangian and Hamiltonian of the system in terms of the displacements q_1 and q_2. Then obtain the normal mode coordinates and frequencies, and express the Hamiltonian in terms of the new coordinates.

3. Quantise the above system in Q.2. Define creation and annihilation operators for the normal mode coordinates in the Hamiltonian formalism, and express the original displacements q_1 and q_2 in terms of them. Find the energy spectrum of the system.

4. Derive the continuity equation for the Klein-Gordon Equation and show that the normalised plane wave solution $\phi(x) = e^{-i(Et-\mathbf{p}.\mathbf{x})}$ corresponds to a probability density of $2E$ particles per unit volume.

Chapter 2

The Klein-Gordon Field

2.1 Introduction

Our goal is to construct a quantum field theory based on a new complex scalar field such that its quantum excitations can be identified with the spinless mass m bosons of the Klein-Gordon equation, in analogy with the way photons are regarded as quanta of the electromagnetic field. This will allow the massive bosons to be created or destroyed in arbitrary numbers, just as in the case of photons. It will also, hopefully, solve some of the other problems associated with the KG equation. In the case of the electromagnetic field, Maxwell has taught us what the field equations of motion are, and it is possible to obtain these from a Lagrangian density (see later). In the case of the new complex scalar field, our only guide is that it reproduce the physics of the KG equation (but without the problems). It turns out that the way to achieve this is to require that the new KG *field* $\hat{\phi}(x)$ obeys the same equation as the KG *wavefunction* $\phi(x)$,

$$(\partial^2 + m^2)\hat{\phi}(x) = 0 \tag{2.1.1}$$

The hat as usual denotes the fact that the field is an operator, and we have at this stage simply assumed that the correct field equation is of the same form as the KG equation, with for example the same mass parameter m as before, although it is not clear at this stage what the interpretation of the parameter m in Eq.2.1.1 is.

Why do we regard the KG field as an operator? The answer is that we wish the energy of the field to be quantised into packets of energy which we can identify with bosons of mass m. If we regarded it as a classical field then the energy of the field could take any value it wished, and we would lose the particle interpretation of the field excitations. Nevertheless for your erudition, it may be instructive to consider the case of a classical field to begin with.

2.2 Classical Scalar Field Theory

Perhaps we should begin by defining what we mean by a field. At the most basic level a field is simply a number, or set of numbers, attached to each point in space. For example every day on the weather map you see a scalar field in which a single number (the temperature) is associated with each point on a weather map of the U.K. at a particular time. Likewise you see a vector field in which three numbers (the components of the wind velocity) is associated with each point at a particular time. These are called scalar and vector fields because they have particular transformation properties under space rotations,

$$x^i \rightarrow x^{i'} = R^i_j x^j$$

where R^i_j describes some rotation about some axis. The transformation of the scalar field is:

$$T'(x^{i'}) = T(x^i)$$

which simply says that the temperature at the same physical point (e.g. Southampton) is invariant under rotations of the coordinate system (where Southampton has coordinates x^i in the original system and $x^{i'}$ in the rotated frame.) For the vector field we have the transformation:

$$v^{j'}(x^{i'}) = R^j_k v^k(x^i)$$

which again simply says that once the same physical point has been correctly identified in the two frames (e.g. Southampton) then the components of the field transform as a vector (so that the wind arrow at Southampton physically points in the same direction).

The above results for scalar and vector fields are easily generalised to the relativistic domain. The rotations of space coordinates become Lorentz boosts of spacetime coordinates,

$$x^\mu \rightarrow x^{\mu'} = \Lambda^\mu_\nu x^\nu$$

and the scalar field transforms as

$$T'(x^{\mu'}) = T(x^\mu)$$

while the vector field transforms as

$$v^{\nu'}(x^{\mu'}) = \Lambda^\nu_\rho v^\rho(x^\mu)$$

In quantum theory there are also spinor fields which have well defined transformation properties under Lorentz boosts (see later). When these Lorentz boosts are supplemented by spacetime translations the Lorentz group becomes the Poincare group. If the scalar field changes sign under space inversion then we refer to it as a pseudoscalar field.

So far we have just discussed the transformation properties of scalar and vector fields. The fields of interest to us are dynamical fields, i.e. fields which

involve energy, and are more than just a number attached to each point in space and time. Thus it will come as no surprise that we are not particularly interested in temperature fields. The kind of fields of interest to us are those fields which describe the motion of the atoms in a crystal lattice, for example, and the abstraction of such fields to the case of the Klein-Gordon field. In such cases the classical field can be regarded as the continuum limit of a set of coupled classical oscillators. Such fields are described by the continuum limit of the finite classical system of N coupled oscillators with generalised coordinates q_i.

At each point in space we can imagine a little oscillator (think of an atom in a crystal for example) described by the coordinate $q_i(t)$ where i refers to the i-th point in space, and $q_i(t)$ is the assumed single degree of freedom associated with this point (the atom is only allowed to vibrate in one direction for example). In three space dimensions the i-th point in space is just labelled by the space coordinates (x, y, z) and so **we must consider a single generalised coordinate at each point in space.** This is impossible for real atoms of course since each atom occupies a finite volume of space but the Klein-Gordon field requires exactly such an abstraction. This is the basic idea of field theory. If there were a four-vector of generalised coordinates associated with each spacetime point (corresponding to each little oscillator being described by a three-vector of generalised coordinates plus time) then we would have a vector field.

According to the above discussion, to arrive at (classical) field theory we need to consider the continuum limit of the system of coupled (classical) oscillators considered previously. The limit can be made precise by dividing up space into little cubes of volume τ which may be thought of as the finite volume occupied by the i-th oscillator with generalised coordinate $q_i(t)$. The contribution to the Lagrangian from the i-th oscillator is given by Eq.1.1.10,

$$L_i = (m_i/2)\dot{q}_i^2 + \dots$$

where $\dots$ denotes terms which are independent of $\dot{q}_i$, and m_i is the mass of the i-th little oscillator. Physically we might expect m_i to decrease with the volume τ so we can write $m_i = const.\tau$ or $m_i = \tau$, setting the constant to unity. Then the contribution to the Lagrangian from the i-th oscillator is

$$L_i = (\tau/2)\dot{q}_i^2 + \dots$$

One can define a Lagrangian per unit volume in the region of the i-th oscillator as

$$\mathcal{L}_i = L_i/\tau = (1/2)\dot{q}_i^2 + \dots$$

which is valid at any point in the space occupied by the i-th oscillator. The total Lagrangian is then given by

$$L = \sum_i \tau \mathcal{L}_i$$

summed over all the oscillators. As $\tau \to 0$ the contribution to the Lagrangian from each oscillator goes to zero, but there are infinitely many of them in a given volume of space, so the sum becomes an integral,

$$L = \int d^3x \mathcal{L} \tag{2.2.1}$$

where $\mathcal{L}$ is the limiting form of $\mathcal{L}_i$ and is called the Lagrangian density.

The KG field theory is precisely the abstraction in which the volume of each little oscillator goes to zero and so a generalised coordinate $q_i(t)$ becomes associated with every point in space. We thus identify the label i with the spacetime point:

$$i \to (x, y, z)$$

so that the coordinate is now written

$$q_i(t) \to q(x, y, z, t)$$

The generalised coordinate at each spacetime point is called a scalar field with the more usual notation:

$$\phi(x^\mu) \equiv q(x, y, z, t)$$

Thus we have finally in the continuum limit

$$q_i(t) \to \phi(x^\mu) \tag{2.2.2}$$

The Lagrangian density in the continuum limit is therefore

$$\mathcal{L} = (1/2)\dot{\phi}^2 + \dots$$

In general the Lagrangian density is given by the generalisation:

$$L(q_i, \dot{q}_i) \to \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.2.3}$$

where $\phi = \phi(x^\mu)$ and the time derivative is replaced by the spacetime derivative, as required for relativistic invariance.

Returning to the finite case, the generalised momentum of the i-th oscillator is just

$$p_i = \frac{\partial L_i}{\partial \dot{q}_i} = \frac{\tau \partial \mathcal{L}_i}{\partial \dot{q}_i}$$

One may define a generalised momentum per unit volume as

$$\pi_i = p_i/\tau = \frac{\partial \mathcal{L}_i}{\partial \dot{q}_i}$$

In the $\tau \to 0$ limit the generalised momentum per unit volume becomes itself a field

$$\pi_i \to \pi(x), \tag{2.2.4}$$

so we have the continuum result

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \tag{2.2.5}$$

For example since $L_i = (m_i/2)\dot{q}_i^2 + \ldots$ then

$$p_i = (m_i)\dot{q}_i = \tau \dot{q}_i$$

In the limit $\tau \to 0$, the generalised momentum per unit volume is

$$\pi(x) = \dot{\phi}(x) \tag{2.2.6}$$

In the finite case the Hamilton is defined by the Legendre transformation

$$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i)$$

$$= \sum_i \tau \pi_i \dot{q}_i - \tau \mathcal{L}(q_i, \dot{q}_i)$$

One may define a Hamiltonian density at a particular point in space within the confines of the i-th oscillator as

$$\mathcal{H}_i = H_i/\tau = \pi_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i)$$

In the $\tau \to 0$ limit we have

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} \tag{2.2.7}$$

The total Hamiltonian is

$$H = \int d^3x \mathcal{H} \tag{2.2.8}$$

Explicitly we find,

$$H = \sum_i (m_i)\dot{q}_i^2 - (m_i/2)\dot{q}_i^2 + \ldots$$

$$= \sum_i (m_i/2)\dot{q}_i^2 + \ldots$$

$$= \sum_i (\tau/2)\dot{q}_i^2 + \ldots$$

$$= \sum_i (\tau/2)\pi_i^2 + \ldots$$

where is the last expression we have used the generalised momenta per unit volume, as appropriate for a Hamiltonian. The contribution to the Hamiltonian from each oscillator is clearly

$$H_i = (\tau/2)\pi_i^2 + \ldots$$

The Hamiltonian density is

$$\mathcal{H}_i = H_i/\tau = (1/2)\pi_i^2 + \ldots$$

which again is valid at points in the space occupied by the i-th oscillator, and in the continuum limit this becomes,

$$\mathcal{H} = (1/2)\pi^2 + \ldots \tag{2.2.9}$$

In the continuum the action is still a single (real) number but is given by

$$S[q_i(t)] \to S[\phi(x^\mu)] = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x$$

where the three space integrations can be thought of as yielding the Lagrangian L from the Lagrangian density $\mathcal{L}$, although this statement is not relativistically invariant. The classical field solutions are those which extremise the action S where we can imagine that the Lagrangian is calculated by integrating a Lagrangian density over some rectangular box of volume Ω, and then integrated from time t_1 to t_2. This is done for each choice of field functions $\phi(x^\mu)$, subject to certain boundary conditions on the surface of the box at t_1 and t_2, and those which extremise S are the correct classical solutions. Given a particular candidate field solution $\phi(x^\mu)$ we may vary it at each spacetime point according to

$$\phi(x^\mu) \to \phi'(x^\mu) = \phi(x^\mu) + \delta\phi(x^\mu)$$

subject to $\delta\phi(x^\mu) = 0$ at the boundary. This is equivalent in the finite case to varying the functions from $q_i(t)$ to $q_i(t) + \delta q_i(t)$ subject to $\delta q_i(t) = 0$ at t_1 and t_2. As in the finite case, the action principle leads to Lagrange equations of motion, in this case given by

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{2.2.10}$$

For example a careful choice of classical Lagrangian density

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \tag{2.2.11}$$

leads to the KG equation for the real classical field

$$(\partial^2 + m^2)\phi(x) = 0 \tag{2.2.12}$$

This classical Lagrangian density corresponds to the continuum limit of Eq.1.1.10, with the obvious generalisations:

$$\dot{q_i}^2 \to (\partial_0 \phi)(\partial^0 \phi) \tag{2.2.13}$$

$$\Omega^2 (q_i - q_{i+1})^2 \to (\partial_j \phi)(\partial^j \phi) \tag{2.2.14}$$

$$\Omega_0^2 q_i^2 \to m^2 \phi^2 \tag{2.2.15}$$

Note how the nearest neighbour potential term becomes part of the Lorentz invariant kinetic term. Nevertheless it is convenient to think of the kinetic energy of the field as

$$T = \int d^3\mathbf{x} \frac{1}{2} (\frac{d\phi(\mathbf{x}, t)}{dt})^2 \tag{2.2.16}$$

and the potential energy as

$$V = \frac{m^2}{2}\int d^3\mathbf{x}\phi^2(\mathbf{x},t) + \int d^3\mathbf{x}\frac{1}{2}\sum_{i=1}^{3}(\frac{d\phi(\mathbf{x},t)}{dx_i})^2 \tag{2.2.17}$$

The Lagrangian L and Hamiltonian H are then given by

$$L = T - V, \quad H = T + V \tag{2.2.18}$$

where H is expressed in terms of

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \dot{\phi}(x) \tag{2.2.19}$$

We can obviously perform a normal mode expansion of the classical field which is just the continuum limit of the normal mode expansion of the oscillators in Eq.1.2.41, in other words the Fourier transform of the fields.

$$\phi(\mathbf{x},t) = \int \frac{d^3k}{\sqrt{(2\pi)^3}}e^{i\mathbf{k}.\mathbf{x}}\tilde{\phi}(\mathbf{k},t) \tag{2.2.20}$$

In terms of the FT'd fields which the Hamiltonian looks like a sum of uncoupled oscillators of different frequencies as in Eq.2.3.14

$$H = \int d^3\mathbf{k}(\frac{1}{2}\tilde{\pi}\tilde{\pi}^\star(\mathbf{k},t) + \frac{\omega_k^2}{2}\tilde{\phi}\tilde{\phi}^\star(\mathbf{k},t)) \tag{2.2.21}$$

This normal mode expansion just corresponds to plane wave solutions to the wave equation, of the form,

$$\phi(\mathbf{x},t) = Ae^{i(\mathbf{k}.\mathbf{x}-\omega t)} \tag{2.2.22}$$

where $A = 1$ corresponds to a KG probability density of 2ω particles per unit volume. Substituting this trial solution into Eq.2.2.12 gives,

$$A[-\omega^2 + \mathbf{k}^2 + m^2]e^{i(\mathbf{k}.\mathbf{x}-\omega t)} = 0 \tag{2.2.23}$$

The trial form is a solution if

$$\omega(k) = \pm\sqrt{\mathbf{k}^2 + m^2} \tag{2.2.24}$$

Notice that there are two solutions, thus the general solution is,

$$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a(\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}-\omega t)} + b(\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}+\omega t)}] \tag{2.2.25}$$

where we have defined $\omega \equiv +|\sqrt{\mathbf{k}^2 + m^2}|$. Since a is a general function, we are allowed to insert any normalisation factor we wish into the expansion; the

factor $\frac{1}{(2\pi)^3 2\omega}$ is chosen for later convenience. Now using the condition $\phi^* = \phi$ we find

$$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a(\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}-\omega t)} - a^*(-\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}+\omega t)}] \tag{2.2.26}$$

Similarly

$$\pi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2}[-ia(\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}-\omega t)} - ia^\star(-\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}+\omega t)}] \tag{2.2.27}$$

These field expansions will become the continuum versions of Eqs.1.2.44,1.2.45, when we quantise the system. So far however there are no operators in sight.

Another feature of classical field theory worthy of note is Noether's theorem: if the action is invariant under some group of continuous transformations on x^μ or ϕ (the latter case being applicable to more complicated situations in which there are several fields ϕ present) then there exists a conserved quantity for each generator of the transformation. From finite classical mechanics it is known that time translation invariance leads to conservation of energy, space translation invariance leads to conservation of momentum, rotational invariance leads to conservation of angular momentum, and Noether's theorem simply generalises and extends such ideas to field theory. Given a continuous symmetry it is always possible to find the Noether current density J^μ_ν which is conserved

$$\partial_\mu J^\mu_\nu = 0 \tag{2.2.28}$$

where the corresponding Noether charge

$$Q_\nu = \int J^0_\nu d^3x \tag{2.2.29}$$

is conserved

$$\frac{dQ_\nu}{dt} = 0 \tag{2.2.30}$$

For example invariance under spacetime translations yields

$$J^\mu_\nu = -\theta^\mu_\nu \tag{2.2.31}$$

where θ^μ_ν is the energy-momentum tensor:

$$\theta^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \tag{2.2.32}$$

and the conserved charge is then the 4-momentum of the field given by $P_\nu = -Q_\nu$.

In the case of the KG Lagrangian the energy-momentum tensor is

$$\theta^{\mu\nu} = (\partial^\mu\phi)(\partial^\nu\phi) - g^{\mu\nu}\mathcal{L} \tag{2.2.33}$$

The field momentum is

$$P^\mu = \int d^3\mathbf{x}\theta^{\mu 0} \tag{2.2.34}$$

The energy-momentum tensor for the Klein-Gordon field happens to be symmetric in μ and ν (required by both Einstein's equation and angular momentum conservation). In general it is always possible to define a symmetric energy-momentum tensor $T^{\mu\nu}$ called the canonical energy-momentum tensor (canonical means authoritative as in the Scriptures!) in terms of which Einstein's field equations are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^2}T_{\mu\nu} \tag{2.2.35}$$

and the angular momentum of the field is

$$M^{\mu\nu} = \int (T^{0\mu}x^\nu - T^{0\nu}x^\mu)d^3x \tag{2.2.36}$$

which is conserved

$$\frac{d}{dt}M^{\mu\nu} = 0 \tag{2.2.37}$$

To take a slightly more complicated example we could consider the complex KG field $\phi \neq \phi^*$ with two real components ϕ_1 and ϕ_2 where

$$\phi = (\phi_1 + i\phi_2)/\sqrt{2} \tag{2.2.38}$$

$$\phi^* = (\phi_1 - i\phi_2)/\sqrt{2} \tag{2.2.39}$$

Since the action is real we guess the Lagrangian

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi^*\phi \tag{2.2.40}$$

and treating ϕ and ϕ^* as two independent fields the Lagrange equations give the two KG field equations

$$(\partial^2 + m^2)\phi(x) = 0 \tag{2.2.41}$$

$$(\partial^2 + m^2)\phi^*(x) = 0 \tag{2.2.42}$$

The Lagrangian (density) is invariant under the transformations

$$\phi \to e^{-i\Lambda}\phi \tag{2.2.43}$$

$$\phi^* \to e^{i\Lambda}\phi^* \tag{2.2.44}$$

It is an example of a continuous symmetry involving the fields rather than the spacetime point, and according to Noether's theorem we expect an additional conserved current and charge which are found to be:

$$J^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) \tag{2.2.45}$$

$$Q = i\int(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t})d^3x \tag{2.2.46}$$

which are of course the field theory versions of the KG probability density and current density found earlier. However we have arrived at these quantities in the context of classical field theory to be compared to the quantum particle theory previously. Thus the interpretation of the current and charge here are quite different from the previous case. In the classical field case it is possible to interpret the charge Q above as electric charge with the correct couplings to the electromagnetic field (see Ryder), with J^μ as the electric current. When the field is quantised these interpretations will survive and supercede the problematic single particle interpretations of the similar quantities.

Finally we note that the energy of the KG field in the case of the real field is:

$$H = \int \theta^{00} d^3x \tag{2.2.47}$$

$$H = \frac{1}{2}\int (\partial_0\phi)^2 + \partial_i\phi\partial^i\phi + m^2\phi^2 d^3x \tag{2.2.48}$$

which is clearly positive. Similarly for the complex scalar case, the energy is

$$H = \int (\partial_0\phi^*)(\partial_0\phi) + \partial_i\phi^*\partial^i\phi + m^2\phi^*\phi d^3x \tag{2.2.49}$$

which also consists of a sum of positive terms. Thus the KG field theory does not suffer from problems with negative energy, even at the classical level.

2.3 Quantum Scalar Field Theory

Having learnt a little about classical field theory, we are now in a position to achieve our objective of quantising the KG field and interpreting the field quanta as massive spinless bosons which the KG particle equation fails to correctly describe. Although the previous results concerning a reinterpretation of the charge and current, and the fact that the energy of the field is positive, the classical discussion of the KG field does not correspond to any known physical phenomena so it remains to be seen how these results are affected by quantisation. However it is clear that quantisation looks promising because the KG field involves an independent generalised coordinate $\phi(x)$ at each spacetime point x so we have in effect a harmonic oscillator at each spacetime point. When quantise these (admittedly rather abstract) oscillators, each of them becomes capable of raising or lowering its energy by a quantum amount, so we have a mechanism for raising or lowering the energy of the field as a whole by discrete amounts – which is exactly what we desire if we wish to describe the appearance or disappearance of arbitrary numbers of bosons.

In fact we are in even better shape because we have gained experience with the quantised coupled oscillator system, and we know that the quantum excitations resemble relativistic particles. All we need to do is simply take the continuum limit of the finite coupled oscillator system and we are home and dry.

We now consider the KG field as an operator $\hat{\phi}(x)$, although we shall follow the popular convention of not bothering to write the hat on the operator. The quantum analogue of the real scalar field is the Hermitian quantum field, which we shall therefore consider first,

$$\phi(x) = \phi^\dagger(x) \tag{2.3.1}$$

The Lagrangian density is just that considered previously

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 \tag{2.3.2}$$

where now we have invisible hats everywhere! In order to quantise the theory we shall go to the Hamiltonian formulation in which the derivatives of fields are traded in for canonical momenta.

The generalised momentum (per unit volume) is given by Eq.2.2.5

$$\pi(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

In the Heisenberg picture the commutation relations for the finite case are at equal times,

$$[\hat{q}_i(t), \hat{p}_j(t)] = iI\delta_{ij}, \quad [\hat{q}_i(t), \hat{q}_j(t)] = [\hat{p}_i(t), \hat{p}_j(t)] = 0 \tag{2.3.3}$$

Remember that in field theory $q_i(t)$ is identified with $\phi(x)$ and $p_i(t)$ is identified with $\pi(x)d^3x$ (see previous section), then it is straightforward to see that these results just become the equal time commutation relations

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = iI\delta(\mathbf{x} - \mathbf{x}') \tag{2.3.4}$$

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0 \tag{2.3.5}$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the three dimensional Dirac delta function defined by:

$$\delta(\mathbf{x} - \mathbf{x}') = 0 \quad (\mathbf{x} \neq \mathbf{x}') \tag{2.3.6}$$

$$\int \delta(\mathbf{x} - \mathbf{x}')d\mathbf{x} = 1 \tag{2.3.7}$$

The Hamiltonian density is given by the the Legendre transformation discussed in the previous section:

$$\mathcal{H}(\phi, \pi) = \pi\dot{\phi} - \mathcal{L} \tag{2.3.8}$$

$$\mathcal{H}(\phi, \pi) = \frac{1}{2}(\pi^2 + \partial_i\phi\partial^i\phi + m^2\phi^2) \tag{2.3.9}$$

The first term was derived previously as the continuum limit of the finite oscillator theory. The full result, when integrated over space, agrees with the classical form of the field energy given previously.

Since the fields are Heisenberg operators they must obey the Heisenberg equations of motion for the field ϕ

$$[H, \phi] = -i\dot{\phi} \tag{2.3.10}$$

from which we find, using the result for the Hamiltonian H and the commutation relations,

$$\pi = \dot{\phi} \tag{2.3.11}$$

a result which we also found in the previous section. Similarly using the Heisenberg equations for the field π

$$[H, \pi] = -i\dot{\pi} \tag{2.3.12}$$

from which one can again deduce the Lagrange equation of motion for the field

$$(\partial^2 + m^2)\phi(x) = 0 \tag{2.3.13}$$

which of course comes as no surprise, except that now it is an operator equation.

In order to pursue our goal of studying the quantum excitations of the KG field, we must find the complete set of energy eigenstates and their eigenvalues of this theory, using only our knowledge of the field commutation relations. To achieve this we must Fourier expand the Heisenberg operators. The proceedure is just the continuum limit of the discussion of the quantised coupled oscillators. Thus we begin by introducing normal mode (tilded) fields as the Fourier transform of the original fields, as we did earlier. In terms of the FT'd fields which the Hamiltonian looks like a sum of uncoupled oscillators of different frequencies as in Eq.2.3.14

$$H = \int d^3\mathbf{k}(\frac{1}{2}\tilde{\pi}\tilde{\pi}^\star(\mathbf{k}, t) + \frac{\omega_k^2}{2}\tilde{\phi}\tilde{\phi}^\star(\mathbf{k}, t)) \tag{2.3.14}$$

To proceed we introduce a raising and lowering operator for each of the tilded fields,

$$\tilde{\phi}(\mathbf{k}) = \frac{1}{\sqrt{2\omega_k}}(a(\mathbf{k}) + a^\dagger(-\mathbf{k})) \tag{2.3.15}$$

$$\tilde{\pi}(\mathbf{k}) = \sqrt{\frac{\omega_k}{2i}}(a(-\mathbf{k}) - a^\dagger(\mathbf{k})) \tag{2.3.16}$$

Finally we may express the original fields in terms of the Fourier transform and the raising and lowering operators. The result of all this is obvious operator version of Eq.2.2.26

$$\phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a(\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}-\omega t)} - a^\dagger(-\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}+\omega t)}] \tag{2.3.17}$$

The expansion of the momentum field is easily obtained by analogy with Eq.2.2.27,

$$\pi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2}[-ia(\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}-\omega t)} - ia^\dagger(-\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}+\omega t)}] \tag{2.3.18}$$

By reversing the sign of some of the $\mathbf{k}$ integrations these results can be written as

$$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a(\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}-\omega t)} + a^\dagger(\mathbf{k})e^{i(-\mathbf{k}.\mathbf{x}+\omega t)}] \tag{2.3.19}$$

$$\pi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2}[-ia(\mathbf{k})e^{i(\mathbf{k}.\mathbf{x}-\omega t)} + ia^\dagger(\mathbf{k})e^{i(-\mathbf{k}.\mathbf{x}+\omega t)}] \tag{2.3.20}$$

The coefficients $a(k), a^\dagger(k)$, are clearly continuum versions of the raising and lowering operators in Eq.1.2.33 which destroy and create particles of energy

$$\omega_k = k_0 = \sqrt{\mathbf{k^2} + m^2} \tag{2.3.21}$$

which is just the continuum version of Eq.**??**. We can see from Eq.2.3.21 why the quantised KG field with the parameter m in the Lagrangian gives rise to quantum excitations corresponding to relativistic particles of mass m; thus our goal has been achieved!

Note that the field expansions may be inverted by integrating over $\mathbf{x}$,

$$\int d^3\mathbf{x} e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}\phi(\mathbf{x},t) = \frac{1}{2\omega_k}[a(\mathbf{k}) - a^\dagger(-\mathbf{k})e^{i(\omega t)}]$$

$$\int d^3\mathbf{x} e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}\dot{\phi}(\mathbf{x},t) = \frac{-i}{2}[a(\mathbf{k}) + a^\dagger(-\mathbf{k})e^{2i(\omega t)}]$$

This gives,

$$a(\mathbf{k}) = \int d^3\mathbf{x} e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}[\omega_k\phi(\mathbf{x},t) + i\dot{\phi}(\mathbf{x},t)]$$

$$a(\mathbf{k}) = i\int d^3\mathbf{x}[e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}\frac{\partial\phi(\mathbf{x},t)}{\partial t} - \left(\frac{\partial e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}}{\partial t}\right)\phi(\mathbf{x},t)]$$

$$a(\mathbf{k}) = i\int d^3\mathbf{x} e^{-i(\mathbf{k}.\mathbf{x}-\omega t)} \overleftrightarrow{\partial_0}\, \phi(\mathbf{x},t) \tag{2.3.22}$$

where the rhs is independent of time since ϕ is a solution to the KG equation and we define

$$a(t) \overleftrightarrow{\partial_0}\, b(t) \equiv a(t)\frac{\partial b}{\partial t} - \frac{\partial a}{\partial t}b(t)$$

What about Lorentz invariance? The measure in the integrand has been judiciously chosen to ensure that it is relativistically invariant, writing $\omega = k_0 = +\sqrt{\mathbf{k}^2 + m^2}$,

$$\frac{d^3\mathbf{k}}{2k_0} = \int_0^\infty d^3\mathbf{k}\frac{dk_0^2}{2k_0}\delta(k_0^2 - \mathbf{k}^2 - m^2)$$

$$= \int_0^\infty d^3\mathbf{k}dk_0\delta(k_0^2 - \mathbf{k}^2 - m^2)$$

$$= \int_{-\infty}^\infty d^3\mathbf{k}dk_0\delta(k_0^2 - \mathbf{k}^2 - m^2)\theta(k_0)$$

$$= \int_{-\infty}^\infty d^4k\delta(k^2 - m^2)\theta(k_0)$$

The expansion can be written in manifestly invariant form as

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}\delta(k^2 - m^2)\theta(k_0)(a(k)e^{-ik.x} + a^\dagger(k)e^{ik.x}) \tag{2.3.23}$$

The commutation relations of the raising and lowering operators are obtained by inserting the field expansions into the equal time commutation relations for the fields. The result is:

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\delta^3(\mathbf{k} - \mathbf{k}') \tag{2.3.24}$$

Also,

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0, \quad [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0 \tag{2.3.25}$$

The factor $(2\pi)^3 2\omega$ in the commutation relation is a direct result of our choice of factor in the field expansion.

The number operator is clearly just the continuum version of Eq.1.2.40 and is the particle number density operator is

$$N(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{2.3.26}$$

where

$$[N(\mathbf{k}), N(\mathbf{k}')] = 0 \tag{2.3.27}$$

The Hamiltonian can be expressed in terms of the raising and lowering operators, just as in Eq.1.2.36,

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} k_0[N(\mathbf{k}) + 1/2] \tag{2.3.28}$$

with $k_0 = \omega_k$. This result, which shows that the quantum excitations have energy given by k_0, can be proven in directly in the continuum theory by using Eq.**??**, 2.3.9, and performing the space integrals to yield delta functions.

Employing normal ordering to remove the zero point energy (which in this case is infinite)

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} k_0[N(\mathbf{k})] \tag{2.3.29}$$

Should one worry about removing an infinite constant from the vacuum energy? Possibly, but we shall not here.

It is already clear without any more work that the quantised KG field leads to packets of energy which correspond to the KG bosons of mass m. Everything just goes through in the continuum case just as for the coupled quantum oscillators considered previously. The raising and lowering operators $a^\dagger(k)$ and $a(k)$ create and destroy KG bosons of four-momentum k, and the normal ordered vacuum state has zero energy and contains zero bosons. To be totally explicit, suppose we start with some vacuum state $|0>$ normalised so that,

$$< 0|0 >= 1$$

Now consider the state obtained by acting on the vacuum with a creation operator

$$|\mathbf{k}>= a^\dagger(\mathbf{k})|0>$$

We can check that the action of the Hamiltonian on this state is

$$Ha^\dagger(\mathbf{k})|0>= \omega a^\dagger(\mathbf{k})|0>$$

so that $|\mathbf{k}>$ corresponds to a single particle state with energy ω. The innner product of two single particle states is

$$<\mathbf{k}'|\mathbf{k}>=<0|a(\mathbf{k}')a^\dagger(\mathbf{k})|0>$$

$$=<0|[a(\mathbf{k}'),a^\dagger(\mathbf{k})]|0>= (2\pi)^3 2\omega\delta^3(\mathbf{k}-\mathbf{k}')$$

Physically this corresponds to a single particle state being normalised to 2ω particles per unit volume; the Lorentz invariant normalisation. [1] Note that

$$a(\mathbf{k})]|\mathbf{k}'>= (2\pi)^3 2\omega\delta^3(\mathbf{k}-\mathbf{k}')|0>$$

In a similar way the two particle state is given by $a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0>$. Because of the commutation relations, the two particle states are even under exchange, and states which involve identical bosons will involve symmetry factors. In general, the state which contains $n(k_1)$ bosons with momentum k_1, $n(k_2)$ bosons with momentum k_2, and so on is given by

$$|n(k_1)n_(k_2)\ldots>= \frac{1}{\sqrt{n(k_1)n(k_2)\ldots}}[a^\dagger(k_1)]^{n(k_1)}[a^\dagger(k_2)]^{n(k_2)}\ldots|0> \qquad (2.3.30)$$

In order to complete our goal of interpreting KG bosons in terms of a quantised KG field, we must finally extend our discussion to include the complex KG field. In fact we now have two fields whose expansions are:

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k}(a(\mathbf{k})e^{-ik.x} + b^\dagger(\mathbf{k})e^{ik.x}) \qquad (2.3.31)$$

$$\phi^\dagger(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k}(b(\mathbf{k})e^{-ik.x} + a^\dagger(\mathbf{k})e^{ik.x}) \qquad (2.3.32)$$

We have introduced creation and annihilation operators for each field so that the dagger of the first expansion is equal to the second expansion. The equal time commutation relations on the fields lead to

$$[a(k),a^\dagger(k')] = (2\pi)^3 2\omega_k\delta^3(\mathbf{k}-\mathbf{k}') \qquad (2.3.33)$$

$$[b(k),b^\dagger(k')] = (2\pi)^3 2\omega_k\delta^3(\mathbf{k}-\mathbf{k}') \qquad (2.3.34)$$

[1] It is interesting to compare relativistic normalisation to the non-relativistic normalisation $<\mathbf{k}'|\mathbf{k}>= (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$ which is not Lorentz invariant since $\int d^3\mathbf{k}\delta^3(\mathbf{k}-\mathbf{k}') = 1$ is not a Lorentz invariant expression.

with all other commutators vanishing.

We now have two sets of creation and annihilation operators a, b. What do they correspond to? Recall Eq.2.2.46 for the charge of the classical KG field. In the quantised theory the same expression for the charge applies except that we must replace ϕ^* by $\phi^\dagger$ and use normal ordering. Thus the quantum version of the charge is now the operator

$$Q = i \int : \phi^\dagger \frac{\partial \phi}{\partial t} - \frac{\partial \phi^\dagger}{\partial t} \phi : d^3x \tag{2.3.35}$$

Using the field expansions in Eqs.3.4.11 and 3.4.12 we find,

$$Q = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (a^\dagger(k)a(k) - b^\dagger(k)b(k)) \tag{2.3.36}$$

The Hamiltonian is a similar expression but has positive terms:

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (a^\dagger(k)a(k) + b^\dagger(k)b(k)) \tag{2.3.37}$$

The interpretation is now clear: $a^\dagger$ and a create and destroy particles of positive charge, while $b^\dagger$ and b create and destroy particles of negative charge, in other words antiparticles. Both particles and antiparticles have positive energy. Thus, finally we have arrived at the resolution of the problems which plauged the KG equation. The KG equation survives, but must be interpreted as a field equation, where the field is quantised and the quantum excitations are interpreted as KG bosons which may be created and destroyed in arbitrary numbers.

2.4 Problems Set 2

1. Verify that the commutation relations between the creation and annihilation operators

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 2E_{\mathbf{p}} \delta^3(\mathbf{p} - \mathbf{p}')$$

leads to the commutaion relations between the field $\phi(x)$ and its canonical momentum at equal times

$$[\pi(x), \phi(y)]_{x^0=y^0} = -i\delta^3(\mathbf{x} - \mathbf{y})$$

N.B.

$$\int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{i\mathbf{p}.\mathbf{x}} = \delta^3(\mathbf{p})$$

2. Verify that the expansion of the field $\phi(x)$ in terms of creation and annihilation operators is consistent with the Euler-Lagrange equations of motion.

3. Show that the momentum operator for the Klein-Gordon field is

$$P_\mu = \int d^3\mathbf{x} : \pi(x)\partial_\mu \phi(x) :$$

By expanding the fields in terms of creation and annihilation operators and using the commutation relations between them show that the momentum operator generates infinitesimal translations on the field $\phi(x)$, i.e.:

$$[P_\mu, \phi(x)] = -i\partial_\mu \phi(x)$$

4. Calculate the matrix element $< 0|\phi(x)|p >$ of the field operator between the vacuum and a single particle state with momentum p.

5. By writing the complex Klein-Gordon field χ as a pair of real fields

$$\chi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

starting from the Lagrangian density for the real KG field, derive the Lagrangian density of the complex KG field in terms of χ and $\chi^\dagger$.

Chapter 3

Interacting Klein-Gordon Fields

3.1 Introduction

So far we have "quantised" the scalar field, both the neutral and the charged cases, and we have shown that the field quanta correspond to bosons of mass m, where m is the parameter in the field equation. Formally the KG field equation and KG single particle equation look identical, with the same mass parameter appearing in both, however the physical wavefunction and the physical field are two quite different beasts, as we have emphasised.

In this chapter we discuss interactions in which the quanta of the KG fields interact with each other. For example we shall consider the case of an electrically neutral KG boson coupling to a pair of charged KG bosons, as for example occurs physically in $K_S \to \pi^+\pi^-$ decay. This is achieved by adding extra terms to the Lagrangian and Hamiltonian density. As we shall discover such extra terms render the theory insoluble, and we shall need to resort to approximation methods. In anticipation of this we shall begin by constructing a perturbation theory formalism suitable for calculating scattering and transition amplitudes.

3.2 Perturbation and Scattering Theory

Consider a system with Hamiltonian H which can be written

$$H = H_0 + V \tag{3.2.1}$$

We assume that the eigenstates and eigenvalues of H_0 are known and that V is a small time independent perturbation to the Hamiltonian. The equation of motion of the system is,

$$i\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \left(H_0 + V\right)\left|\psi(t)\right\rangle . \tag{3.2.2}$$

If V vanished, we could calculate the time evolution of $|\psi(t)\rangle$ by expanding it as a linear combination of energy eigenstates. When V does not vanish, the

eigenstates of H_0 are no longer eigenstates of the full Hamiltonian so when we expand in terms of H_0 eigenstates, the coefficients of the expansion become time dependent. To develop a perturbation theory in V we will change our basis of states from the Schrödinger picture to the *interaction* or Dirac picture, where we hide the time evolution due to H_0 and concentrate on that due to V. Thus we define the interaction picture states and operators by,

$$|\psi_I(t)\rangle \equiv e^{iH_0 t}\,|\psi(t)\rangle\,, \qquad \mathcal{O}_I(t) \equiv e^{iH_0 t}\mathcal{O}e^{-iH_0 t}, \tag{3.2.3}$$

where unsubscripted states and operators refer to the Schrodinger picture. The interaction picture and Schrödinger picture states agree at time $t = 0$, $|\psi_I(0)\rangle = |\psi(0)\rangle$, with a similar relation for the operators. In the new basis, the equation of motion becomes,

$$i\frac{\partial}{\partial t}\,|\psi_I(t)\rangle = V_I(t)\,|\psi_I(t)\rangle\,. \tag{3.2.4}$$

We note that the time independent perturbation V in the Schrodinger picture has become a time dependent perturbation $V_I(t)$ in the Dirac interaction picture. The operators in the interaction picture, on the other hand, have a time dependence determined only by H_0,

$$i\frac{d}{dt}\mathcal{O}_I(t) = [\mathcal{O}_I(t), H_0] \tag{3.2.5}$$

Eq.3.2.4 can be iterated to yield an infinite series in V,

$$|\psi_I(t)\rangle = \left[1+\sum_{n=1}^{\infty}\frac{1}{i^n}\int_{-T/2}^{t} dt_1 \int_{-T/2}^{t_1} dt_2 \cdots \int_{-T/2}^{t_{n-1}} dt_n\, V_I(t_1)V_I(t_2)\cdots V_I(t_n)\right]|\psi_I(-T/2)\rangle\,. \tag{3.2.6}$$

The integration involves formally integrating Eq.3.2.4 by writing

$$\int_{-T/2}^{t} d\,|\psi_I(t_1)\rangle = \int_{-T/2}^{t}\frac{1}{i}V_I(t_1)\,|\psi_I(t_1)\rangle\,dt$$

To first order we insert $|\psi_I(t_1)\rangle \approx |\psi_I(-T/2)\rangle$ into the rhs so that we have,

$$|\psi_I(t)\rangle \approx |\psi_I(-T/2)\rangle + \int_{-T/2}^{t}\frac{1}{i}V_I(t_1)dt_1\,|\psi_I(-T/2)\rangle \tag{3.2.7}$$

Then this solution is used in the rhs of the original equation to improve the approximation, and so on. This process of iteration is useful only if the perturbation V_I is small and so a small number of terms in the series (i.e. a small number of iterations) may be taken.

The time evolution is being done by a unitary operator,

$$|\psi_I(t)\rangle = U_I(t, -T/2)\,|\psi_I(-T/2)\rangle \tag{3.2.8}$$

where $U_I(t, -T/2)$ is identified with the square bracket in Eq.3.2.6. Let us recall at this stage our earlier definition of the time translation operator in the Schrodinger picture,

$$|\psi(t)>= U(t,0)|\psi(0)> \tag{3.2.9}$$

where

$$U(t,0) = e^{-iHt} \tag{3.2.10}$$

Going to the interaction picture we find the equivalent results,

$$|\psi_I(t)>= U_I(t,0)|\psi_I(0)> \tag{3.2.11}$$

where

$$U_I(t,0) = e^{iH_0t}e^{-iHt} \tag{3.2.12}$$

We can write

$$|\psi_I(-T/2)\rangle = U_I(-T/2,0)|\psi_I(0)> \tag{3.2.13}$$

hence

$$|\psi_I(t)>= U_I(t,0)U_I^\dagger(-T/2,0)\,|\psi_I(-T/2)\rangle \tag{3.2.14}$$

and we identify the square bracket in Eq.3.2.6 as

$$U_I(t,-T/2) = U_I(t,0)U_I^\dagger(-T/2,0) = e^{iH_0t}e^{-iHt}e^{-iHT/2}e^{iH_0T/2}. \tag{3.2.15}$$

Now consider the calculation of the probability of a transition to an eigenstate $|\Phi_\beta\rangle$ at time t. The amplitude is,

$$\begin{aligned}\langle\Phi_\beta|\psi(t)\rangle &= \langle\Phi_{\beta I}|\psi_I(t)\rangle \\ &= \langle\Phi_\beta|\,e^{-iH_0t}\,|\psi_I(t)\rangle \\ &= e^{-iE_\beta t}\langle\Phi_\beta|\psi_I(t)\rangle,\end{aligned}$$

so $\left|\langle\Phi_\beta|\psi(t)\rangle\right|^2 = \left|\langle\Phi_\beta|\psi_I(t)\rangle\right|^2$. We let V be time independent and consider the amplitude for a transition from an eigenstate $|\Phi_\alpha>$ of H_0 at $t=-T/2$ to an orthogonal eigenstate $|\Phi_\beta>$ at $t=T/2$. The idea is that at very early or very late times H_0 describes some set of free particles. We allow some of these particles to approach each other and scatter under the influence of V, then look again a long time later when the outgoing particles are propagating freely under H_0 again. To first order in V, using Eq.3.2.7 we find

$$\langle\Phi_\beta\,|\psi_I(T/2)\rangle = -i\int_{-T/2}^{T/2}\langle\Phi_\beta|\,V_I(t)\,|\Phi_\alpha\rangle\,dt = -i\langle b|V|a\rangle\int_{-T/2}^{T/2}e^{i\omega_{ba}t}dt,$$

$$\langle\Phi_\beta\,|\psi_I(T/2)\rangle = -i\langle b|V|a\rangle\frac{2}{\omega_{\beta\alpha}}\sin(\omega_{\beta\alpha}T/2) \tag{3.2.16}$$

where $\omega_{\beta\alpha} = E_\beta - E_\alpha$.

Strictly the "in" states at early times, and the "out" states at late times are not eigenstates of H_0, but of the full Hamiltonian H. The states are of course states in the interaction picture, so they have no time dependence in the limit $V \to 0$. In fact this lack of time dependence is the reason why we cannot glibly identify the "in" and "out" states with the eigenstates of the free Hamiltonian. If we are close to the scattering centre then we clearly cannot

disregard the potential: the "centre" states must be eigenstates of the full Hamiltonian. Now as time goes on and the wavepackets leave the scattering centre the potential term gets switched off, but as it does so the "centre" states evolve less and less in time (because in the interaction picture all the time dependence of states is coming from the potential). The "centre" states get frozen in time as the potential is switched off, so the "in" and "out" states must be identified with the "centre" states which are eigenstates of the full Hamiltonian. This is a by-product of being in the interaction picture where it makes no sense to identify asymptotic states with eigenstates of H_0. However there is one piece of good news: the "in" and "out" states have the same energy eigenvalues as the free particle eigenstates of H_0, which is a simple physical consequence of the potential becoming ineffectual at early and late times. The "in" and "out" states are a subset of the eigenstates of the full Hamiltonian $H = H_0 + V$, namely those which have the same eigenvalues as the eigenstates of the free Hamiltonian H_0. They are not the same eigenvectors, but are related by a unitary transformation. To be specific suppose the free states are

$$H_0|\Phi_\alpha >= E_\alpha|\Phi_\alpha > \tag{3.2.17}$$

and the asymptotic states at time $\pm\infty$ are

$$H|\Psi_\alpha^\pm >= E_\alpha|\Psi_\alpha^\pm > \tag{3.2.18}$$

The relation between the states is,

$$|\Psi_\alpha^\pm >= U_I^\dagger(\pm\infty, 0)|\Phi_\alpha > \tag{3.2.19}$$

Note that the operator above is the dagger of the time translation operator and not the time translation operator itself. This condition is a result of the requirement that wavepackets of "in" and "out" states smoothly evolve into wavepackets of free particle states at early and late times. The basic idea is that at time $t = 0$ we have some "centre" state which is an eigenstate of the full Hamiltonian, but which may be expanded in terms of eigenstates of the free Hamiltonian. The above relation then applies to each of the components of such an expansion which is time translated to plus or minus infinity to give the out or in state. The reason for the dagger is the usual reason to do with dealing with wavepackets or wavefunctions rather than state vectors. It is discussed further in Weinberg's book, for example.

The S-matrix is defined to be

$$S_{\beta\alpha} \equiv< \Psi_\beta^+|\Psi_\alpha^- > \tag{3.2.20}$$

Using Eq.3.2.19 we have

$$S_{\beta\alpha} =< \Phi_\beta|U_I(\infty, 0)U_I^\dagger(-\infty, 0)|\Phi_\alpha > \tag{3.2.21}$$

one can define a scattering operator S such that

$$S_{\beta\alpha} =< \Phi_\beta|S|\Phi_\alpha > \tag{3.2.22}$$

so we identify

$$S = U_I(\infty, 0)U_I^\dagger(-\infty, 0) = U_I(\infty, -\infty) \tag{3.2.23}$$

After the above considerations we can return to our first attempt at calculating a scattering amplitude in Eq.3.2.16 and reassess it. The last few results above show that the physical quantity of interest is the collection of S-matrix elements $S_{\beta\alpha}$ which give the transition amplitude from some "in" state to some "out" state. Eqs.3.2.22 and 3.2.23 tell us how to calculate these transition amplitudes: all we need to do is sandwich the operator $U_I(\infty, -\infty)$ between the free particle states which have the same energy as the in and out states of interest. This is precisely what we did in Eq.3.2.16, and this result is therefore vindicated in the limit that $T \to \infty$.

It is clear from Eqs.3.2.22 and 3.2.23 that we need the operator $U_I(\infty, -\infty)$ in our calculations. Let us therefore return to our expansion of $U_I(t, -T/2)$ in Eq.3.2.6

$$U_I(t, -T/2) = \left[1 + \sum_{n=1}^{\infty} \frac{1}{i^n} \int_{-T/2}^{t} dt_1 \int_{-T/2}^{t_1} dt_2 \cdots \int_{-T/2}^{t_{n-1}} dt_n \, V_I(t_1)V_I(t_2)\cdots V_I(t_n)\right]. \tag{3.2.24}$$

The expansion is of the form

$$U_I(t, -T/2) = I + U_I^{(1)} + U_I^{(2)} + \cdots \tag{3.2.25}$$

where we have already considered the first order term

$$U_I^{(1)} = (-i) \int_{-T/2}^{t} V_I(t_1) dt_1 \tag{3.2.26}$$

To second order we have

$$U_I^{(2)} = (-i)^2 \int_{-T/2}^{t} dt_2 \int_{-T/2}^{t_2} dt_1 V_I(t_2)V_I(t_1) \tag{3.2.27}$$

To nth order we have

$$U_I^{(n)} = (-i)^n \int_{-T/2}^{t} dt_n \int_{-T/2}^{t_n} dt_{n-1} \cdots \int_{-T/2}^{t_2} dt_1 V_I(t_n)V_I(t_{n-1})\cdots V_I(t_2)V_I(t_1)$$

Notice that there is a time ordering hierarchy in the above expression

$$t_n > t_{n-1} > \cdots t_2 > t_1$$

and the interaction picture interaction operators $V_I(t)$ are ordered with earlier times to the right as a result of the iteration expansion. This time ordering is important and must be preserved at all times. Now it is somewhat inconvenient that the upper limits of integration are all different, so let us remedy this. The problem first arises in the second order term which can be written

$$U_I^{(2)} = (-i)^2 \int_{-T/2}^{t} dt_2 \int_{-T/2}^{t} dt_1 \theta(t_2 - t_1) V_I(t_2)V_I(t_1) \tag{3.2.28}$$

where a $\theta(t_2 - t_1)$ function has been inserted to ensure that $t_2 > t_1$. Now comes the hard part: we want to remove the $\theta(t_2 - t_1)$ function. We first observe that we can write

$$U_I^{(2)} = (-i)^2 \int_{-T/2}^{t} dt_2 \int_{-T/2}^{t} dt_1 \theta(t_1 - t_2) V_I(t_1) V_I(t_2) \tag{3.2.29}$$

where we have exchanged the dummy integration variables t_1 and t_2. Therefore we may average Eqs.3.2.28 and 3.2.29 to give,

$$U_I^{(2)} = \frac{1}{2}(-i)^2 \int_{-T/2}^{t} dt_2 \int_{-T/2}^{t} dt_1 \theta(t_2 - t_1) V_I(t_2) V_I(t_1)$$

$$+\frac{1}{2}(-i)^2 \int_{-T/2}^{t} dt_2 \int_{-T/2}^{t} dt_1 \theta(t_1 - t_2) V_I(t_1) V_I(t_2)$$

$$= \frac{1}{2}(-i)^2 \int_{-T/2}^{t} dt_2 \int_{-T/2}^{t} dt_1 \left[\theta(t_2 - t_1) V_I(t_2) V_I(t_1) + \theta(t_1 - t_2) V_I(t_1) V_I(t_2)\right]$$

So finally,

$$U_I^{(2)} = \frac{1}{2}(-i)^2 \int_{-T/2}^{t} dt_2 \int_{-T/2}^{t} dt_1 T(V_I(t_2), V_I(t_1)) \tag{3.2.30}$$

where we have defined the time ordered product of the two operators as,

$$T(V_I(t_2), V_I(t_1)) = [\theta(t_2 - t_1) V_I(t_2) V_I(t_1) + \theta(t_1 - t_2) V_I(t_1) V_I(t_2)] \tag{3.2.31}$$

In words, the time ordered product ensures that the earlier time operator is to the right.

A similar procedure may be repeated at subsequent orders to yield,

$$U_I^{(n)} = \frac{1}{n!}(-i)^2 \int_{-T/2}^{t} \prod_i dt_i T(V_I(t_n), V_I(t_{n-1}), \cdots, V_I(t_2), V_I(t_1)) \tag{3.2.32}$$

with an obvious generalisation of the definition of the time ordered product of n fields, and a corresponding permutation averaging factor of $\frac{1}{n!}$.

Using Eq.3.2.32 and Eq.3.2.25 we therefore have

$$U_I(T/2, -T/2) = T\left(\exp\left[-i \int_{-T/2}^{T/2} dt V_I(t)\right]\right) \tag{3.2.33}$$

Finally let us consider the folowing Heisenberg picture matrix element:

$$G = < \psi_H | T(O_H(t_1) O_H(t_2) \cdots O_H(t_n)) | \psi_H > \tag{3.2.34}$$

where the states and operators are all in the Heisenberg picture. Let us now relate this matrix element to one involving Interaction picture operators and states. Both the Heisenberg picture and the interaction picture may be related to the Schrodinger picture, and hence they may be related to each other. Their relationship is:

$$|\psi_H > = e^{iHt} e^{-iH_0 t} |\psi_I(t) > \tag{3.2.35}$$

$$O_H(t) = e^{iHt}e^{-iH_0t}O_I(t)e^{iH_0t}e^{-iHt} \tag{3.2.36}$$

Making these substitutions the matrix element G becomes

$$G =< \psi_I(T/2)|e^{iH_0(T/2)}e^{-iH(T/2)}T(e^{iHt_1}e^{-iH_0t_1}O_I(t_1)e^{iH_0t_1}e^{-iHt_1}$$

$$\times e^{iHt_2}e^{-iH_0t_2}O_I(t_2)e^{iH_0t_2}e^{-iHt_2}\cdots$$

$$\cdots e^{iHt_n}e^{-iH_0t_n}O_I(t_n)e^{iH_0t_n}e^{-iHt_n}e^{iH(-T/2)}e^{-iH_0(-T/2)}|\psi_I(-T/2) >$$

which may be written in terms of the interaction picture time translation operator in Eq.3.2.12 and Eq.3.2.15,

$$G =< \psi_I(T/2)|T(U_I(T/2,t_1)U_I(t_1,t_2)\cdots$$

$$\cdots U_I(t_n,-T/2)O_I(t_1)O_I(t_2)\cdots O_I(t_n)|\psi_I(-T/2) >$$

Combining all the (interaction picture) time translation operators we have

$$G =< \psi_I(T/2)|T(U_I(T/2,-T/2)O_I(t_1)O_I(t_2)\cdots O_I(t_n)|\psi_I(-T/2) > \tag{3.2.37}$$

And using our expression for $U_I(T/2,-T/2)$ in Eq.3.2.33 we have

$$G =< \psi_I(T/2)|T\left(\exp\left[-i\int_{-T/2}^{T/2}dtV_I(t)\right]O_I(t_1)O_I(t_2)\cdots O_I(t_n)\right)|\psi_I(-T/2) > \tag{3.2.38}$$

Compare Eqs.3.2.34 and 3.2.38. They represent the same matrix element G but expressed in the Heisenberg and Interaction picture respectively.

3.3 The Interaction Hamiltonian

Let us consider the Lagrangian density describing the real scalar field:

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 \tag{3.3.1}$$

This theory is regarded as free because the field quanta do not influence each other. They each behave like a single particle solution to the free-particle KG equation. The only difference is that in principle they may be created or destroyed in arbitrary numbers, although as yet we have not provided a physical mechanism for them to do so because the Lagrangian is quadratic in the fields and so each normal ordered term contains one creation and one annihilation operator.

To include interactions we must add to the Lagrangian terms which are of higher degree in the fields. Let us begin by considering adding an interaction Lagrangian density so that the full Lagrangian is

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_{\rangle\backslash\sqcup} \tag{3.3.2}$$

where the interaction Lagrangian density $\mathcal{L}_{\rangle\backslash\sqcup}$ given by

$$\mathcal{L}_{int} = -\frac{g}{3!}\phi^3(x) \tag{3.3.3}$$

corresponding to an interaction term in the Hamiltonian density of

$$\mathcal{H}_{int} = \frac{g}{3!}\phi^3(x) \tag{3.3.4}$$

Now ϕ^3 contains terms with one annihilation and two creation operators and so it can describe the branching of one particle into two.

Before proceeding we recall that in chapter 2 we quantised the free KG field in the Heisenberg picture where the states did not have any time dependence, and the fields had time dependence and obeyed Heisenberg equations which were equivalent to the Lagrange equations for the field. For example the fields obeyed the free KG equation. Now, including the interaction term, if we remain in the Heisenberg picture the states remain time independent but the fields obey a more complicated equation of motion, now non-linear:

$$(\partial^2 + m^2)\phi_H = -\frac{g}{2}\phi_H^2 \tag{3.3.5}$$

It is the presence of the non-linear term which renders the theory insoluble and forces us to resort to approximate techniques. We emphasise that everything we have done in chapter 2 is in the Heisenberg picture: all our fields and Hamiltonians and so on should have a little subscript H for Heisenberg on them. The full Hamiltonian is thus schematically:

$$H_H = H_0 + V_H$$

where each part of the Hamiltonian is expressed in terms of a space integration over products of time dependent Heisenberg fields which obey the equal time commutation relations.

Now we have just seen in the preceding section that the approximation methods are best phrased in the interaction picture where the Hamiltonian is now

$$H_I = H_0 + V_I$$

It is straightforward to show that in the interaction picture the states now have a time dependence governed by V_I, and the fields have the same time dependence as in the Heisenberg picture as seen in Eq.3.2.5, in other words governed by H_0. This means that the fields in the interaction picture satisfy the same field equations as in chapter 2, i.e. the free field equations

$$(\partial^2 + m^2)\phi_I = 0. \tag{3.3.6}$$

This means that we can replace ϕ_I everywhere it appears by its free field Heisenberg expansion given in chapter 2. In particular we may write

$$V_I = \int d^3\mathbf{x}\frac{g}{3!}\phi_I^3(x) \tag{3.3.7}$$

and hence express V_I in terms of *free field* creation and annihilation operators. Note the word *free*. This is the beauty of the interaction picture!

We can now envisage a procedure for calculating S-matrix elements to a given order in perturbation theory (lowest order in this course!). We use the expression in Eq.3.2.32 with $t = T/2$, and eventually take $T \to \infty$. The operator V_I is expressed in terms of free field creation and annihilation operators as just described. It is then sandwiched between *free particle states* according to Eq. 3.2.22 and 3.2.23 which are again constructed by using free field creation and annihilation operators in the free field theory. The application of perturbation theory in the interaction picture has yielded a perturbation expansion for S-matrix elements each of which can be completely evaluated by using the *free* field expansions in terms of creation and annihilation operators. This is little short of a miracle, and caused great excitement when it was discovered by Dyson in 1949.

3.4 Example: $K_S \to \pi^+\pi^-$

Let us consider an interacting theory consisting of a real scalar ϕ and a complex scalar χ. The Lagrangian density is thus

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\chi + \mathcal{L}_{\rangle\backslash\sqcup} \tag{3.4.1}$$

where

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{{m_\phi}^2}{2}\phi^2 \tag{3.4.2}$$

$$\mathcal{L}_\chi = (\partial_\mu\chi^\dagger)(\partial^\mu\chi) - {m_\chi}^2\chi^\dagger\chi \tag{3.4.3}$$

The interaction term is taken to be

$$\mathcal{L}_{\rangle\backslash\sqcup} = -g\chi^\dagger\chi\phi \tag{3.4.4}$$

In the Heisenberg picture the field equations are

$$(\partial^2 + {m_\phi}^2)\phi_H = -g\chi_H^\dagger\chi_H \tag{3.4.5}$$

$$(\partial^2 + {m_\chi}^2)\chi_H = -g\phi_H\chi_H \tag{3.4.6}$$

Again we see no alternative but to use the approximate method. By the way this is only useful if g is small.

As in the preceding section we go to the interaction picture in which we have free field equations

$$(\partial^2 + {m_\phi}^2)\phi_I = 0 \tag{3.4.7}$$

$$(\partial^2 + {m_\chi}^2)\chi_I = 0 \tag{3.4.8}$$

We write

$$V_I = \int d^3\mathbf{x} g\chi_I^\dagger\chi_I\phi_I \tag{3.4.9}$$

where the fields have the same expansions as considered in the free case,

$$\phi_I(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a(\mathbf{k})e^{-ik.x} + a^\dagger(\mathbf{k})e^{ik.x}] \tag{3.4.10}$$

$$\chi_I(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k}[b(\mathbf{k})e^{-ik.x} + d^\dagger(\mathbf{k})e^{ik.x}] \tag{3.4.11}$$

$$\chi_I^\dagger(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k}[d(\mathbf{k})e^{-ik.x} + b^\dagger(\mathbf{k})e^{ik.x}] \tag{3.4.12}$$

With the commutation relations,

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_k \delta^3(\mathbf{k}-\mathbf{k}') \tag{3.4.13}$$

$$[b(\mathbf{k}), b^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_k \delta^3(\mathbf{k}-\mathbf{k}') \tag{3.4.14}$$

$$[d(\mathbf{k}), d^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_k \delta^3(\mathbf{k}-\mathbf{k}') \tag{3.4.15}$$

Now suppose we are interested in an "in" state with a single ϕ particle with momentum $\mathbf{k}$ and an "out" state with a χ of momentum $\mathbf{p}$ and a $\chi^\dagger$ of momentum $\mathbf{q}$. We can think of this as $K_S \to \pi^+\pi^-$ (pretending that we know nothing about quarks!) According to Eq.3.2.22, we need to consider the corresponding free states in the interaction picture, which are constructed by the action of free particle creation operators on the vacuum,

$$|in> \to a^\dagger(\mathbf{k})|0> \tag{3.4.16}$$

$$|out> \to b^\dagger(\mathbf{p})d^\dagger(\mathbf{q})|0> \tag{3.4.17}$$

The S-operator is given in Eq.3.2.23 as

$$S = U_I(\infty, -\infty) \tag{3.4.18}$$

Where U_I is expanded to lowest order as

$$U_I(T/2, -T/2) = I + U_I^{(1)} + \cdots \tag{3.4.19}$$

where

$$U_I^{(1)} = (-i)\int_{-T/2}^{T/2} V_I(t)dt \tag{3.4.20}$$

Thus we have to this order

$$S = I + (-i)\int dt \int d^3\mathbf{x} g \chi_I^\dagger \chi_I \phi_I \tag{3.4.21}$$

where the field expansions are given above. Thus the S-matrix element is

$$S_{fi} = (-ig) < 0|b(\mathbf{p})d(\mathbf{q})\left(\int d^4x \chi_I^\dagger \chi_I \phi_I\right) a^\dagger(\mathbf{k})|0> \tag{3.4.22}$$

The S-matrix element can now be evaluated by inserting field expansions and using the commutation relations for the creation and annihilation operators in

order to put the annihilation operators to the right and the creation operators to the left so that these terms hit against the vacuum and vanish, leaving a slew of delta functions. The final result is

$$S_{fi} = -ig \int d^4x e^{-i(k-p-q)} < 0|0 >= -ig(2\pi)^4\delta^4(k-p-q) \tag{3.4.23}$$

where we have used $< 0|0 >= 1$. The factor $(2\pi)^4\delta^4(k-p-q)$ is common to all calculations of matrix elements and just expresses conservation of energy and momentum between the initial and final state. For this reason we write the S-matrix element generically as

$$S_{fi} = i\mathcal{M}_{fi}(2\pi)^4\delta^4(k-p-q) \tag{3.4.24}$$

where $i\mathcal{M}_{fi}$ is called the "matrix element" or the "amplitude". In this case

$$i\mathcal{M}_{fi} = -ig \tag{3.4.25}$$

As we shall see in the next section it is conventional to represent $\mathcal{M}_{fi}$ by a sum of Feynman diagrams, although the one in this case is rather simple! It is just a Y-shaped vertex of three lines with a "Feynman rule" : $Vertex = -ig$.

3.5 Wick's Theorem, Feynman Propagator, Feynman Diagrams

The procedure we have just followed can be repeated for any initial and final state, and to any order in perturbation theory. In general we need to consider the operator in Eq.3.2.32 reproduced below

$$U_I^{(n)} = \frac{1}{n!}(-i)^2 \int \prod_i d^4x_i T(V_I(x_n), V_I(x_{n-1}), \cdots, V_I(x_2), V_I(x_1)) \tag{3.5.1}$$

This operator is sandwiched between the initial and final states and the object we need to evaluate is of the form

$$S_{\beta\alpha}^{(n)} =< \Phi_\beta|U_I^{(n)}|\Phi_\alpha > \tag{3.5.2}$$

where the free states have the form

$$|\Phi_\alpha >= a^\dagger(\mathbf{k})b^\dagger(\mathbf{q})\cdots|0> \tag{3.5.3}$$

In the above example the basic strategy was to commute all the annihilation operators to the right. This results in *normal ordering* which we defined earlier. Any normal ordered string of operators acting on the vacuum gives zero. In general the vacuum expectation value (VEV) of a string of normal ordered operators is zero,

$$< 0| : ABC\cdots : |0 >= 0 \tag{3.5.4}$$

which is the reason why we consider normal ordering. In more complicated examples than that considered above, the procedure in going from time ordered operators to normal ordered operators can be very tedious. Fortunately there is a very neat relationship between the time ordered product of operators and the normal ordered product of operators called Wick's theorem.

We begin by considering the product of two fields at two different spacetime points $\phi(x)\phi(y)$. We write

$$T(\phi(x)\phi(y)) =: \phi(x)\phi(y) : \; + \; something \tag{3.5.5}$$

where the *something* depends on the details of the two fields, but in general we expect it to be a c-number rather than an operator. We call this *something* the *contraction* of the two fields and write it as $\phi^{\cdot}(x)\phi^{\cdot}(y)$. [1] Thus the contraction of two fields is defined as

$$\phi^{\cdot}(x)\phi^{\cdot}(y) \equiv T(\phi(x)\phi(y)) - : \phi(x)\phi(y) : \tag{3.5.6}$$

so that

$$T(\phi(x)\phi(y)) =: \phi(x)\phi(y) : + \phi^{\cdot}(x)\phi^{\cdot}(y) \tag{3.5.7}$$

Thus the VEV of the time ordered product of two fields is just equal to the contraction of the two fields

$$< 0|T(\phi(x)\phi(y))|0 >= \phi^{\cdot}(x)\phi^{\cdot}(y) \tag{3.5.8}$$

Wick's theorem deals with the general case of the time ordered product of n operators. For example for the case of three operators it states

$$T(\phi(x)\phi(y)\phi(z)) = \; : \phi(x)\phi(y)\phi(z) : \; + \; : \phi(x) : \phi^{\cdot}(y)\phi^{\cdot}(z)$$

$$+ \; : \phi(y) : \phi^{\cdot}(x)\phi^{\cdot}(z) + \; : \phi(z) : \phi^{\cdot}(x)\phi^{\cdot}(y)$$

For the case of four operators it states

$$\begin{aligned} T(\phi(w)\phi(x)\phi(y)\phi(z)) = {} & : \phi(w)\phi(x)\phi(y)\phi(z) : + : \phi(w)\phi(x) : \phi^{\cdot}(y)\phi^{\cdot}(z) \\ & + \; : \phi(w)\phi(y) : \phi^{\cdot}(x)\phi^{\cdot}(z) + \; : \phi(w)\phi(z) : \phi^{\cdot}(x)\phi^{\cdot}(y) \\ & + \; : \phi(x)\phi(y) : \phi^{\cdot}(w)\phi^{\cdot}(z) + \; : \phi(x)\phi(z) : \phi^{\cdot}(w)\phi^{\cdot}(y) \\ & + \; : \phi(y)\phi(z) : \phi^{\cdot}(w)\phi^{\cdot}(x) + \phi^{\cdot}(w)\phi^{\cdot}(x)\phi^{\cdot\cdot}(y)\phi^{\cdot\cdot}(z) \\ & + \phi^{\cdot}(w)\phi^{\cdot}(y)\phi^{\cdot\cdot}(x)\phi^{\cdot\cdot}(z) \; + \; \phi^{\cdot}(w)\phi^{\cdot}(z)\phi^{\cdot\cdot}(x)\phi^{\cdot\cdot}(y) \end{aligned} \tag{3.5.9}$$

For the case of n fields the rhs involves a sum over all permutations of terms involving zero pairs of contractions, one pair of contractions, and so on up the the maximum number of pairs of contractions. Wick's theorem is proved by

[1] Many books write the contraction as $\underline{\phi(x)\phi(y)}$ or some other way but it means the same thing.

induction. The results also apply to different fields. For fermion fields some modification is required.

For fields having equal times (as is the case when the interaction Hamiltonian is a product of fields at the same spacetime point) the time ordered product of operators is defined so as to not change the operator ordering. The same applies to the case when the operators have no time dependence. A good example of this is the calculation performed in the previous section. There we had the matrix element

$$S_{fi}^{(1)} = (-ig) < 0|b(\mathbf{p})d(\mathbf{q}) \left(\int d^4x \chi_I^\dagger(x)\chi_I(x)\phi_I(x) \right) a^\dagger(\mathbf{k})|0 > \tag{3.5.10}$$

which involves an operator string

$$b(\mathbf{p})d(\mathbf{q})\chi_I^\dagger(x)\chi_I(x)\phi_I(x)a^\dagger(\mathbf{k}) \tag{3.5.11}$$

Here three operators are fields at the same spacetime point and three operators are creation and annihilation operators with no spacetime dependence at all. In this case Wick's theorem still applies providing we define

$$T(b(\mathbf{p})d(\mathbf{q})\chi_I^\dagger(x)\chi_I(x)\phi_I(x)a^\dagger(\mathbf{k}))$$
$$\equiv b(\mathbf{p})d(\mathbf{q})\chi_I^\dagger(x)\chi_I(x)\phi_I(x)a^\dagger(\mathbf{k})$$

It is a useful excercise to evaluate the matrix element using Wick's theorem and show that the same result is obtained.

Using Wick's theorem matrix elements are reduced to products of pairs of field contractions. For example in the next order correction above we are left with contractions such as

$$\phi^{\cdot}(x)\phi^{\cdot}(y) =< 0|T(\phi(x)\phi(y))|0 > \tag{3.5.12}$$

Such contractions may be represented diagrammatically as a line joining the two spacetime points x and y. In the above example the third order matrix element involves three spacetime points and after the normal ordering terms have been dropped we are left with different permutations of pairs of contractions, where each permutation may be represented by diagrams involving three spacetime points x, y, z, and three lines joining the three dots in all possible ways, together with three external lines corresponding to the contraction of a field with a creation or annihilation operator. Feynman recognised that each part of the diagram corresponded to a universal factor, and that all one needed to do to evaluate the matrix element was to draw all possible diagrams and multiply each of the universal factors together to obtain the contribution to the matrix element from a particular diagram. These universal factors are called the Feynman rules. In the previous section we already encountered an example of the Feynman rule corresponding to the intersection of three lines at vertex which we saw gave a factor of $-ig$. Another Feynman rule corresponds to the contraction of two fields between two spacetime points which we represented by a line joining the

points. Henceforth we shall call such a contraction a *Feynman Propagator* and designate it by the symbol $\Delta_F(x,y)$. To be precise we define the Feynman propagator for the ϕ field as

$$i\Delta_F(x,y) \equiv \phi^{\cdot}(x)\phi^{\cdot}(y) =< 0|T(\phi(x)\phi(y))|0 > \tag{3.5.13}$$

where a factor of i is conventionally included in the definition of the Feynman propagator.

The Feynman propagator may be evaluated as follows. From the definition of the Feynman propagator we have

$$i\Delta_F(x_2,x_1) = \theta(t_2-t_1) < 0|\phi(x_2)\phi(x_1)|0 >$$

$$+ \ \ \theta(t_1-t_2) < 0|\phi(x_1)\phi(x_2)|0 >$$

Then we consider the piece

$$< 0|\phi(x_2)\phi(x_1)|0 >$$

and insert the field expansions

$$\phi(x_1) = \int \frac{d^3\mathbf{k}_1}{(2\pi)^3 2\omega_1}[a(\mathbf{k}_1)e^{-ik_1.x_1} + a^\dagger(\mathbf{k}_1)e^{ik_1.x_1}] \tag{3.5.14}$$

$$\phi(x_2) = \int \frac{d^3\mathbf{k}_2}{(2\pi)^3 2\omega_2}[a(\mathbf{k}_2)e^{-ik_2.x_2} + a^\dagger(\mathbf{k}_2)e^{ik_2.x_2}] \tag{3.5.15}$$

Thus, dropping the terms which give zero,

$$< 0|\phi(x_2)\phi(x_1)|0 >=< 0| \int \frac{d^3\mathbf{k}_2}{(2\pi)^3 2\omega_2} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3 2\omega_1}[a(\mathbf{k}_2)e^{-ik_2.x_2}][a^\dagger(\mathbf{k}_1)e^{ik_1.x_1}]|0 >$$

Commuting the two operators using

$$[a(\mathbf{k}_2), a^\dagger(\mathbf{k}_1)] = (2\pi)^3 2\omega\delta^3(\mathbf{k_2} - \mathbf{k_1})$$

and integrating over the delta function gives,

$$< 0|\phi(x_2)\phi(x_1)|0 >= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} e^{-i(\omega(t_2-t_1)-\mathbf{k}.(\mathbf{x}_2-\mathbf{x}_1))}$$

Similarly we find for the second term

$$< 0|\phi(x_1)\phi(x_2)|0 >= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} e^{-i(\omega(t_1-t_2)-\mathbf{k}.(\mathbf{x}_1-\mathbf{x}_2))}$$

Thus, putting the two terms together,

$$i\Delta_F(x_2,x_1) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} \left[\theta(t_2-t_1)e^{-i(\omega(t_2-t_1)-\mathbf{k}.(\mathbf{x}_2-\mathbf{x}_1))} + \ \ \theta(t_1-t_2)e^{-i(\omega(t_1-t_2)-\mathbf{k}.(\mathbf{x}_1-\mathbf{x}_2))}\right]$$

The two terms can be combined into an explicitly Lorentz invariant form as follows,

$$i\Delta_F(x_2,x_1) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik.(x_2-x_1)} \tag{3.5.16}$$

where $\epsilon > 0$ and $\epsilon \to 0$. Eq.3.5.16 will prove extremely useful. In momentum space the Feynman rule for the real scalar propagator is thus:

$$i\Delta_F(k) = \frac{i}{k^2-m^2+i\epsilon} \tag{3.5.17}$$

where k is the four momentum flowing through the line.

Proof of Eq.3.5.16:

To prove Eq.3.5.16 consider the following integral

$$I = \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - E^2 + i\epsilon}$$

where $\epsilon > 0$ and $\epsilon \to 0$. In the complex plane we have

$$I = \oint dz \frac{e^{-izt}}{z^2 - E^2 + i\epsilon}$$

where the contour is along the entire real axis must be closed in the upper half of the complex plane for $t < 0$ and in the lower half for $t > 0$. There are simple poles at $z = -E + i\epsilon$ and $z = E - i\epsilon$. The residue theorem states that

$$\oint f(z) = 2\pi i \sum Res f(z)$$

where the contour must be closed in an anticlockwise sense. In this case $f(z) = \frac{p(z)}{q(z)}$ where $p(z) = e^{-izt}$ and $q(z) = z^2 - E^2 + i\epsilon$, and

$$Res f(z) = \frac{p(z = pole)}{q'(z = pole)}$$

Thus for $t < 0$ we close in the upper half plane and capture the upper pole which gives

$$I = -i\frac{(2\pi)}{2E}e^{iEt}$$

while for $t > 0$ we close in the lower half with a clockwise contour and capture the lower pole to give

$$I = -i\frac{(2\pi)}{2E}e^{-iEt}$$

Thus we have the result

$$\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t_2-t_1)}}{\omega^2 - E^2 + i\epsilon} = -i\frac{2\pi}{2E}[\theta(t_2-t_1)e^{-iE(t_2-t_1)} + \theta(t_1-t_2)e^{-iE(t_1-t_2)}]$$

and the desired result follows after a change of sign in one of the $\mathbf{k}$ integrations.

3.6 The LSZ Reduction Formula

The interaction picture is so nice why should anyone wish to return to the Heisenberg picture where the field operators obey the horribly complicated non-linear field equations:

$$(\partial^2 + m_0^2)\phi(x) = j(x) \tag{3.6.1}$$

where $j(x)$ is some combination of Heisenberg fields $\hat{\phi}(x)$. In fact that is exactly what we wish to do in this section. The reason for going back to the Heisenberg picture is partly to do with rigour, partly to do with symmetry and partly to do with calculability. In the Heisenberg picture we can rigorously define what we mean by the S-matrix in quantum field theory; we can exploit the time invariance of the Heisenberg states under translation symmetry and so on to help our arguments; and we can relate the S-matrix elements to quantities which we can readily calculate from field theory, namely the *Green's functions* which we begin by defining.

The vacuum expectation value of a time ordered product of n Heisenberg field operators is referred to as the *n particle Green's function* and is written,

$$\mathcal{G}(x_1, x_2, \cdots, x_n) \equiv < 0|T(\phi(x_1)\phi(x_2)\cdots\phi(x_n))|0 > \tag{3.6.2}$$

The definition reminds us of similar quantities considered in the previous section. However we emphasise that the operators in the Green's function definition are Heisenberg operators not free field operators. For example the two particle Green's function is not equal to the Feynman propagator in the full interacting theory.

In fact in defining Green's functions we are getting ahead of ourselves because we have yet to discuss the Heisenberg vacuum which appears in this definition. We are not even sure that physical states exist in the interacting theory! However we know that physical states must exist because we see them around, so we begin by postulating the existence of physical Heisenberg states. Then we exploit the translational invariance of Heisenberg states to conclude that such states must be eigenstates of energy and momentum. From here on everything is based on assumption. We assume that the eigenvalues of the physical states have non-negative mass and energy. A ground state or vacuum is postulated, and is denoted $|0>$, with its energy conventionally taken to be zero. Stable single particle states $|p>$ whose momentum p satisfies $p^2 = m^2$ are postulated, and for every type of particle of different mass and so on a field is invented.

Having defined states, we can now define in and out states in the Heisenberg picture. To do this we define in and out operators in the Heisenberg picture which which can create particle states with the physically observed mass m, analagously to the free field operators with which we are familiar. Note that the physical mass m is not the same as the parameter m_0 in the field equation 3.6.3. The in field $\phi_{in}(x)$ (and out field) is required to meet certain criteria. It must be able to create single physical particle states from the vacuum with

$p^2 = m^2$. It must be a functional of Heisenberg fields $\phi(x)$ (and any other fields which may appear in $j(x)$). It must transform like $\phi(x)$. It must satisfy the free KG equation for the physical mass m:

$$(\partial^2 + m^2)\phi_{in}(x) = j(x) \tag{3.6.3}$$

It must be expandable in terms of *in* creation and annihilation operators:

$$\phi_{in}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a_{in}(\mathbf{k})e^{-ik.x} + a^\dagger_{in}(\mathbf{k})e^{ik.x}] \tag{3.6.4}$$

with $a_{in}|0>=0$ and so on. Finally we might expect $\phi_{in}(x)$ (and out) to be identified as the asymptotic limit of the full Heisenberg field, up to a normalisation factor,

$$\phi(x) \to \sqrt{Z}\phi_{in}(x) \tag{3.6.5}$$

as $x_0 \to -\infty$ where $\sqrt{Z}$ is to ensure that $\phi_{in}(x)$ is normalised like a free field. However it is impossible to prevent the full interacting field $\phi(x)$ from interacting with itself even at large times, and so this asymptotic condition is too strong. The correct asymptotic condition is to relate the matrix elements of the respective operators, rather than the operators themselves. When expressed in terms of smeared wavepackets this is known as the weak asymptotic condition, and is due to Lehmann, Symanzik and Zimmermann (LSZ) in 1955. A consequence of the weak asymptotic condition is that:

$$<0|\phi(x)|p>=\sqrt{Z}<0|\phi_{in}(x)|p>=\sqrt{Z}\frac{e^{-ip.x}}{(2\pi)^3 2\omega_p} \tag{3.6.6}$$

after inserting Eq.3.6.4, with a similar result for out states.

The weak asymptotic condition, or more particularly Eq.3.6, can be used to derive a formula which relates S-matrix elements to Green's functions, called the LSZ reduction formula. To begin with we recall the definition of the S-matrix for a $2 \to 2$ scattering process,

$$S_{\beta\alpha} \equiv <\beta\ out|\alpha\ in> \tag{3.6.7}$$

where we have used a slightly different notation in which the Heisenberg states given by:

$$|\alpha\ in>=|\mathbf{p}_1, \mathbf{p}_2\ in>, \quad |\beta\ out>=|\mathbf{q}_1, \mathbf{q}_2\ out> \tag{3.6.8}$$

The scattering amplitude is

$$<\mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in>=<\mathbf{q}_1, \mathbf{q}_2\ out|a^\dagger_{in}(\mathbf{p}_1)|\mathbf{p}_2\ in>$$

$$=<\mathbf{q}_1, \mathbf{q}_2\ out|a^\dagger_{out}(\mathbf{p}_1)|\mathbf{p}_2\ in>$$

$$+<\mathbf{q}_1, \mathbf{q}_2\ out|a^\dagger_{in}(\mathbf{p}_1) - a^\dagger_{out}(\mathbf{p}_1)|\mathbf{p}_2\ in>$$

Now we use the analogue of Eq.2.3.22 to replace $a^\dagger_{in}$ by the in field,

$$a^\dagger_{in}(\mathbf{p}_1) = \int d^3\mathbf{x}\phi_{in}(\mathbf{x}, t)\ i\overleftrightarrow{\partial_0}\ e^{i(\mathbf{p}_1.\mathbf{x}-\omega t)} \tag{3.6.9}$$

and we define as usual

$$a(t) \overset{\leftrightarrow}{\partial_0} b(t) \equiv a(t)\frac{\partial b}{\partial t} - \frac{\partial a}{\partial t}b(t)$$

We insert Eq.3.6.9 and a similar expression for the out fields to give

$$< \mathbf{q}_1, \mathbf{q}_2 \; out|\mathbf{p}_1, \mathbf{p}_2 \; in >=< \mathbf{q}_1, \mathbf{q}_2 \; out|a^\dagger_{out}(\mathbf{p}_1)|\mathbf{p}_2 \; in >$$

$$- \int d^3\mathbf{x} e^{i(\mathbf{p}_1.\mathbf{x}-\omega t)} \; i\overset{\leftrightarrow}{\partial_0} < \mathbf{q}_1, \mathbf{q}_2 \; out|[\phi_{in}(\mathbf{x},t) - \phi_{out}(\mathbf{x},t)]|\mathbf{p}_2 \; in >$$

The first term is zero unless $\mathbf{q}_1 = \mathbf{p}_1$ or $\mathbf{q}_2 = \mathbf{p}_1$ corresponding to forward elastic scattering. Let us henceforth ignore this term. For $t \to -\infty$ we can replace the field ϕ_{in} by ϕ using

$$< a|\phi_{in}(x)|b >= \frac{1}{\sqrt{Z}} < a|\phi(x)|b >$$

Similarly for $t \to \infty$ we can replace the field ϕ_{out} by ϕ using

$$< a|\phi_{out}(x)|b >= \frac{1}{\sqrt{Z}} < a|\phi(x)|b >$$

Thus we can write

$$< \mathbf{q}_1, \mathbf{q}_2 \; out|\mathbf{p}_1, \mathbf{p}_2 \; in >=$$

$$[(t \overset{lim}{\to} \infty) - (t \overset{lim}{\to} -\infty)] \int d^3\mathbf{x} e^{i(\mathbf{p}_1.\mathbf{x}-\omega t)} \; i\overset{\leftrightarrow}{\partial_0} < \mathbf{q}_1, \mathbf{q}_2 \; out|\phi(\mathbf{x},t)|\mathbf{p}_2 \; in >$$

For a more convenient covariant form we use the identity,

$$[(t \overset{lim}{\to} \infty) - (t \overset{lim}{\to} -\infty)] \int d^3\mathbf{x} g_1(x) \; i\overset{\leftrightarrow}{\partial_0} \; g_2(x) = \int_{-\infty}^{\infty} d^4x \partial_0 [g_1(x) \; \overset{\leftrightarrow}{\partial_0} \; g_2(x)]$$

$$= \int_{-\infty}^{\infty} d^4x [g_1(x)\partial_0^2 g_2(x) - (\partial_0^2 g_1(x))g_2(x)]$$

Using this result and

$$(-\partial_0^2)e^{-ip_1.x} = (-\nabla^2 + m^2)e^{-ip_1.x}$$

we have

$$< \mathbf{q}_1, \mathbf{q}_2 \; out|\mathbf{p}_1, \mathbf{p}_2 \; in >=$$

$$\frac{i}{\sqrt{Z}} \int d^4x [(-\nabla^2 + m^2)e^{-ip_1.x} < \mathbf{q}_1, \mathbf{q}_2 \; out|\phi(x)|\mathbf{p}_2 \; in >$$

$$+e^{-ip_1.x}(-\partial_0^2) < \mathbf{q}_1, \mathbf{q}_2 \; out|\phi(x)|\mathbf{p}_2 \; in >]$$

Integrating the $-\nabla^2$ by parts twice onto $\phi(x)$, and discarding the surface term, we have

$$< \mathbf{q}_1, \mathbf{q}_2 \; out|\mathbf{p}_1, \mathbf{p}_2 \; in >= \frac{i}{\sqrt{Z}} \int d^4x e^{-ip_1.x}(\partial_x^2 + m^2) < \mathbf{q}_1, \mathbf{q}_2 \; out|\phi(x)|\mathbf{p}_2 \; in > \tag{3.6.10}$$

The above procedure which led to Eq.3.6.10 can be repeated by removing one of the particles from the initial state to give,

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= \frac{i}{\sqrt{Z}} \int d^4x e^{-ip_1.x}(\partial_x^2+m^2) < \mathbf{q}_2\ out|a_{out}(\mathbf{q}_1)\phi(x)|\mathbf{p}_2\ in >$$

The next step leads to a piece of a term like

$$a_{in}(\mathbf{q}_1)\phi(x)|\mathbf{p}_2\ in >$$

We want to be able to reject this term on the grounds that $\mathbf{p}_2 \neq \mathbf{q}_1$ but we could only do this if we had the term

$$\phi(x)a_{in}(\mathbf{q}_1)|\mathbf{p}_2\ in >$$

We can achieve this by replacing

$$< \mathbf{q}_2\ out|\phi(y)\phi(x)|\mathbf{p}_2\ in >$$

by the time ordered product,

$$< \mathbf{q}_2\ out|T(\phi(y)\phi(x))|\mathbf{p}_2\ in >$$

as can be verified explicitly. The final result is then:

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= \frac{i}{\sqrt{Z}}\frac{-i}{\sqrt{Z}} \int d^4x d^4y e^{-ip_1.x} e^{iq_1.y}(\partial_x^2 + m^2)(\partial_y^2 + m^2)$$

$$< \mathbf{q}_2\ out|T(\phi(y)\phi(x))|\mathbf{p}_2\ in >$$

Proceeding in the same way for the particles with momentum $\mathbf{q}_2$ and $\mathbf{p}_2$ we end up with the LSZ reduction formula:

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= (\frac{i}{\sqrt{Z}})^2(\frac{-i}{\sqrt{Z}})^2 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{-i(p_1.x_1+p_2.x_2-q_1.y_1-q_2.y_2)}$$

$$\times(\partial_{x_1}^2 + m^2)(\partial_{x_2}^2 + m^2)(\partial_{y_1}^2 + m^2)(\partial_{y_2}^2 + m^2) < 0|T(\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2))|0 > \tag{3.6.11}$$

The above formula relates the four particle Green's function (for the interacting theory in the Heisenberg picture) to the $2 \rightarrow 2$ S-matrix element. The four particle Green's function involves a sum of Feynman diagrams involving the four spacetime points x_1, x_2, y_1, y_2 with propagators attached to these points. The diagrams look like four legs attached to some interaction blob. The factors of $(\partial^2 + m^2)$ remove the propagators for the external legs leading into the interaction blob and put the external momenta onto the mass shell $p_i^2 = q_i^2 = m^2$.

It is straightforward to generalise the LSZ reduction formula to any number of in and out particles:

$$< \mathbf{q}_1, \cdots, \mathbf{q}_m\ out|\mathbf{p}_1, \cdots, \mathbf{p}_n\ in >= (\frac{i}{\sqrt{Z}})^n(\frac{-i}{\sqrt{Z}})^m \int d^4x_1 \cdots d^4x_n d^4y_1 \cdots d^4y_m$$

$$e^{-i\sum_{j=1}^{n} p_j.x_j} e^{i\sum_{k=1}^{m} q_k.y_k} (\partial_{x_1}^2 + m^2)\cdots(\partial_{x_n}^2 + m^2)(\partial_{y_1}^2 + m^2)\cdots(\partial_{y_m}^2 + m^2)$$
$$\times < 0|T(\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m))|0 > \qquad (3.6.12)$$

This is the general LSZ reduction formula which relates things that experimenters can actually measure (S-matrix elements) to things that can be calculated in field theory (Green's functions). In fact earlier in this chapter we short-cutted this proceedure by calculating the S-matrix elements directly in the interaction picture, without ever talking about Green's functions. However it turns out that the Green's functions are essential for a systematic rigorous treatment of field theory, and this is the reason why we introduced them here. In the next sections we shall learn new powerful techniques for calculating the Green's functions based on path integral techniques. But to get back to physics we shall always return to the LSZ reduction formula in Eq.3.6.12.

Finally we observe that the LSZ reduction formula relates the S-matrix elements to the full interacting theory Green's functions in Eq.3.6.2,

$$\mathcal{G}(x_1, x_2, \cdots, x_n) \equiv < 0_H|T(\phi_H(x_1)\phi_H(x_2)\cdots\phi_H(x_n))|0_H > \qquad (3.6.13)$$

where we have inserted subscripts H to emphasise that all the states and operators in this expression are in the Heisenberg picture. We can alternatively express the Green's functions in the interaction picture and hence make the connection with our analysis in the previous part of this chapter. Recall Eqs.3.2.34 and 3.2.38

$$G =< \psi_H|T(O_H(t_1)O_H(t_2)\cdots O_H(t_n))|\psi_H > \qquad (3.6.14)$$

$$G =< \psi_I(T/2)|T\left(\exp\left[-i\int_{-T/2}^{T/2} dt V_I(t)\right] O_I(t_1)O_I(t_2)\cdots O_I(t_n)\right)|\psi_I(-T/2) > \qquad (3.6.15)$$

Eqs.3.6.14 and 3.6.15 represent the same matrix element G but expressed in the Heisenberg and Interaction picture respectively. Using these results we may express the Green's functions in the interaction picture,

$$\mathcal{G}(x_1, x_2, \cdots, x_n) \equiv < 0_I(\infty)|T(\exp\left[-i\int_{-\infty}^{\infty} dt V_I(t)\right]$$
$$\times\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n))|0_I(-\infty) > \qquad (3.6.16)$$

This expression for the Green's functions in the interaction picture is very close to our previous expressions for the S-matrix element in the example of section 3.4. In that example V_I was given by Eq.3.4.9 and the lowest order S-matrix element was given by Eq.3.4.22 which can be identified as the lowest order term in the expansion of the exponential in Eq.3.6.16. Of course in the previous S-matrix element we had the external states generated by interaction picture creation and annihilation operators whereas the corresponding interaction picture three point Green's function in Eq.3.6.16 involves the three interaction picture fields for the external lines. Thus when we evaluate the Green's function we will get field contractions for the external lines corresponding to external *propagators*, whereas the S-matrix element gave contractions of

field with creation/annihilation operator resulting in external *wavefunctions* . To get to the S-matrix element from the Green's function, all we need to do is therefore remove the external propagators and replace them by the appropriate wavefunction (corresponding to replacing the external field by the external creation or annihilation operator). This is exactly what the LSZ reduction formula in Eq.3.6.12 does.

3.7 Problems Set 3

1. Show that the matrix element in Eq.3.2.16 can be written in covariant form

$$\langle \Phi_\beta \, | \psi_I(\infty) \rangle = -i \int d^4x \phi^*_\beta(x) V \phi_\alpha(x)$$

2. Show that as $T \to \infty$,

$$\frac{1}{2\pi T} \frac{\sin^2(\omega_{\beta\alpha} T/2)}{(\omega_{\beta\alpha}/2)^2} \to \delta(\omega_{\beta\alpha}).$$

3. Suppose we have a single real field $\phi(x)$ and

$$\mathcal{L} = -\lambda \phi^n$$

$$H_I = \int d^3\mathbf{x} \lambda : \phi^n(x) :$$

Consider an initial state with 2 particles of momenta $\mathbf{k}_1$,$\mathbf{k}_2$,

$$|t = -\infty >= a^\dagger(\mathbf{k}_1) a^\dagger(\mathbf{k}_2) |0>$$

and a final state with m particles,

$$|t = \infty >= a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_m) |0>$$

To leading order ($U_I^{(1)}$) what is the value of m such that the transition is non-zero?

4. Consider the case of the interaction between the real scalar ϕ and the complex scalar χ in the notes. Suppose that the initial state is again

$$|t = -\infty >= a^\dagger(\mathbf{k}_1) a^\dagger(\mathbf{k}_2) |0>$$

and the final state is

$$|t = \infty >= b^\dagger(\mathbf{p}_1) \cdots b^\dagger(\mathbf{p}_m) d^\dagger(\mathbf{q}_1) \cdots d^\dagger(\mathbf{q}_n) |0>$$

(a) Show that to first order that the matrix element is zero.
(b) What is the lowest order where the matrix element is non-zero and what is the value of m and n?

5. Show that the matrix element

$$S_{fi}^{(1)} = (-ig) < 0 | b(\mathbf{p}) d(\mathbf{q}) \left(\int d^4x \chi_I^\dagger(x) \chi_I(x) \phi_I(x) \right) a^\dagger(\mathbf{k}) |0>$$

leads to Eq.3.4.23 by inserting field expansions and using the commutation relations for the creation and annihilation operators in order to put the annihilation operators to the right and the creation operators to the left so that these terms hit against the vacuum and vanish.

6. Using Wick's theorem evaluate the same matrix element as in Q.5,

$$S_{fi}^{(1)} = (-ig) < 0|b(\mathbf{p})d(\mathbf{q}) \left(\int d^4x \chi_I^\dagger(x)\chi_I(x)\phi_I(x) \right) a^\dagger(\mathbf{k})|0 >$$

[Note that $\phi^{\cdot}(x)\chi^{\cdot}(y) = 0$, $\chi^{\cdot}(x)\chi^{\cdot}(y) = 0$, $\chi^{\dagger\cdot}(x)\chi^{\dagger\cdot}(y) = 0$, and $\phi^{\cdot}(x)a^{\dagger\cdot}(\mathbf{k}) = e^{ik.x}$.]

7.Show that the next order correction

$$S_{fi}^{(2)} = \frac{(-ig)^2}{2} \int d^4x \int d^4y$$

$$< 0|T \left(b(\mathbf{p})d(\mathbf{q})\chi_I^\dagger(x)\chi_I(x)\phi_I(x)\chi_I^\dagger(y)\chi_I(y)\phi_I(y)a^\dagger(\mathbf{k}) \right) |0 >$$

is zero.

Then use Wick's theorem to evaluate

$$S_{fi}^{(3)} = \frac{(-ig)^3}{2!} \int d^4x \int d^4y \int d^4z$$

$$< 0|T \left(b(\mathbf{p})d(\mathbf{q})\chi_I^\dagger(x)\chi_I(x)\phi_I(x)\chi_I^\dagger(y)\chi_I(y)\phi_I(y)\chi_I^\dagger(z)\chi_I(z)\phi_I(z)a^\dagger(\mathbf{k}) \right) |0 > .$$

8. By repeating the proceedure which led to Eq.3.6.10 verify explicitly that the LSZ formula Eq.3.6.11 is obtained.

Chapter 4

Transition Rates and Cross-Sections

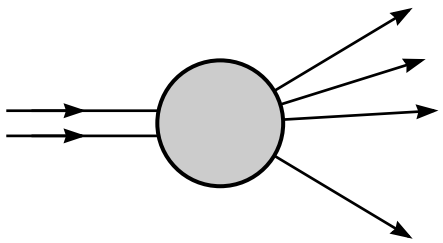
4.1 Transition Rates

In quantum field theory you calculate the S-matrix element to go from state $|i\rangle$ to state $|f\rangle$ to be of the form in Eq.3.4.24,

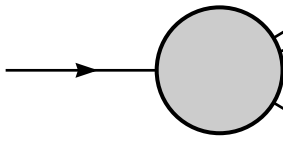
$$S_{fi} = i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i), \tag{4.1.1}$$

where $i\mathcal{M}_{fi}$ is the scattering amplitude or matrix element obtained from a Feynman diagram calculation, and the overall energy-momentum delta function has been factored out (so when you draw your Feynman diagrams you conserve energy-momentum at every vertex). We have in mind processes where two particles scatter, or one particle decays, as shown in Figure 4.1.1.

Attempting to take the squared modulus of this amplitude produces a meaningless square of a delta function. This is a technical problem because our amplitude is expressed between non-normalisable plane wave states. These states extend throughout space-time so the scattering process occurs everywhere all the time. To deal with this properly you can construct normalised wavepacket states which do become well separated in the far past and the far



(a)



(b)

Figure 4.1.1 Scattering (a) and decay (b) processes.

future. We will be low-budget and put our system in a box of volume $V = L^3$ [1]. We also imagine that the interaction is restricted to act only over a time of order T. The final answers come out independent of V and T, reproducing the luxury wavepacket ones. We are in good company here: Nobel Laureate Steven Weinberg says in his recent book, when discussing cross sections and decay rates, "... (as far as I know) no interesting open problems in physics hinge on getting the fine points right regarding these matters."

In infinite spacetime with plane wave states the transition amplitude from i to f is given by (4.1.1). However in our box of finite size L and for our finite time T the amplitude is given by Eq.4.1.1 but with the Dirac delta functions replaced by well behaved functions:

$$(2\pi)^4\delta^4(P_f - P_i) \to I(E_f - E_i, T)I^3(\mathbf{P}_f - \mathbf{P}_i, L) \tag{4.1.2}$$

where for example,

$$I(E_f - E_i, T) = \frac{1}{(\frac{E_f - E_i}{2})} \sin\left(\frac{(E_f - E_i)T}{2}\right) \tag{4.1.3}$$

which is familiar from Eq.3.2.16. This function has the property that, as $T \to \infty$,

$$I(E_f - E_i, T) \to 2\pi\delta(E_f - E_i) \tag{4.1.4}$$

and also

$$I^2(E_f - E_i, T) \to 2\pi T\delta(E_f - E_i) \tag{4.1.5}$$

with analagous results for $I(P_f - P_i, L)$. Thus in our spacetime box we have the approximate result,

$$\left|(2\pi)^4\delta^4(P_f - P_i)\right|^2 \simeq VT\,(2\pi)^4\delta^4(P_f - P_i). \tag{4.1.6}$$

The second ingredient in the amplitude is a factor of $1/(2E_iV)^{1/2}$ for every particle in the initial or final state (here I am using E_i synonymously with ω_{k_i}). This comes from converting between relativistic and box normalisations for the states.

> The box states are normalised to one particle in volume V and the relativistic states have $2\omega_k$ particles per unit volume, thus the states which occur in the amplitude are related by
>
> $$|k\rangle_{\rm rel} \longleftrightarrow \sqrt{2\omega_k}\sqrt{V}|\mathbf{k}\rangle_{\rm box}.$$
>
> We shall henceforth use box normalisation for the final states which we simply label by $|f\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle$ and similarly for the initial states which we write as
>
> $$|i\rangle = \begin{cases} |\mathbf{p}\rangle & \text{one particle} \\ |\mathbf{p}_1, \mathbf{p}_2\rangle & \text{two particles} \end{cases} \tag{4.1.7}$$

[1] Please do not confuse the volume of the cube $V = L^3$ with the potential V introduced earlier

Allowing for one or two particles in the initial state and N in the final state,

$$\text{box amp} = i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i)\prod_{f=1}^{N}\left[\frac{1}{\sqrt{2E_fV}}\right]\prod_{\text{in}}\left[\frac{1}{\sqrt{2E_iV}}\right] ,$$

The squared matrix element is thus:

$$|\text{box amp}|^2 = |\mathcal{M}_{fi}|^2VT(2\pi)^4\delta^4(P_f - P_i)\prod_{f=1}^{N}\left[\frac{1}{2E_fV}\right]\prod_{\text{in}}\left[\frac{1}{2E_iV}\right] , \quad (4.1.8)$$

where we have used Eq.4.1.6.

4.2 The Number of Final States

For a single particle final state, the number of available states dn in some momentum range $\mathbf{k}$ to $\mathbf{k} + d\mathbf{k}$ is, in the box normalisation,

$$dn = \frac{d^3\mathbf{k}}{(2\pi)^3}\,V. \quad (4.2.1)$$

This result is proved by recalling that the allowed momenta in the box have components which can only take on discrete values such as $k_x = 2\pi n_x/L$ where n_x is an integer. Thus $dn = dn_x dn_y dn_z$ and the result follows.

For a two particle final state we have

$$dn = dn_1 dn_2$$

where

$$dn_1 = \frac{d^3\mathbf{k}_1}{(2\pi)^3}\,V, \quad dn_2 = \frac{d^3\mathbf{k}_2}{(2\pi)^3}\,V,$$

where dn is the number of final states in some momentum range $\mathbf{k}_1$ to $\mathbf{k}_1 + d\mathbf{k}_1$ for particle 1 and $\mathbf{k}_2$ to $\mathbf{k}_2 + d\mathbf{k}_2$ for particle 2. There is an obvious generalisation to an N particle final state,

$$dn = \prod_{f=1}^{N}\frac{d^3\mathbf{k}_f\,V}{(2\pi)^3}. \quad (4.2.2)$$

4.3 Lorentz Invariant Phase Space (LIPS)

The differential transition rate is given by

$$dW = \frac{|\text{box amp}|^2 dn}{T} \quad (4.3.1)$$

Using Eqs.4.2.2 and Eq.4.1.8 we find,

$$dW = |\mathcal{M}_{fi}|^2(2\pi)^4\delta^4(P_f - P_i)V\prod_{f=1}^{N}\left[\frac{1}{2E_fV}\right]\prod_{\text{in}}\left[\frac{1}{2E_iV}\right]\prod_{f=1}^{N}\frac{d^3\mathbf{k}_f\,V}{(2\pi)^3}. \quad (4.3.2)$$

$$dW = |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(P_f - P_i) V \prod_{\text{in}} \left[\frac{1}{2E_i V}\right] \prod_{f=1}^{N} \frac{d^3\mathbf{k}_f}{(2\pi)^3 2E_f}. \tag{4.3.3}$$

This can be written as

$$dW = S\,|\mathcal{M}_{fi}|^2 V \prod_{\text{in}} \left[\frac{1}{2E_i V}\right] \times (dLIPS)\ , \tag{4.3.4}$$

where the LIPS is,

$$dLIPS = (2\pi)^4 \delta^4(P_f - P_i) \prod_{f=1}^{N} \frac{d^3\mathbf{k}_f}{(2\pi)^3\, 2E_f}. \tag{4.3.5}$$

Observe that everything in the transition rate is Lorentz invariant save for the initial energy factor and the factors of V (using $d^3k\,/2E = d^4k\,\delta^4(k^2 - m^2)\theta(k^0)$, which is manifestly Lorentz invariant, where $E = (\mathbf{k}^2 + m^2)^{1/2}$). For a one particle initial state the factor of V cancels, and we can breath a sigh of relief (after all we would not expect physical quantities to depend on the size of our artificial box). For a two initial particle scattering situation the factors of V will also cancel in the physical cross-section as we will show in the next section. I have smuggled in one extra factor, S, in equation (4.3.1) for the transition probability. If there are some identical particles in the final state, we will overcount them when integrating over all momentum configurations. The symmetry factor S takes care of this. If there n_i identical particles of type i in the final state, then

$$S = \prod_i \frac{1}{n_i!}. \tag{4.3.6}$$

4.4 Cross Sections

The total cross-section for a static target and a beam of incoming particles is defined as the total transition rate for a single target particle and a unit beam flux. The differential cross-section is similarly related to the differential transition rate. We have calculated the differential transition rate with a choice of normalisation corresponding to a single "target" particle in the box, and a "beam" corresponding also to one particle in the box. A beam consisting of one particle per volume V with a velocity v has a flux N_0 given by

$$N_0 = \frac{v}{V}$$

particles per unit area per unit time. Thus the differential cross-section σ is related to the differential transition rate in Eq.4.3.4 by

$$d\sigma = \frac{dW}{N_0} = dW \times \frac{V}{v} \tag{4.4.1}$$

where as promised the factors of V cancel in the cross-section.

Now let us generalise to the case where in the frame where you make your measurements the "beam" has a velocity v_1 but the "target" particles are also moving with a velocity v_2. In a colliding beam experiment for example v_1 and v_2 will point in opposite directions in the laboratory. In this case the definition of the cross-section is retained as above, but now the beam flux of particles N_0 is effectively increased by the fact that the target particles are moving towards it. The effective flux in the laboratory in this case is given by

$$N_0 = \frac{|\vec{v}_1 - \vec{v}_2|}{V}$$

which is just the total of particles per unit area which run past each other per unit time. I denote the velocities with arrows to remind you that they are vector velocities which must be added using the vector law of velocity addition not the relativistic law. In the general case, then, the differential cross-section is given by

$$d\sigma = \frac{dW}{N_0} = \frac{1}{|\vec{v}_1 - \vec{v}_2|}\,\frac{1}{4E_1E_2} S|\mathcal{M}_{fi}|^2 \times dLIPS \tag{4.4.2}$$

where we have used Eq.4.3.4 for the transition rate, and the box volume V has again cancelled (phew!). [2] We re-emphasise that the velocities in the flux factor, $1/|\vec{v}_1 - \vec{v}_2|$, are subtracted *nonrelativistically*. The amplitude-squared and phase space factors are manifestly Lorentz invariant. What about the initial velocity and energy factors? Observe that

$$E_1E_2(\vec{v}_1 - \vec{v}_2) = E_2\mathbf{p}_1 - E_1\mathbf{p}_2.$$

In a frame where $\mathbf{p}_1$ and $\mathbf{p}_2$ are collinear,

$$|E_2\mathbf{p}_1 - E_1\mathbf{p}_2|^2 = (p_1{\cdot}p_2)^2 - m_1^2m_2^2,$$

and the last expression is manifestly Lorentz invariant. Hence the differential cross section is Lorentz invariant, as is the total cross section,

$$\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2|}\,\frac{1}{4E_1E_2}\,S\sum\int_{\text{final states}}|\mathcal{M}_{fi}|^2 \times dLIPS. \tag{4.4.3}$$

4.5 Two-body Scattering

An important special case is $2 \to 2$ scattering (see Figure 4.5.1),

$$a(p_a) + b(p_b) \to c(p_c) + d(p_d).$$

[2] Because the result is independent of the dimensions of the box, you can think of making the box as large as you like – say as large as CERN or perhaps as large as the Earth, or the Universe! This means that there is no reason to worry about the box.

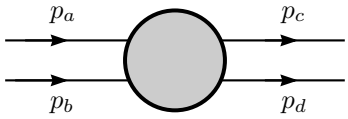


Figure 4.5.1 $2 \to 2$ scattering.

Invariant $2 \to 2$ scattering amplitudes are frequently expressed in terms of the *Mandelstam variables*, defined by,

$$\begin{aligned} s &\equiv (p_a + p_b)^2 = (p_c + p_d)^2 \\ t &\equiv (p_a - p_c)^2 = (p_b - p_d)^2 \\ u &\equiv (p_a - p_d)^2 = (p_b - p_c)^2 \end{aligned} \tag{4.5.1}$$

In fact there are only two independent Lorentz invariant combinations of the available momenta in this case, so there must be some relation between s, t and u (see exercise).

4.6 Decay Rates

With one particle in the initial state the total transition rate is

$$W = \frac{1}{2E} S \sum \int_{\text{final states}} |\mathcal{M}_{fi}|^2 \times dLIPS$$

Only the factor $1/2E$ is not manifestly Lorentz invariant. In the rest frame, for a particle of mass m, we have

$$\Gamma \equiv \frac{1}{2m} \sum \int_{\text{final states}} |\mathcal{M}_{fi}|^2 \times dLIPS. \tag{4.6.1}$$

This is the "decay rate." In an arbitrary frame we find, $W = (m/E)\Gamma$, which has the expected Lorentz dilatation factor. In the master formula (equation 4.3.4) this is what the product of $1/2E_i$ factors for the initial particles does.

4.7 Optical Theorem

Recall the S-operator in Eq.3.2.23

$$S \equiv U_I(\infty, -\infty)$$

Since the S-operator is unitary, we can write,

$$(S - I)(S^\dagger - I) = -\Big((S - I) + (S - I)^\dagger\Big). \tag{4.7.1}$$

Note that $S-I$ is the quantity of interest, since we generally ignore cases where there is no interaction (the "I" piece of S). In terms of the invariant amplitude,

$$\begin{aligned} \langle f|\, S-I\,|i\rangle &= i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f-P_i) \\ \langle f|\,(S-I)^\dagger\,|i\rangle &= -i\mathcal{M}^*_{if}(2\pi)^4\delta^4(P_f-P_i) \end{aligned}$$

Sandwiching the above unitarity relation (equation 4.7.1) between states $|i\rangle$ and $|f\rangle$, and inserting a complete set of states between the factors on the left hand side,

$$\begin{aligned} &\sum_m \langle f|\, S-I\,|m\rangle\,\langle m|\, S^\dagger - I\,|i\rangle \\ &\quad= \sum_m \mathcal{M}_{fm}\mathcal{M}^*_{im}(2\pi)^8\delta^4(P_f-P_m)\delta^4(P_i-P_m)\prod_{j=1}^{r_m}\frac{d^3\mathbf{k}_j}{(2\pi)^3\,2E_j} \\ &\quad= \sum_m \int \mathcal{M}_{fm}\mathcal{M}^*_{im}(2\pi)^4\delta^4(P_f-P_i)D_m \end{aligned}$$

where D_m is the LIPS for the state labelled by m, containing r_m particles, $D_m \equiv D_{r_m}(P_i;k_1,\ldots,k_{r_m})$. Hence,

$$\sum_m \int \mathcal{M}_{fm}\mathcal{M}^*_{im}D_m = i(\mathcal{M}^*_{if}-\mathcal{M}_{fi}).$$

If the intermediate state m contains n_i identical particles of type i, there is an extra symmetry factor S, with,

$$S=\prod_i \frac{1}{n_i!}$$

on the left hand side of the above equation to avoid overcounting. The same factor (see equation 4.3.6) appears in the cross section formula (equation 4.4.2) when some of the final state particles are identical.

If $|i\rangle$ and $|f\rangle$ are the same two particle state, corresponding to two particles scattering elastically in the forward direction, then

$$2\,\mathrm{Im}\,\mathcal{M}_{ii} = 4E_T p_i\,\sigma. \tag{4.7.2}$$

This is the *optical theorem* which relates the forward part of the scattering amplitude to the total cross-section. If particles of masses m_1 and m_2 scatter, then $E_T = s^{1/2}$ and $4sp_i^2 = \lambda(s,m_1^2,m_2^2)$, where λ is the function defined in equation (4.8.2). Then the optical theorem reads, $\mathrm{Im}\,\mathcal{M}_{ii} = \lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)\,\sigma$.

4.8 Problems Set 4

▷ **Exercise 4.8.1**
Show that the expression for two-body phase space in the centre of mass frame is given by

$$\frac{d^3k_1}{(2\pi)^3\,2\omega_{k_1}}\,\frac{d^3k_2}{(2\pi)^3\,2\omega_{k_2}}\,(2\pi)^4\delta^4(P-k_1-k_2)=\frac{1}{32\pi^2 s}\lambda^{1/2}(s,m_1^2,m_2^2)d\Omega^*,\quad(4.8.1)$$

where $s = P^2$ is the centre of mass energy squared, $d\Omega^*$ is the solid angle element for the angle of one of the outgoing particles with respect to some fixed direction, and

$$\lambda(a,b,c)=a^2+b^2+c^2-2ab-2bc-2ca.\quad(4.8.2)$$

▷ **Exercise 4.8.2**
Show that in the centre of mass frame the differential cross section is,

$$\frac{d\sigma}{d\Omega^*}=\frac{S\,\lambda^{1/2}(s,m_c^2,m_d^2)}{64\pi^2 s\,\lambda^{1/2}(s,m_a^2,m_b^2)}|\mathcal{M}_{fi}|^2.\quad(4.8.3)$$

The result of equation (4.8.3) is valid for any $|\mathcal{M}_{fi}|^2$, but if $|\mathcal{M}_{fi}|^2$ is a constant you can trivially get the total cross section.

▷ **Exercise 4.8.3**
Show that

$$s+t+u=m_a^2+m_b^2+m_c^2+m_d^2.$$

▷ **Exercise 4.8.4**
Show that for two body scattering of particles of equal mass m,

$$s\geq 4m^2,\qquad t\leq 0,\qquad u\leq 0.$$

Chapter 5

Path Integrals in Quantum Mechanics

5.1 Introduction

The first thing you need to know about "path integral quantisation" is that it is sometimes called "functional methods". Essentially path integral quantisation is an alternative formulation of quantum mechanics which was developed by Feynman in 1948, based on some early ideas by Dirac in 1933. The Schrodinger formulation of quantum mechanics, which we reviewed in section 1.2, is based on the Hamiltonian approach, i.e. we take the Hamiltonian from classical mechanics and promote the generalised positions and momenta to the status of operators. We used this approach to quantise the KG field, which we regarded as the continuum limit of a system of coupled oscillators. By contrast Feynman's path integral approach to quantum mechanics is based on the Lagrangian approach, and uses the action S which is related to the Lagrangian by Eq.1.1.2. We have already seen in Eq.1.2.10 that in quantum mechanics, knowledge of the wavefunction at time t_1, together with the propagator $< q_2|U(t_2,t_1)|q_1 >$, is sufficient to determine the wavefunction at some later time t_2, providing that the integral in Eq.1.2.10 can be performed. Providing we know the propagator, we don't need Schrodinger's equation at all. What Feynman's path integral approach does is give the propagator directly, as a sum over all paths between the two points q_i, t_i and q_f, t_f, where each path is ascribed an amplitude $e^{iS[q(t)]}$ where S is in units of $h/(2\pi)$. Thus schematically,

$$< q_2|U(t_2,t_1)|q_1 >= const. \sum_{allpaths} e^{iS[q(t)]} \tag{5.1.1}$$

Notice that the sum is over all paths, not just the classical path. Thus a classical ball falling under gravity between two spacetime points is, from the quantum point of view, taking all possible paths across the Universe, rather like a Star Trek transporter. Classically the Newtonian trajectory is preferred not because the other paths are not taken, but because all the amplitudes associated with the other paths tend to accurately cancel out in the sum. We are left only

with the classical path and paths which are very close to the classical path, where very close means that the action is equal to the classical action to an accuracy of $h/(2\pi)$. The classical path is of course the one which extremises the action, which is why paths very close to it do not cancel and this provides an understanding or derivation of the principle of least action in the classical limit.

The path integral approach to quantum mechanics clearly has great aethetic value. It answers deep philosophical questions like "which of the two slits does the particle really pass through in the two slit experiment?" with "the particle takes both paths through the slits". In fact the particle really takes two infinite classes of paths. However, from the point of view of quantum field theory, you may wonder why we are bothering to consider this approach when the standard or "canonical" approach to quantisation seems perfectly serviceable. Why are we wasting time considering an alternative approach to quantum field theory when we already have achieved our goal of quantising the KG field? The answer is twofold. Firstly the path integral approach based on the notion of a propagator provides a more intuitive understanding of Feynman diagrams and Feynman rules which are the nuts and bolts of scattering and decay calculations. Indeed, Feynman himself introduced these diagrams from the point of view of the path integral approach, and there is no higher recommendation than that. The second reason is that the Hamiltonian formalism of canonical quantisation does not manifestly preserve Lorentz invariance, while the Lagrangian formalism of the path integral approach does. When we consider more complicated field theories such as QED, and gauge theories in general, it will prove to be technically convenient to maintain Lorentz invariance in a manifest way all through the calculations. Thus, in the following QFT courses, we will rely almost exclusively on path integral methods – so it is as well to learn these methods as early as possible.

5.2 The Point to Point Transition Amplitude

Our starting point is the expression for the propagator in the Heisenberg basis as given in Eq.5.2.1

$$< q_f|U(t_f,t_i)|q_i >=< q_f,t_f|q_i,t_i > \tag{5.2.1}$$

which shows very clearly that the propagator is just the amplitude that the particle is at some final (f) point q_f, t_f given that it was at the initial (i) point q_i, t_i where $t_f > t_i$. In the Schrodinger picture the propagator is

$$< q_f|U(t_f,t_i)|q_i >=< q_f|e^{-iH(t_f-t_i)}|q_i > \tag{5.2.2}$$

For convenience we also reproduce Eq.1.2.10 which shows how the propagator may be used to determine the future value (at $t_f > t_i$) of the wavefunction,

$$\psi(q_f,t_f) = \int dq_i iG(q_f,t_f;q_i,t_i)\psi(q_i,t_i) \tag{5.2.3}$$

where we have defined

$$iG(q_f, t_f; q_i, t_i) \equiv < q_f|U(t_f, t_i)|q_i > \tag{5.2.4}$$

again valid only for $t_f > t_i$. The function $G(q_f, t_f; q_i, t_i)$ is known as the Green's function. It has the property that if the initial wavefunction at $t_i = 0$ is just a delta function at $q_i = 0$, $\psi(q_i, t_i) = \delta(q_i)$, then the subsequent wavefunction is just equal to i times the Green's function

$$\psi(q_f, t_f) = iG(q_f, t_f; 0, 0) \tag{5.2.5}$$

Eq.5.2.5 is only valid for $t_f > t_i$, but can be written in a form valid for all times as

$$\theta(t_f - t_i)\psi(q_f, t_f) = \int dq_i iG_(q_f, t_f; q_i, t_i)\psi(q_i, t_i) \tag{5.2.6}$$

where the Green's function is defined to be zero for $t_f < t_i$,

$$G_(q_f, t_f; q_i, t_i) \equiv 0, \quad for \;\; t_f < t_i$$

and is called a retarded Green's function or propagator.

The name of the game in this approach is therefore to find the retarded propagator. We do this by converting $< q_f, t_f|q_i, t_i >$ to a path integral as follows.

Let us begin by dividing the time interval between t_i and t_f into two, with t_1 as the intermediate time and q_1 as the intermediate point in space. Inserting a complete set of Heisenberg states at time t_1 we have,

$$< q_f, t_f|q_i, t_i >= \int dq_1 < q_f, t_f|q_1, t_1 >< q_1, t_1|q_i, t_i > \tag{5.2.7}$$

which corresponds to the product of the propagator from q_i, t_i to q_1, t_1 with the propagator from q_1, t_1 to q_f, t_f, integrated over all possible positions q_1 at the intermediate time t_1. Mathematically Eq.5.2.7 displays a sort of group property of propagators. Physically we are performing a very crude sum over paths between the inital and final points which allow the particle to be anywhere at the intermediate time t_1. Of course we haven't gained anything yet because the expression still involves a product of propagators of the type we wish to evaluate. Nevertheless Eq.5.2.2 shows that there may be some merit in continuing the process of dividing up the time interval into smaller and smaller steps. If the time steps τ are small enough then we can write,

$$e^{-iH\tau} = 1 - iH\tau + O(H\tau 2)^2 \tag{5.2.8}$$

where $H\tau$, in units of $h/(2\pi)$, is small compared to unity.

Motivated by the above discussion, let us now divide the time interval between t_i and t_f into $n + 1$ equal steps $\tau = (t_f - t_i)/(n + 1)$. Inserting a

complete set of Heisenberg position eigenstates at each time step we have the obvious generalisation of Eq.5.2.7,

$$< q_f, t_f | q_i, t_i >= \int \dots \int dq_1 \dots dq_n < q_f, t_f | q_n, t_n >< q_n, t_n | \dots | q_1, t_1 >< q_1, t_1 | q_i, t_i > \tag{5.2.9}$$

Physically we are now performing a more refined sum over all paths by allowing the particle to be in any position at each of the specified time slices. Even though the time slices may be infinitesimally close together the particle is allowed to be infinitely far apart between the two times, so this cannot be regarded as a sum over trajectories in the usual sense (these "paths" are strictly called "Markov chains").

Now consider one of the propagators between q_j, t_j and q_{j+1}, t_{j+1},

$$< q_{j+1}, t_{j+1} | q_j, t_j >=< q_j | e^{-iH\tau} | q_j > \tag{5.2.10}$$

Using Eq.5.2.8 we have,

$$\begin{aligned} < q_{j+1}, t_{j+1} | q_j, t_j > &= < q_j | 1 - iH\tau | q_j > \\ &= \delta(q_{j+1} - q_j) - i\tau < q_{j+1} | H | q_j > \end{aligned} \tag{5.2.11}$$

In the first term the Dirac delta function can be written as

$$\delta(q_{j+1} - q_j) = \frac{1}{2\pi} \int dp e^{ip(q_{j+1} - q_j)} \tag{5.2.12}$$

In the second term the Hamiltonian matrix element can be written

$$< q_{j+1} | H | q_j >=< q_{j+1} | \frac{p^2}{2m} | q_j > + < q_{j+1} | V(q) | q_j > \tag{5.2.13}$$

where we have assumed a simple kinetic plus potential form for the Hamiltonian. We now indulge in a little operator expungation. Inserting complete sets of momentum eigenfuctions in the first (kinetic) term and remembering that p is an operator gives,

$$\begin{aligned} < q_{j+1} | \frac{p^2}{2m} | q_j > &= \int \int dp dp' < q_{j+1} | p' >< p' | \frac{p^2}{2m} | p >< p | q_j > \\ &= \int \int \frac{dp dp'}{2\pi} e^{i(p' q_{j+1} - p q_j)} \frac{p^2}{2m} \delta(p - p') \\ &= \int \frac{dp}{2\pi} e^{ip(q_{j+1} - q_j)} \frac{p^2}{2m} \end{aligned} \tag{5.2.14}$$

where p^2 in the last line is a number, and we have used the plane wave result,

$$< q | p >= \frac{1}{\sqrt{2\pi}} e^{ipq} \tag{5.2.15}$$

Similarly the operators may be expunged from the potential term, which may also be expressed as a momentum integral,

$$\begin{aligned} < q_{j+1} | V(q) | q_j > &= V(\overline{q_j}) < q_{j+1} | q_j > \\ &= V(\overline{q_j}) \int \frac{dp}{2\pi} e^{ip(q_{j+1} - q_j)} \end{aligned} \tag{5.2.16}$$

where we define the average value as

$$\overline{q_j} \equiv \frac{1}{2}(q_{j+1} + q_j) \tag{5.2.17}$$

Putting Eqs.5.2.14 and 5.2.16 together yields the Hamiltonian matrix element with all operators expunged,

$$\begin{aligned} < q_{j+1}|H|q_j > &= < q_{j+1}|\frac{p^2}{2m}|q_j > + < q_{j+1}|V(q)|q_j > \\ &= \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)}(\frac{p^2}{2m} + V(\overline{q_j})) \\ &= \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)} H(p, \overline{q_j}) \end{aligned} \tag{5.2.18}$$

Inserting Eqs.5.2.18 and 5.2.12 into 5.2.11

$$\begin{aligned} < q_{j+1}, t_{j+1}|q_j, t_j > &= < q_j|1 - iH\tau|q_j > \\ &= \delta(q_{j+1} - q_j) - i\tau < q_{j+1}|H|q_j > \\ &= \frac{1}{2\pi}\int dp e^{ip(q_{j+1}-q_j)} - i\tau \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)} H(p, \overline{q_j}) \\ &= \frac{1}{2\pi}\int dp e^{ip(q_{j+1}-q_j)}(1 - i\tau H(p_j, \overline{q_j})) \\ &= \frac{1}{2\pi}\int dp e^{ip(q_{j+1}-q_j)} e^{-iH(p_j,\overline{q_j})\tau} \\ &= \frac{1}{2\pi}\int dp_j e^{i((p_j(q_{j+1}-q_j) - H(p_j,\overline{q_j})\tau)} \end{aligned} \tag{5.2.19}$$

where we have used Eq.5.2.8 in reverse, have written the dummy momentum variable as p_j to remind us that we are doing this near the time t_j. Eq.5.2.19 is an expression for the propagator between q_j, t_j and q_{j+1}, t_{j+1} which only involves an integration over c-numbers (no q-numbered operators), but is only valid in the small τ limit. The full propagator in Eq.5.2.9 is merely a product of such little propagators,

$$< q_f, t_f|q_i, t_i > = n \overset{lim}{\rightarrow} \infty \int \prod_{j=1}^{n} dq_j \prod_{j=0}^{n} \frac{dp_j}{2\pi} e^{i\sum_{j=0}^{n}((p_j(q_{j+1}-q_j) - H(p_j,\overline{q_j})\tau)} \tag{5.2.20}$$

with $q_0 = q_i$ and $q_{n+1} = q_f$. In the continuum limit we write $q_i = q(t_i)$ and $q_f = q(t_f)$ and Eq.5.2.20 may be written in symbollic form as

$$< q_f, t_f|q_i, t_i > = \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q} - H(p,q))\right)} \tag{5.2.21}$$

We stress again that this expression only involves c-numbers. Eq.5.2.21 means no more and no less than Eq.5.2.20, but is easier to write down. In practical calculations we must return to 5.2.20.

We can expunge the Hamiltonian by performing the momentum integrations in Eq.5.2.20, using the standard Gaussian integral,

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c} \tag{5.2.22}$$

Using this result for each of the little propagators yields the full propagator,

$$\begin{aligned} < q_f, t_f|q_i, t_i > &= n \stackrel{lim}{\rightarrow} \infty \int \prod_{j=1}^{n} dq_j \prod_{j=0}^{n} \frac{dp_j}{2\pi} e^{i\sum_{j=0}^{n}((p_j(q_{j+1}-q_j)-\frac{p_j^2}{2m}\tau - V(\overline{q_j})\tau)} \\ &= n \stackrel{lim}{\rightarrow} \infty (\frac{m}{2\pi i\tau})^{\frac{n+1}{2}} \int \prod_{j=1}^{n} dq_j e^{i\tau\sum_{j=0}^{n}(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\tau}\right)^2 - V(\overline{q_j})} \end{aligned} \tag{5.2.23}$$

The schematic for the continuum limit is thus,

$$< q_f, t_f|q_i, t_i >= N \int \mathcal{D}q(t) e^{i\int_{t_i}^{t_f} dtL(q,\dot{q})} = N \int \mathcal{D}q(t) e^{iS[q(t)]} \tag{5.2.24}$$

where N is the constant of proportionality in the continuum limit. We have proven our original assertion that the propagator is just equal to the sum over all possible paths of a bunch of phases, where the phase of each path is given by $e^{iS[q(t)]}$. Even more importantly we have given a practical way of calculating the sum over all paths of these phases, as the continuum limit of a lot of time slices, where the particle is allowed to be infinitely far apart in its coordinates between infinitesimally close time slices. In no way are these to be imagined as possible classical paths - in fact they are not paths at all since over an infinitesimal time the particle is allowed to travel from one end of the Universe to another which makes no physical sense for a path. Perhaps we should not call this the path integral approach since the word path is a misnomer. A better name would be the Markov chain approach or even the Star Trek transporter approach, but I shall stick with tradition and use the conventional name!

5.3 Imaginary Time

In the previous section we considered the point to point transition amplitude [1] and showed how it could be expressed as a path integral. In this section we shall *analytically continue* the time coordinate t from the real axis to the complex plane:

$$t \rightarrow z$$

In other words we shall take all the results obtained in the previous section and make the above analytic continuation. In particular we shallbe interested in analytic continuation of time from the real axis to the imaginary axis:

$$t \rightarrow -i\tau$$

[1] i.e. the amplitude that given the particle is at a point q_i at time t_i that it will be found at the point q_f at time t_f where $t_f > t_i$.

where τ is a real number. In other words we set $t = -i\tau$ in all the previous results. This implies that

$$x^2 - c^2t^2 = x^2 + c^2\tau^2 \tag{5.3.1}$$

which is familiar in relativity as Euclidean space.

The whole of quantum mechanics can be re-formulated in imaginary time just by replacing t by $-i\tau$ everywhere. The Schrodinger equation becomes:

$$i\frac{\partial}{\partial t}|\psi(t)>= H|\psi(t)> \longrightarrow -\frac{\partial}{\partial \tau}|\psi(\tau)>= H|\psi(\tau)>$$

However the energy eigenfunctions and eigenvalues do not change since H is time independent:

$$H|E_n>= E_n|E_n> \longrightarrow H|E_n>= E_n|E_n>$$

We shall exploit this fact. The time translation operator becomes:

$$U(t) = e^{-iHt} \longrightarrow U(\tau) = e^{-H\tau}$$

The point to point transition amplitude becomes

$$< q_f|U(t_f, t_i)|q_i > \longrightarrow < q_f|U(\tau_f, \tau_i)|q_i >$$

Note that the position eigenstates $|q>$ do not change. Its path integral evaluation

$$< q_f|U(t_f, t_i)|q_i >= N \int \mathcal{D}q(t) e^{i\int_{t_i}^{t_f} dtL}$$

becomes

$$< q_f|U(\tau_f, \tau_i)|q_i >= N \int \mathcal{D}q(\tau) e^{-\int_{\tau_i}^{\tau_f} d\tau L_E}$$

where the measure

$$N \int \mathcal{D}q(t) = n \stackrel{lim}{\rightarrow} \infty \left(\frac{m}{2\pi i\delta t}\right)^{\frac{n+1}{2}} \int \prod_{j=1}^{n} dq_j$$

becomes the measure

$$N \int \mathcal{D}q(\tau) = n \stackrel{lim}{\rightarrow} \infty \left(\frac{m}{2\pi \delta \tau}\right)^{\frac{n+1}{2}} \int \prod_{j=1}^{n} dq_j$$

Thus the derivation proceeds exactly as before, dividing up τ into small slices, $\delta\tau$ etc. The Euclidean Lagrangian is defined as

$$L_E \equiv \frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)$$

which is the negative of the Lagrangian in Euclidean space, causing the potential to appear in an upside down way.

A useful feature of the Euclidean formulation is that for large Euclidean times any initial state evolves into the ground state, providing it is not orthogonal to it. In particular if the particle is in a position eigenstate $|q_i>$ at time τ_i then at a much later time $\tau_f >> \tau_i$ it will have evolved totally into the ground state $|E_0>$. To prove this we write $\tau = \tau_f - \tau_i$ and consider

$$\tau \stackrel{lim}{\rightarrow} \infty < q_f|U(\tau_f, \tau_i)|q_i >$$

$$= \tau \stackrel{lim}{\rightarrow} \infty \sum_n < q_f|E_n >< E_n|q_i > e^{-E_n\tau}$$

$$\approx < q_f|E_0 >< E_0|q_i > e^{-E_0\tau} = \psi_0(q_f)\psi_0^*(q_i)e^{-E_0\tau}$$

and the result follows.

If we set $q_i = q_f$ then the above result gives us the ground state wavefunction (or rather its modulus squared) in terms of a limit of the propagator and the energy E_0.

$$\psi_0(q)\psi_0^*(q) = \tau \stackrel{lim}{\rightarrow} \infty \frac{< q|U(\tau_f, \tau_i)|q >}{e^{-E_0\tau}}$$

The normalisation condition

$$\int_{-\infty}^{\infty} \psi_0(q)\psi_0^*(q)dq = 1$$

allows us to eliminate the exponential term since it implies,

$$e^{-E_0\tau} \approx \int_{-\infty}^{\infty} dq < q|U(\tau_f, \tau_i)|q >$$

hence we can write

$$\psi_0(q)\psi_0^*(q) = \tau \stackrel{lim}{\rightarrow} \infty \frac{< q|U(\tau_f, \tau_i)|q >}{\int_{-\infty}^{\infty} dq < q|U(\tau_f, \tau_i)|q >}$$

It is worth pointing out that it is not necessary to analytically continue to imaginary time to obtain useful results. Suppose that we analytically continue,

$$t \rightarrow e^{-i\delta}\tilde{t}$$

where $\tilde{t}$ is a real variable and δ is a real constant. For example if we set $\delta = \pi/2$ then we can identify $\tilde{t}$ as τ considered above. Clearly if $\delta = 0$ then $\tilde{t}$ is identical to t, so we see that by adjusting δ we can analytically continue results from real time to imaginary time and back again. In fact a give choice of delta corresponds to a rotation of the real time axis through an angle δ in the complex plane. Of course for general δ the Schrodinger equation looks a little odd, but the important point is that the energy eigenvalues and eigenvectors are fixed since the Hamiltonian has not explicit time dependence and so the energy eigenvalue problem is unchanged for any choice of δ. The important

point is that a non-zero choice of δ is sufficient to allow the ground state to be isolated since in real time we have

$$t \overset{lim}{\to} \infty < q_f|U(t_f,t_i)|q_i >$$

$$= t \overset{lim}{\to} \infty \sum_n < q_f|E_n >< E_n|q_i > e^{-iE_nt}$$

and we see that after analytic continuation

$$e^{-iE_nt} \to e^{-iE_n\tilde{t}\cos\delta}e^{-E_n\tilde{t}\sin\delta}$$

so that the ground state is isolated provided $0 < \delta < \pi$ for real positive $\tilde{t}$. For such δ we have,

$$\tilde{t} \overset{lim}{\to} \infty < q_f|U(\tilde{t_f},\tilde{t_i})|q_i >$$

$$\approx < q_f|E_0 >< E_0|q_i > e^{-iE_0\tilde{t}\cos\delta}e^{-E_0\tilde{t}\sin\delta}$$

and again the higher energy states are exponentially damped for large times, leaving only the ground state in the sum.

Of course if we choose $\delta = 0$ then the approximation is no longer valid, but we may choose a small positive value of δ,

$$\delta = \epsilon, \quad 0 < \epsilon \ll 1$$

providing we consider sufficiently large values of $\tilde{t}$. In this case the orientation of the complex time axis is very close to the real axis and we shall be rather sloppy and drop the tilde to write,

$$t \overset{lim}{\to} \infty < q_f|U(t_f,t_i)|q_i >$$

$$= t \overset{lim}{\to} \infty < q_f|E_0 >< E_0|q_i > e^{-iE_0t}e^{-E_0t\epsilon}$$

Although we are being sloppy we must always remember that the lhs is not strictly the physical point to point transition amplitude, but an analytic continuation of it slightly off the real axis. For example in the case of the harmonic oscillator propagator considered above we must analytically continue this result according to

$$t \to (1 - i\epsilon)t \tag{5.3.2}$$

then take $\epsilon \to 0$ to avoid any ambiguities.

5.4 Transition Amplitudes With an External Driving Force

Now let us return to real time (at least for the moment!) and consider adding an external time dependent driving force to the particle so that the Lagrangian becomes

$$L \to L + J(t)q(t) \tag{5.4.1}$$

where $J(t)$ is the driving force. For example if the particle is in a harmonic oscillator potential then the new term coresponds to forced harmonic motion, where $J(t)$ may be a sinusoidally time dependent driving term for example.

Why are we considering such a driving term? The answer has to do with quantum field theory where the field is equivalent to a continnum of coupled harmonic oscillators, as we have seen. Consider the field to be an electromagnetic field for example, whose quanta are photons. Photons may be created or destroyed by atoms which in some sense act as a souce or sink of photons. From the point of view of the e.m. field the electron in the atom acts like an external driving force for the oscillators of the e.m. field which may excite or de-excite the e.m. field oscillators, corresponding to photon emmission or absorption. In fact the electron is itself described by a field, and it is the interaction of the electron field with the e.m. field which is responsible, but we may choose to consider the e.m. field only, and describe the effects of the interaction by an external term driving the photon field oscillators. The idea of an external driving term is therefore a useful way of describing the creation or destruction of field quanta, where the driving term acts locally on all the field oscillators. In this field context the oscillator driving term becomes a spacetime dependent driving term and we have in the case of the KG field

$$J(t)q(t) \rightarrow J(x)\phi(x) \tag{5.4.2}$$

where x is the spacetime point. In this case $J(x)$ is referred to as a field *source*.

Having motivated the consideration of a driving term, let us now suppose it is only operative between the two times t and t' where $t < t'$. For example at time t an electron in an atom may begin to drive the e.m. field and cause a photon to be created. The photon may later be absorbed by another atom such that its aborption is complete at time t'. The process of emission and absorption therefore takes place over a time interval between t and t', and this is the interval over which we may consider the external driving force to be operative. Suppose that T is an earlier time than t and T' is a later time than t', and let us consider the point to point transition amplitude of a single particle.

We shall now see that it is possible to relate such a point to point transition amplitude to the *ground state to ground state* transition amplitude in the presence of the external driving force. This is the analogue of finding the ground state wavefunction in the absence of the driving force as considered in the previous section. As in the previous section we shall need to analytically continue to complex time in order to achieve this.

The amplitude that a particle is at the point Q' at time T' given that it was at Q at time T, and taking account of the driving term J between t and t' where $T < t < t' < T'$ is then (in real time),

$$< Q', T'|Q, T >^J = N \int \mathcal{D}q(t) e^{i \int_T^{T'} dt L(q,\dot{q}) + Jq} \tag{5.4.3}$$

where we have put a superscript J on the lhs to remind us about the driving term dependence of the matrix element. We now insert a complete set of Heisenberg

position states at times t and t' at which the driving term switches on and off,

$$< Q', T'|Q, T >^J = \int dqdq' < Q', T'|q', t' >< q', t'|q, t >^J < q, t|Q, T > \quad (5.4.4)$$

where we have a superscript J only on the middle matrix element. Now the two matrix elements which do not depend on J can each be treated as in the previous section:

$$< Q'|U(T', t')|q' > = \sum_n < Q'|E_n >< E_n|q' > e^{-iE_n(T'-t')}$$

$$< q|U(t, T)|Q > = \sum_n < q|E_n >< E_n|Q > e^{-iE_n(t-T)}$$

In each case the ground state is isolated by analytically continuing to complex time and taking the time differences to be large. An ϵ rotation of the time axis as in Eq.5.3.2 yields the results

$$< Q'|U(T', t')|q' > \approx < Q'|E_0 >< E_0|q' > e^{-iE_0(T'-t')}e^{-E_0(T'-t')\epsilon}$$

$$< q|U(t, T)|Q > \approx < q|E_0 >< E_0|Q > e^{-iE_0(t-T)}e^{-E_0(t-T)\epsilon}$$

valid for very large (infinite) time differences and as before we use a slightly sloppy notation for the lhs's which must strictly be evaluated slightly off the real axis if any ambiguity arises. Inserting these results Eq.5.4.4 gives,

$$< Q', T'|Q, T >^J \approx \int dqdq' < Q'|E_0 >< E_0|q' > e^{-iE_0(T'-t')}e^{-E_0(T'-t')\epsilon}$$

$$\times < q', t'|q, t >^J < q|E_0 >< E_0|Q > e^{-iE_0(t-T)}e^{-E_0(t-T)\epsilon}$$

Thus we find

$$< Q', T'|Q, T >^J \approx < Q'|E_0 >< E_0|Q > e^{-iE_0(T'-T)}e^{-E_0(T'-T)\epsilon}$$

$$\times \int dqdq' < E_0|q', t' > \times < q', t'|q, t >^J < q, t|E_0 >$$

Or, rearranging,

$$\int dqdq' < E_0|q', t' >< q', t'|q, t >^J < q, t|E_0 >$$

$$\approx \frac{< Q', T'|Q, T >^J}{< Q'|E_0 >< E_0|Q > e^{-iE_0(T'-T)}e^{-E_0(T'-T)\epsilon}}$$

The LHS is just the amplitude that a particle in the ground state wavefunction at time t will be found in the ground state wavefunction at time t'. [2] The result is only true in the limit

$$T' \to \infty, \quad T \to -\infty$$

[2] Actually it is not strictly this because we must always remember that our times have been analytically continued off the real axis. But this is only by a small amount ϵ, so unless ambiguities arise we can think of the LHS as indicated.

Taking the subsequent limits

$$t' \to \infty, \quad t \to -\infty$$

allows J to act over all times. We write the *ground state to ground state* transition amplitude in this limit as

$$< 0, \infty | 0, -\infty >^J = \int dq dq' < E_0 | q', \infty >< q', \infty | q, -\infty >^J < q, -\infty | E_0 >$$

This amplitude would of course be unity in the absence of the external driving force J, by energy conservation. We write the *point to point* transition amplitude in the above limit in the presence of the driving force as

$$< Q', \infty | Q, -\infty >^J \tag{5.4.5}$$

The evaluation of the above matrix element requires some care. The staring point for its evaluation is Eq.5.4.3 for $< Q', T' | Q, T >^J$ for real finite times. We must then analytically continue off the real axis, and take the limit of large times to obtain the desired amplitude. With these provisos we write,

$$< Q', \infty | Q, -\infty >^J = N \int \mathcal{D}q(t) e^{i \int_{-\infty}^{\infty} dt L(q,\dot{q}) + Jq} \tag{5.4.6}$$

where in the above expression we should write $t \to (1 - i\epsilon)t$. Providing we stay off the real time axis we see a nice relationship between the point to point amplitude and the ground state to ground state amplitude $< 0, \infty | 0, -\infty >^J$,

$$< 0, \infty | 0, -\infty >^J = \frac{< Q', \infty | Q, -\infty >^J}{< Q' | E_0 >< E_0 | Q > e^{-iE_0(T'-T)} e^{-E_0(T'-T)\epsilon}} \tag{5.4.7}$$

The denominator is just the product of two wavefunctions; it is a constant independent of J. It is conventional to define the RHS of the above expressiion as the functional $Z[J]$,

$$Z[J] \equiv \frac{< Q', \infty | Q, -\infty >^J}{< Q' | E_0 >< E_0 | Q > e^{-iE_0(T'-T)} e^{-E_0(T'-T)\epsilon}} \tag{5.4.8}$$

Then we have simply

$$Z[J] =< 0, \infty | 0, -\infty >^J \tag{5.4.9}$$

where

$$Z[J] = \frac{N \int \mathcal{D}q(t) e^{i \int_{-\infty}^{\infty} dt L(q,\dot{q}) + Jq}}{< Q' | E_0 >< E_0 | Q > e^{-iE_0(T'-T)} e^{-E_0(T'-T)\epsilon}} \tag{5.4.10}$$

Up to a constant, $Z[J]$ is just the point to point transition amplitude, in the presence of J in the limit of large time differences, and evaluated off the real time axis. In field theory $Z[J]$ is known as the *generating functional*.

5.5 Expectation Values of Heisenberg Position Operators

Let us again return to real time but instead of considering the amplitude $< q_f, t_f | q_i, t_i >$ between the Heisenberg position states $< q_f, t_f |$ and $| q_i, t_i >$ we consider the matrix element

$$< q_f, t_f | q(t_{n_1}) | q_i, t_i > \tag{5.5.1}$$

where $q(t_{n_1})$ is a Heisenberg position operator and $t_f > t_{n_1} > t_i$.

Let us now proceed as in Eq.5.2.9 and choose t_{n_1} to be one of the times $t_1 \cdots t_n$. Then the matrix element in Eq.5.5.1 becomes

$$< q_f, t_f | q(t_{n_1}) | q_i, t_i >= \int \ldots \int dq_1 \ldots dq_n < q_f, t_f | q_n, t_n >< q_n, t_n | q_{n-1}, t_{n-1} >$$

$$\ldots < q_{n_1}, t_{n_1} | q(t_{n_1}) | q_{n_1-1}, t_{n_1-1} > \ldots | q_1, t_1 >< q_1, t_1 | q_i, t_i >$$

We can now write

$$< q_{n_1}, t_{n_1} | q(t_{n_1}) | q_{n_1-1}, t_{n_1-1} >= q(t_{n_1}) < q_{n_1}, t_{n_1} | q_{n_1-1}, t_{n_1-1} >$$

where $q(t_{n_1})$ is henceforth a number not an operator. The rest of the argument follows that below Eq.5.2.9, and we obtain

$$< q_f, t_f | q(t_1) | q_i, t_i >= \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} q(t_1) e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q} - H(p,q))\right)} \tag{5.5.2}$$

where as before the RHS involves no operators.

Next consider the matrix element

$$< q_f, t_f | q(t_{n_1}) q(t_{n_2}) | q_i, t_i > \tag{5.5.3}$$

where $q(t_{n_1}), q(t_{n_2})$ are Heisenberg position operators and $t_f > t_{n_1} > t_{n_2} > t_i$.

$$< q_f, t_f | q(t_{n_1}) q(t_{n_2}) | q_i, t_i >= \int \ldots \int dq_1 \ldots dq_n < q_f, t_f | q_n, t_n >$$

$$\ldots < q_{n_1}, t_{n_1} | q(t_{n_1}) | q_{n_1-1}, t_{n_1} - 1 > \ldots < q_{n_2}, t_{n_2} | q(t_{n_2}) | q_{n_2-1}, t_{n_2} - 1 > \ldots < q_1, t_1 | q_i, t_i >$$

We obtain

$$< q_f, t_f | q(t_1) q(t_2) | q_i, t_i >= \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} q(t_1) q(t_2) e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q} - H(p,q))\right)} \tag{5.5.4}$$

valid for $t_1 > t_2$. If however $t_2 > t_1$ then the RHS is equal to

$$< q_f, t_f | q(t_2) q(t_1) | q_i, t_i > .$$

In general then the RHS is equal to the time ordered product of the operators,

$$< q_f, t_f | T(q(t_1) q(t_2)) | q_i, t_i >= \int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} q(t_1) q(t_2) e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q} - H(p,q))\right)} \tag{5.5.5}$$

where time ordering was defined earlier (earlier times are to the right).

The result generalises to,

$$< q_f, t_f|T(q(t_1)q(t_2)\cdots q(t_n))|q_i, t_i >= N\int \mathcal{D}q(t)q(t_1)q(t_2)\cdots q(t_n)e^{i\int_{t_i}^{t_f} dtL} \tag{5.5.6}$$

Now we recall from Eq.5.4.10 that

$$Z[J] \propto N\int \mathcal{D}q(t)e^{i\int_{-\infty}^{\infty} dtL(q,\dot{q})+Jq} \tag{5.5.7}$$

where the RHS is evaluated off the real time axis. The *functional derivative* (defined in the appendix) of $Z[J]$ with respect to J is

$$\frac{\delta Z[J]}{\delta J(t_1)} \propto iN\int \mathcal{D}q(t)q(t_1)e^{i\int_{-\infty}^{\infty} dtL(q,\dot{q})+Jq} \tag{5.5.8}$$

and generally

$$\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)} \propto i^n N\int \mathcal{D}q(t)q(t_1)\cdots q(t_n)e^{i\int_{-\infty}^{\infty} dtL(q,\dot{q})+Jq} \tag{5.5.9}$$

If we set $J = 0$ we obtain,

$$\left[\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right]_{J=0} \propto i^n N\int \mathcal{D}q(t)q(t_1)\cdots q(t_n)e^{i\int_{-\infty}^{\infty} dtL(q,\dot{q})} \tag{5.5.10}$$

If we compare the RHS of Eq.5.5.10 to Eq.5.5.6 we see that there is a strong similarity. If we analytically continue time in Eq.5.5.6 in the same way as in the definition of $Z[J]$ and insert complete sets of energy eigenstates, then we see that as usual the ground state is isolated,

$$< q_f, t_f|T(q(t_1)q(t_2)\cdots q(t_n))|q_i, t_i >$$

$$= \sum_{n,m} < q_f|E_n >< E_n|e^{-iE_n t_f}T(q(t_1)q(t_2)\cdots q(t_n))|E_m >< E_m|q_i > e^{iE_n t_i}$$

$$\approx < q_f|E_0 > e^{-iE_0 t_f} < E_0|q_i > e^{iE_0 t_i} < E_0|T(q(t_1)q(t_2)\cdots q(t_n))|E_0 >$$

The wavefunction factor which arises is the same constant as in Eq.5.4.10 and they both cancel to leave,

$$i^n < 0|T(q(t_1)q(t_2)\cdots q(t_n))|0 > = \left[\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right]_{J=0} \tag{5.5.11}$$

where we have written $|E_0 >= |0 >$. Eq.5.5.11 relates the ground state expectation value of time ordered products of Heisenberg coordinate operators to some quantity on the RHS which can be calculated. We saw in Eq.3.6.12 that in canonical quantum field theory the S-matrix boils down to calculating the vacuum expectation value of a time ordered product of field operators in the Heisenberg picture, so we can see that an expression such as in Eq.5.5.11 when generalised to field theory could be useful.

5.6 Appendix

5.6.1 Gaussian Integration

Integral $\alpha > 0$:

$$I_0(\alpha) = \int_{-\infty}^{\infty} dx e^{-\alpha x^2} \tag{5.6.1}$$

This is called a Gaussian integral.

Result:

$$I_0(\alpha) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} \tag{5.6.2}$$

Proof:

$$I_0^2 = \int_{-\infty}^{\infty} dx e^{\alpha x^2} \int_{-\infty}^{\infty} dy e^{\alpha y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dy e^{-\alpha(x^2+y^2)}$$

In polar coordinates r, θ we can perform the integral.

Integral ($\alpha > 0$):

$$I_0(\alpha, \beta) = \int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} \tag{5.6.3}$$

Result:

$$I_0(\alpha, \beta) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{\frac{\beta^2}{4\alpha}} \tag{5.6.4}$$

Proof:

We complete the square

$$I_0(\alpha, \beta) = \int_{-\infty}^{\infty} dx e^{-\alpha(x - \frac{\beta}{2\alpha})^2 + \frac{\beta^2}{4\alpha^2}}$$

Change variables to $x - \frac{\beta}{2\alpha}$ and we use the previous result.

Integral:

$$I_2(\alpha) = \int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2} \tag{5.6.5}$$

Result:

$$I_2(\alpha) = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} \tag{5.6.6}$$

Proof:

Differentiate $I_0(\alpha)$ with respect to α. Or differentiate $I_0(\alpha, \beta)$ with respect to β then set $\beta = 0$. The second method shows that $I_1(\alpha) = 0$.

In general we can find $I_{2n}(\alpha)$ this way for any n. The odd powers of x in the integrand $I_{2n+1}(\alpha)$ lead to a zero result (odd integrand).

Note that all these results are also valid for α and β being complex numbers providing $Re(\alpha) > 0$. Next consider the result:

$$\int_0^\infty dr e^{-\alpha r} = \frac{1}{\alpha} \tag{5.6.7}$$

By applying the operator

$$\left(\frac{-d}{d\alpha}\right)^n$$

we obtain

$$\int_0^\infty dr r^n e^{-\alpha r} = \frac{n!}{\alpha^{n+1}} \tag{5.6.8}$$

Setting $\alpha = 1$ and replacing n by $z - 1$ where z is a complex number defines the gamma function,

$$\Gamma(z) = \int_0^\infty dr r^{z-1} e^{-r} \tag{5.6.9}$$

For real positive integer z we have for example $\Gamma(z) = (z-1)!$.

Last but no no means least we come to what some people have described as the most important integral in the world! You will soon see why.

Integral:

$$Z_0 = \int_{-\infty}^\infty \prod_{i=1}^n dq_i e^{-\sum_{i,j=1}^n q_i K_{ij} q_j + \sum_{k=1}^n J_k q_k} \tag{5.6.10}$$

This is an n dimensional integral and K_{ij} is an $n \times n$ symmetric matrix, while J_k is an n component column vector. Amazingly the above integral can be done for all invertible K_{ij}!

Result:

$$Z_0 = \frac{\pi^{n/2}}{\sqrt{det(K)}} e^{+\frac{1}{4}\sum_{i,j=1}^n J_i (K^{-1})_{ij} J_j} \tag{5.6.11}$$

Proof:

The strategy is to reduce Z_0 to a product of n do-able integrals of the type $I_0(\alpha, \beta)$. To do this we diagonalise K as follows. In matrix notation:

$$q^T K q = q^T U^T U K U^T U q = q'^T K' q'$$

where $U^T = U^{-1}$ is an orthogonal matrix (the complex analogue being a unitary matrix) and T of course means transpose, and

$$q' = Uq, \quad K' = UKU^T = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

The fact that K is symmetric (Hermitian) means that we can diagonalise it by a single orthogonal (Unitary) matrix U. Now change integration variables to q_i' with a trivial Jacobian of $detU = 1$,

$$Z_0 = \int \prod_{i=1}^n dq_i' e^{-\sum_{j=1}^n q_i' \lambda_j q_j' + \sum_{k,j=1}^n (J_k U_{jk}) q_j'} \tag{5.6.12}$$

The integral is now reduced to n products of $I_0(\alpha, \beta)$ and the result involves in the denominator $\prod_i \sqrt{\lambda_i}$. Now clearly

$$det(K) = det(K') = \prod_i \lambda_i$$

so we get a factor of $\sqrt{det(K)}$ in the denominator. In addition we have a product of $e^{\frac{\beta_i^2}{4\alpha_i}}$ type factors where

$$\beta_i = (JU^T)_i$$

and

$$\alpha_i = \lambda_i = (K')_{ii}$$

Thus the exponent consists of the sum

$$\sum_i \frac{(JU^T)_i (JU^T)_i}{4\lambda_i}$$

It is straightforward to see that this sum is equal to

$$\frac{1}{4} J^T K^{-1} J = \frac{1}{4} J^T U^T K'^{-1} U J$$

so the result is proved.

5.6.2 Functionals

In mathematics a *function* maps a number into a different number, i.e. $f(x)$ maps a number x into another number $y = f(x)$.

A *function of a function* again maps a number into a different number, i.e. $F(f(x))$ maps a number x into another number $z = F(y)$ where $y = f(x)$.

A *functional* on the other hand maps a function into a number, i.e. $F[f(x)]$ maps the entire function $f(x)$ into a number F. e.g.

$$F[f(x)] = \int_0^1 dx f(x)$$

is a functional; for each choice of function $f(x)$ the result will be a different number.

The action S is clearly a functional since it is defined as

$$S[x(t)] = \int_{t_i}^{t_f} dt L(x(t))$$

i.e. for each path $x(t)$ there is a number S.

Quantities like the path integral

$$< q_f, t_f | q_i, t_i >= N \int \mathcal{D}q(t) e^{iS[q(t)]} \tag{5.6.13}$$

are *functional integrals*: the integrand is a functional and the integration is performed over all functions $q(t)$ to yield a numerical result.

The other aspect of functional calculus is *functional differentiation*. This is the operation of taking the derivative of a functional $F[f(x)]$ with respect to the function $f(y)$, and is defined by

$$\frac{\delta F[f(x)]}{\delta f(y)} \equiv \epsilon \overset{lim}{\rightarrow} 0 \; \frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon} \tag{5.6.14}$$

5.7 Problems Set 5

1. Using Gaussian integration show explicitly that Eq.5.2.23 follows from Eq.5.2.20.

2. By performing the Gaussian integrals in the discretised case, derive the transition amplitude for a free non-relativistic particle of mass m:

$$< q_f, t_f | q_i, t_i >= \theta(t_f - t_i) \left(\frac{m}{i(t_f - t_i)} \right)^{\frac{1}{2}} exp \left[\frac{im(x_f - x_i)^2}{2(t_f - t_i)} \right]$$

3. Show that for the harmonic oscillator we have in real time

$$< q_f | U(t_f, t_i) | q_i >= A(t) exp \left(-\frac{im\omega}{2 \sin \omega t} [(q_f^2 + q_i^2) \cos \omega t - 2 q_f q_i)] \right)$$

where $A(t)$ is some function of $t = t_f - t_i$.

4. Analytically continue the result in Q.3 to imaginary time and show

$$< q_f | U(\tau_f, \tau_i) | q_i >= A(\tau) exp \left(-\frac{m\omega}{2 \sinh \omega \tau} [(q_f^2 + q_i^2) \cosh \omega \tau - 2 q_f q_i)] \right)$$

As $\tau \to \infty$ show that the rhs becomes proportional to a product of ground state wavefunctions.

[This example demonstrates that analytic continuation is a useful device or trick to obtain physically meaningful results, even though the lhs $< q_f | U(\tau_f, \tau_i) | q_i >$ does not have a direct physical interpretation as the point to point transition amplitude, at least not in our universe where time is not imaginary.]

5. Given following functionals $F[f]$ find their first and second functional derivatives with respect to $f(y)$ and $f(z)$:

$$\frac{\delta F[f]}{\delta f(y)}, \quad \frac{\delta^2 F[f]}{\delta f(y) \delta f(z)}$$

(a)

$$F[f] = \int dx f(x)$$

(b)

$$F[f] = \int dx f(x) g(x)$$

(c)

$$F[f] = \int dx e^{f(x)}$$

(d)

$$F[f] = \int dx e^{f(x)+g(x)}$$

(e)

$$F[f] = \int\int dx_1 dx_2 g(x_1, x_2) f(x_1) f(x_2)$$

(f)

$$F[f] = \int\int\int dx_1 dx_2 dx_3 g(x_1, x_2) f(x_1) f(x_2) f(x_3)$$

6. Consider the following functional defined by the series:

$$F[f] = 1 + \int dx_1 f(x_1) g_1(x_1) + \frac{1}{2!}\int\int dx_1 dx_2 g_2(x_1, x_2) f(x_1) f(x_2)$$

$$+\frac{1}{3!}\int\int\int dx_1 dx_2 dx_3 g_3(x_1, x_2, x_3) f(x_1) f(x_2) f(x_3) + \cdots$$

Verify that to order $n = 3$,

$$g_n(x_1, \cdots, x_n) = \left[\frac{\delta^n F[f]}{\delta f(x_1) \cdots \delta f(x_n)}\right]_{f=0}.$$

[This result shows how a generating functional can be Taylor expanded as a power series.]

Chapter 6

Path Integral Quantisation of the Klein-Gordon Field

6.1 Introduction

In the previous chapter we were concerned with the path integral formulation of non-relativistic quantum mechanics of a single particle. We summarise the three key results proved there:

The ground state to ground state transition amplitude:

$$Z[J] =< 0, \infty|0, -\infty >^J \tag{6.1.1}$$

The path integral expression for $Z[J]$:

$$Z[J] \propto N \int \mathcal{D}q(t) e^{i\int_{-\infty}^{\infty} dt L(q,\dot{q})+Jq} \tag{6.1.2}$$

The time ordered vacuum expectation value of Heisenberg operators:

$$i^n < 0|T(\hat{q}(t_1)\hat{q}(t_2)\cdots\hat{q}(t_n))|0 > = \left[\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right]_{J=0} \tag{6.1.3}$$

All these results are only valid if we regard time t as having been analytically continued off the real axis by a small ϵ rotation. This proceedure allowed the ground state to be isolated, and having used this property we cannot ever return to the real time axis. Thus in the evaluation of $Z[J]$ we should replace t by $t(1 - i\epsilon)$ before doing the path integrations. The time ordered product of Heisenberg operators has also been analyticically continued.

In chapter 2 we discussed the free Klein-Gordon field and showed how it could be regarded as a continuum of harmonic oscillators, one at each spacetime point, and each coupled to their nearest neighbours in a Lorentz invariant way. The path integral results in chapter 4 may be taken to apply to each of these little oscillators. The first step in going to field theory is clearly to generalise

the path integral results from a single coordinate $q(t)$ to many coordinates $q_i(t)$, one for each oscillator. We then need to take the continuum limit,

$$q_i(t) \rightarrow \phi(x,y,z,t) \tag{6.1.4}$$

just as we did to arrive at canonical field theory. Thus the Heisenberg operators $\hat{q}(t)$ in Eq.6.1.3 become Heisenberg fields $\hat{\phi}(x,y,z,t)$ whose eigenstates are given by

$$\hat{\phi}(x,y,z,t)|\phi(x,y,z,t)>=\phi(x,y,z,t)|\phi(x,y,z,t)> \tag{6.1.5}$$

Recall the point to point amplitude

$$<q_f,t_f|q_i,t_i>=\int \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} e^{i\left(\int_{t_i}^{t_f} dt(p\dot{q}-H(p,q))\right)} \tag{6.1.6}$$

In field theory this applies separately to each of the little field oscillators

$$<\phi_f(x,y,z,t_f)|\phi_i(x,y,z,t_i)>=\int \frac{\mathcal{D}\phi\mathcal{D}\pi}{2\pi} e^{i\left(\int_{t_i}^{t_f} dt\int d^3\mathbf{x}(\pi\dot{\phi}-\mathcal{H}(\pi,\phi))\right)} \tag{6.1.7}$$

The RHS involves no operators, and is a functional integral over all classical field functions $\phi(x,y,z,t)$ and $\pi(x,y,z,t)$ satisfying some field boundary conditions at t_i and t_f.

In field theory the generalisation of the driving force $J(t)$ is to add a driving force to each of the individual field oscillators, so that

$$J(t) \rightarrow J(x,y,z,t) \tag{6.1.8}$$

where $J(x,y,z,t)$ is called a *source* rather than driving term; the effect is the same however since it is an external "hand" which pushes each of the little oscillators comprising the field. Thus the field transition amplitude in the presence of a source is,

$$<\phi_f(x,y,z,t_f)|\phi_i(x,y,z,t_i)>^J=\int \frac{\mathcal{D}\phi\mathcal{D}\pi}{2\pi} e^{i\left(\int_{t_i}^{t_f} dt\int d^3\mathbf{x}(\pi\dot{\phi}-\mathcal{H}(\pi,\phi)+J\phi)\right)} \tag{6.1.9}$$

The field theory generalisation of the transition amplitude from the ground state at time $-\infty$ to the ground state at time $+\infty$ in the presence of driving term, is then the corresponding *vacuum to vacuum* transition amplitude in the presence of a *source*. The *vacuum* is simply the ground state of the field. The three key results reproduced at the start of this section then become generalised to field theory an the obvious way.

The vacuum to vacuum transition amplitude in field theory is defined to be the *generating functional*:

$$Z[J] \equiv <0,\infty|0,-\infty>^J \tag{6.1.10}$$

The path integral expression for the generating functional $Z[J]$:

$$Z[J] \propto \int \mathcal{D}\phi e^{i\left(\int_{-\infty}^{\infty} dt \int d^3\mathbf{x}(\mathcal{L}+J\phi)\right)} \tag{6.1.11}$$

The time ordered vacuum expectation value of Heisenberg field operators:

$$i^n < 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n))|0> = \left[\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J(x)=0} \tag{6.1.12}$$

where $|0>$ is the time independent vacuum, and we have have allowed the Heisenberg field operators to correspond to the coordinates of different field oscillators at different times, i.e. $x_i = (t_i, \mathbf{x}_i)$. We emphasise again that the time ordered product of Heisenberg operators is not equal to the time ordered product of operators considered previously in the interaction picture. However there will be a relationship between the quantity in Eq.6.1.12 and S-matrix elements as we shall see.

The vacuum expectation value of a time ordered product of n Heisenberg field operators is of course the *n particle Green's function* defined in Eq.3.6.2,

$$\mathcal{G}(x_1, x_2, \cdots, x_n) \equiv < 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n))|0> \tag{6.1.13}$$

Thus

$$i^n\mathcal{G}(x_1, x_2, \cdots, x_n) = \left[\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J(x)=0} \tag{6.1.14}$$

According to the LSZ reduction formula in Eq.3.6.12 the n particle Green's function is related to the S-matrix element involving a total of n particles in initial plus final state. However we shall first consider free field theory and later consider the interacting case.

6.2 The Feynman Propagator (again)

Consider the free field Lagrangian density of the real KG field, from Eq.2.3.2

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 \tag{6.2.1}$$

The path integral expression for the generating functional is:

$$Z_0[J] \propto \int \mathcal{D}\phi \exp\left(i\int d^4x \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 + J\phi\right]\right) \tag{6.2.2}$$

We use the identity

$$\int (\partial_\mu\phi)(\partial^\mu\phi)d^4x = \int \partial_\mu(\phi\partial^\mu\phi)d^4x - \int \phi\partial^2\phi d^4x$$

and convert the first term on the rhs to a surface integral using the four dimensional version of Gauss's divergence theorem. The surface term vanishes if $\phi \to 0$ at infinity, leaving

$$\int (\partial_\mu \phi)(\partial^\mu \phi) d^4x = -\int \phi \partial^2 \phi d^4x$$

and hence

$$Z_0[J] \propto \int \mathcal{D}\phi \exp\left(-i \int d^4x \left[\frac{1}{2}\phi(\partial^2 + m^2)\phi - J\phi\right]\right) \tag{6.2.3}$$

Now we wish to evaluate the functional integral in Eq.6.2.3 explicitly, remembering that we must do so off the real time axis because it is only there that the results involving $Z[J]$ are valid. To begin with we recall the most useful integral in the world from the Appendix of the last chapter, in Eqs.5.6.10 and 5.6.11,

$$Z_0' = \int_{-\infty}^{\infty} \prod_{i=1}^{n} dq_i e^{-\frac{1}{2}\sum_{i,j=1}^{n} q_i K_{ij} q_j + \sum_{k=1}^{n} J_k q_k}$$

$$Z_0' = \frac{(2\pi)^{n/2}}{\sqrt{det(K)}} e^{+\frac{1}{2}\sum_{i,j=1}^{n} J_i (K^{-1})_{ij} J_j}$$

which is an n dimensional integral and K_{ij} is an $n \times n$ symmetric matrix, while J_k is an n component column vector. The field theory limit of this integral is

$$Z_0'[J] = \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int d^4x d^4y \left[\phi(x) K(x,y) \phi(y)\right] + \int d^4x J(x)\phi(x)\right) \tag{6.2.4}$$

$$Z_0'[J] = \frac{1}{\sqrt{det(K)}} \exp\left(+\frac{1}{2}\int d^4x d^4y \left[J(x) K^{-1}(x,y) J(y)\right]\right) \tag{6.2.5}$$

The reason why the factors of 2π have disappeared in the result is related to the definition of $det(K)$. Since this factor multiplying the exponential is a constant as far as J is concerned, and Eq.6.2.2 is only a proportionality we shall not worry about it. If you wish to worry about it see Bailin and Love p.1.

Now we shall write the exponent of Eq.6.2.3 in a way so that it more closely resembles Eq.6.2.4,

$$\left(-\frac{1}{2}\int d^4x d^4y \left[\phi(x) i\delta^4(x-y)(\partial^2 + m^2)\phi(y)\right] + \int d^4x (iJ(x))\phi(x)\right)$$

We can now use the result in Eq.6.2.5 with the transcriptions:

$$K(x,y) \to i\delta^4(x-y)(\partial^2 + m^2)$$

$$J(x) \to iJ(x)$$

Thus we find,

$$Z_0[J] \propto \exp\left(\frac{i}{2}\int d^4x d^4y J(x) \left[\delta^4(x-y)(\partial^2 + m^2)\right]^{-1} J(y)\right) \tag{6.2.6}$$

If we define as

$$\Delta_F(x,y) \equiv -\left[\delta^4(x-y)(\partial^2+m^2)\right]^{-1} \tag{6.2.7}$$

then the final result is:

$$Z_0[J] \propto \exp\left(\frac{-i}{2}\int d^4x d^4y J(x)\Delta_F(x,y)J(y)\right) \tag{6.2.8}$$

What is this inverse operator? Clearly we have

$$\int d^4y [\Delta_F(x,y)]^{-1}\Delta_F(y,z) = \delta(x-z)$$

Hence from its definition we have

$$\int d^4y \left[\delta^4(x-y)(\partial^2+m^2)\right]\Delta_F(y,z) = -\delta(x-z)$$

Integrating over y, we find the equation which must be satisfied by $\Delta_F(x,z)$,

$$\left[(\partial^2+m^2)\right]\Delta_F(x,z) = -\delta(x-z) \tag{6.2.9}$$

The solution to this equation is formally

$$\Delta_F(x,z) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}e^{-ik.(x-z)}$$

as may be verified by substitution. However this result is ambiguous: there are poles at $k_0^2 = \mathbf{k}^2+m^2$.

In order to resolve this ambiguity we only need recall that in all of the results above time must be analytically continued off the real time axis, corresponding to a rotation of ϵ in the complex time plane. Once we remember this fact then we must replace t by $(1-i\epsilon)t$ everywhere and in particular Eq.6.2.6 and Eq.6.2.7 are strictly only valid if analytically continued off the real time axis. This implies that the defining equation for the inverse operator should have been strictly

$$\left[((1+2i\epsilon)\partial_0^2-\partial_i^2+m^2)\right]\Delta_F(x,z) = -\delta(x-z) \tag{6.2.10}$$

and the solution should have been

$$\Delta_F(x,z) = \int \frac{dk_0 d^3\mathbf{k}}{(2\pi)^4}\frac{1}{k_0^2-\mathbf{k}^2-m^2}e^{-i(1-i\epsilon)k_0.(x_0-z_0)+i\mathbf{k}.(\mathbf{x}-\mathbf{z})}$$

After a change of variables to $k_0' = (1-i\epsilon)k_0$, this looks like, dropping the primes,

$$\Delta_F(x,z) = \int \frac{dk_0 d^3\mathbf{k}}{(2\pi)^4}\frac{1}{(1+2i\epsilon)k_0^2-\mathbf{k}^2-m^2}e^{-ik_0.(x_0-z_0)+i\mathbf{k}.(\mathbf{x}-\mathbf{z})}$$

which shows that the previous poles are now avoided. What has happened is that the rotation of the time axis in the complex time plane has led to a rotation

of the energy axis in the complex energy plane corresponding to the change of variables above. From the point of view of the rotated energy variable the poles are shifted off the energy axis and are now at:

$$k_0 = \pm E(1 - i\epsilon) = \pm(E - i\epsilon)$$

where $E = |\sqrt{\mathbf{k}^2 + m^2}|$ and we have exploited the fact that we are going to take ϵ to zero at the end of the calculation. Thus we could equally well write,

$$\Delta_F(x, z) = \int \frac{dk_0 d^3\mathbf{k}}{(2\pi)^4} \frac{1}{k_0^2 - \mathbf{k}^2 - m^2 + i\epsilon} e^{-ik_0.(x_0 - z_0) + i\mathbf{k}.(\mathbf{x}-\mathbf{z})}$$

or simply

$$\Delta_F(x - z) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik.(x-z)} \tag{6.2.11}$$

where in the last line we have written $\Delta_F(x-z)$ to emphasise that the Feynman propagator only depends on the difference between the two points. The factor $1/(k^2 - m^2 + i\epsilon)$ is just the Fourier transform of $\Delta_F(x, z)$ and is written

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon} \tag{6.2.12}$$

Now we see that $\Delta_F(x, z)$ in Eq.6.2.11 is identical to the definition of the Feynman propagator in Eq.3.5.16. The rotation of the time axis in the complex plane through a small angle ϵ which was always implicit in the definition of $Z[J]$ (otherwise none of the results involving $Z[J]$ would have been true) has led to Eq.6.2.8 in which $Z[J]$ is expressed in terms of the Feynman propagator $\Delta_F(x, z)$. The Feynman propagator $\Delta_F(x, z)$ is thus seen to be the inverse operator shown in Eq.6.2.7 where this operator appears in the original expression for $Z[J]$ in Eq.6.2.3. The strategy to find the propagator is thus to express the exponent of the functional integral in the form: $\phi(x)$.OPERATOR.$\phi(y)$ then take the inverse of the OPERATOR.

The rotation of the real time axis in the complex plane is seen to be equivalent to a rotation of the real energy axis in the complex energy plane, and provides a way of avoiding the poles which is equivalent to adding a small quantity ϵ to the denominator of the Feynman propagator. The effect of this is that after the energy integral is performed two θ functions in time are generated, in such a way that propagation is allowed forwards in time both for $x_0 > y_0$ and $y_0 > x_0$. The practical effect of this is that when we draw our Feynman diagrams in position space we do not have to worry about the time ordering of the propagator since both time ordering possibilities are covered.

6.3 Green's Functions in Free Field Theory

In the previous section we managed to perform the functional integrations in the generating functional for the free field theory case, and ended up with the

result in Eq.6.2.8,

$$Z_0[J] \propto \exp\left(\frac{-i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\right) \tag{6.3.1}$$

where $\Delta_F(x-y)$ was shown to be our old friend the Feynman propagator. If it weren't for the awkward constant of proportionality we would now be in a position to calculate the n particle Green's functions for free field theory from Eq.6.1.14

$$i^n\mathcal{G}(x_1,x_2,\cdots,x_n) = \left[\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J(x)=0} \tag{6.3.2}$$

where we set $Z[J] = Z_0[J]$ for free field theory. Recall the definition of the Green's functions from Eq.6.1.13

$$\mathcal{G}(x_1,x_2,\cdots,x_n) \equiv <0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n))|0> \tag{6.3.3}$$

where $\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots$ are Heisenberg field operators.

In order to find the constant of proportionality above, we recall that our original defintion of $Z[J]$ in Eq.6.1.10 shows that it is normalised to

$$Z_0[J=0] = 1 \tag{6.3.4}$$

since in the absence of a source the vacuum will remain the vacuum with unit amplitude. Since the constant of proportionality is independent of J (otherwise we would not call it a constant) it is clear that it must always arrange itself to be equal to unity in order to satisfy Eq.6.3.4. Thus Eq.6.3.1 become simply

$$Z_0[J] = \exp\left(\frac{-i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\right) \tag{6.3.5}$$

Now that $Z_0[J]$ is known exactly there is now no obstacle to us using it to crank out the Green's functions by functional differentiation. This of course is why we call it the generating functional: it may be used to generate the Green's functions. Thus we find the two, three and four point Green's functions to be:

$$\mathcal{G}_0(x_1,x_2) = i\Delta_F(x_1-x_2) \tag{6.3.6}$$

$$\mathcal{G}_0(x_1,x_2,x_3) = 0 \tag{6.3.7}$$

$$\mathcal{G}_0(x_1,x_2,x_3,x_4) = i^2[\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)$$
$$+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)] \tag{6.3.8}$$

In general the odd number of point Green's functions are zero and the even number of point Green's functions involve pairs of Feynman propagators. The astute will realise that the result for the four point Green's function in Eq.**??** is the same as the result of using Wick's theorem for the time ordered product of four field operators in Eq.3.5.9. This is to be expected since the Green's functions are *defined* as the VEV of a time ordered product of field operators,

and there is no distiction between the interaction picture and the Heisenberg picture for free field theory. Note that we have obtained the *same* Green's functions using the path integral formalism as using the canonical formalism. From the point of view of physics only the Green's functions are really important, and it does not matter whether we have calculated the Green's functions using canonical methods or using functional methods since we have arrived at the same results.

We have already introduced Feynman diagrams as a way of representing the Feynman propagator:

$$i\Delta_F(x_1 - x_2) = \quad ----------$$

where the line is drawn between the two spactime points x_1 and x_2. This notation also serves for the two point Green's function since it is just equal to $i\Delta_F(x_1 - x_2)$. Thus the Feynman propagator is also a Green's function. Incidentally this justifies the use of the name "Green's function" to objects of this kind since the Feynman propagator is the solution to the differential equation 6.2.10 which is what classical applied mathematicians would recognise as a Green's function (apart from the appearance of ϵ which would probably cause them some confusion!)

We can similarly write the four point Green's functions using Feynman diagram between four spacetime points x_1, x_2, x_3, x_4:

It will be noticed that the four spacetime points are not all simultaneously connected together by lines. In fact only pairs of points are connected by lines in any particular diagram. Diagrams such as this are called *disconnected* diagrams and the corresponding Green's functions are called *disconnected* Green's functions. The LSZ reduction formula relates the Green's functions to S-matrix elements and we know that the latter always involve connected Feynman diagrams. Thus only the connected Green's functions are the ones of physical interest. It turns out that there is a different generating functional which generates only connected Green's functions which is denoted by $W[J]$. We shall take the definition of $W[J]$ to be:

$$Z[J] \equiv e^{iW[J]} \tag{6.3.9}$$

and then demonstrate that this definition leads to *connected* Green's functions $G(x_1, x_2, \cdots, x_n)$, given by

$$i^n G(x_1, x_2, \cdots, x_n) = \left[i \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right]_{J(x)=0} \tag{6.3.10}$$

Note the extra factor of i in the above result as compared to Eq.6.3.2.

In the case of free field theory it is immediately clear that the definition of $W_0[J]$ means that it can be identified with the exponent of $Z_0[J]$ in Eq.6.3.5:

$$iW_0[J] = \left(\frac{-i}{2} \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \right) \tag{6.3.11}$$

The only non-zero connected Green's function is therefore

$$G_0(x_1, x_2) = \mathcal{G}_0(x_1, x_2) = i\Delta_F(x_1 - x_2) \tag{6.3.12}$$

Thus $W[J]$ has generated the only connected Green's function in the free theory: the two point function. This vindicates the definition of $W_0[J]$ in the case of the free theory at least.

6.4 Green's Functions for $\lambda\phi^4$ Theory

Consider the interacting field Lagrangian density of the real KG field,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{6.4.1}$$

The path integral expression for the generating functional is:

$$Z[J] \propto \int \mathcal{D}\phi \exp\left(i \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi \right] \right) \tag{6.4.2}$$

This may be written as

$$Z[J] \propto \int \mathcal{D}\phi \exp\left(-i \int d^4x \left[\frac{1}{2}\phi(\partial^2 + m^2)\phi + \frac{\lambda}{4!}\phi^4 - J\phi \right] \right) \tag{6.4.3}$$

but, unlike the free field case, the functional integral cannot be performed. This comes as no surprise, since we were unable to solve the field equations in the interacting theory either. Our strategy now as then is to develop a perturbation expansion in the parameter λ.

In the case $\lambda = 0$ the functional integrals can be performed and the result was given earlier:

$$Z_0[J] = \exp\left(\frac{-i}{2} \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \right) \tag{6.4.4}$$

where the Feynman propagator is:

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik.(x-y)} \tag{6.4.5}$$

For non-zero λ the generating functional is given by (see Appendix for proof):

$$Z[J] = exp\left[i \int d^4x \mathcal{L}_I \left(-i \frac{\delta}{\delta J(x)} \right) \right] Z_0[J] \tag{6.4.6}$$

where we have written

$$\mathcal{L}_I(\phi) = -\frac{\lambda}{4!}\phi^4 \tag{6.4.7}$$

and therefore

$$\mathcal{L}_I \left(-i \frac{\delta}{\delta J(x)} \right) = -\frac{\lambda}{4!} \left(-i \frac{\delta}{\delta J(x)} \right)^4 \tag{6.4.8}$$

The perturbation series comes from expanding the exponential as a power series,

$$exp\left[i \int d^4x \mathcal{L}_I \left(-i \frac{\delta}{\delta J(x)} \right) \right] = 1 + i \int d^4x \mathcal{L}_I \left(-i \frac{\delta}{\delta J(x)} \right)$$
$$+ \frac{i^2}{2!} \int d^4x d^4y \mathcal{L}_I \left(-i \frac{\delta}{\delta J(x)} \right) \mathcal{L}_I \left(-i \frac{\delta}{\delta J(y)} \right) + \cdots$$

Thus the lowest non-trivial approximation for the interaction in Eq.6.4.7 is

$$exp\left[i \int d^4x \mathcal{L}_I \left(-i \frac{\delta}{\delta J(x)} \right) \right] \approx 1 + i \int d^4x \frac{-\lambda}{4!} \left(-i \frac{\delta}{\delta J(x)} \right)^4$$

To this approximation we have:

$$Z[J] \approx Z_0[J] - \frac{i\lambda}{4!} \int d^4x \frac{\delta^4 Z_0}{\delta J(x)^4} \tag{6.4.9}$$

We can easily calculate

$$\frac{\delta Z_0}{\delta J(x)} = -\int d^4y \Delta_F(x-y) J(y) Z_0[J]$$

and repeated differentiation gives

$$Z[J] \approx [1 - \frac{i\lambda}{4!} \int d^4x 3(i\Delta_F(0))^2$$

$$+ 6i\Delta_F(0) \frac{i\lambda}{4!} \int d^4x \int d^4y_1 d^4y_2 i\Delta_F(x-y_1) i\Delta_F(x-y_2) J(y_1) J(y_2)$$

$$- \frac{i\lambda}{4!} \int d^4x \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 i\Delta_F(x-y_1) i\Delta_F(x-y_2) i\Delta_F(x-y_3) i\Delta_F(x-y_4)$$
$$\times J(y_1) J(y_2) J(y_3) J(y_4)] Z_0[J] \tag{6.4.10}$$

The Green's functions are as usual given by

$$i^n\mathcal{G}(x_1,x_2,\cdots,x_n)=\left[\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J(x)=0} \tag{6.4.11}$$

In the free theory we found

$$\mathcal{G}_0(x_1,x_2)=i\Delta_F(x_1-x_2) \tag{6.4.12}$$

$$\mathcal{G}_0(x_1,x_2,x_3)=0 \tag{6.4.13}$$

$$\begin{aligned}\mathcal{G}_0(x_1,x_2,x_3,x_4)=i^2[\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)\\ +\Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)]\end{aligned} \tag{6.4.14}$$

In the interacting theory the Green's functions will differ from these. For example the zero point Green's function which in the free theory is just

$$Z_0[J=0]=1 \tag{6.4.15}$$

in the interscting theory to lowest order in λ is

$$Z[J=0]=1-\frac{i}{8}\lambda\int d^4x i\Delta_F(x-x)i\Delta_F(x-x) \tag{6.4.16}$$

Note the normalisation of the full generating functional is still based on the free normalisation at this stage. Of course the "correct" normalisation is $Z[J=0]=1$ corresponding to a properly normalised vacuum. But we shall for the moment stick to free field normalisation.

The two point Green's function to lowest order in λ is

$$\mathcal{G}(x_1,x_2)=\mathcal{G}_0(x_1,x_2)-\frac{i}{2}\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_2) \tag{6.4.17}$$

The four point Green's function to lowest order in λ is

$$\begin{aligned}\mathcal{G}(x_1,x_2,x_3,x_4)=\mathcal{G}_0(x_1,x_2,x_3,x_4)\\ -\frac{i}{2}\mathcal{G}_0(x_1,x_2)\lambda\int d^4x i\Delta_F(x_3-x)i\Delta_F(x-x)i\Delta_F(x-x_4)\\ -\frac{i}{2}\mathcal{G}_0(x_3,x_4)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_2)\\ -\frac{i}{2}\mathcal{G}_0(x_1,x_3)\lambda\int d^4x i\Delta_F(x_2-x)i\Delta_F(x-x)i\Delta_F(x-x_4)\\ -\frac{i}{2}\mathcal{G}_0(x_2,x_4)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_3)\\ -\frac{i}{2}\mathcal{G}_0(x_1,x_4)\lambda\int d^4x i\Delta_F(x_2-x)i\Delta_F(x-x)i\Delta_F(x-x_3)\\ -\frac{i}{2}\mathcal{G}_0(x_2,x_3)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_4)\end{aligned}$$

$$-i\lambda \int d^4x i\Delta_F(x_1 - x) i\Delta_F(x_2 - x) i\Delta_F(x_3 - x) i\Delta_F(x_4 - x) \qquad (6.4.18)$$

Recall the Feynman rule for the propagator:

$$i\Delta_F(x_1 - x_2) = \quad - - - - - - - - - - -$$

We now supplement this with a new Feynman rule for the vertex of four lines which is equal to $-i\lambda$:

We can then draw Feynman diagrams to represent the Green's functions of the interacting theory to lowest order in perturbation theory. The two point Green's function is thus:

where the first term is the free-propagator, and the second term uses the new Feynman rule with an extra factor of a half, and the prescription that we must integrate over the coordinates x of any internal vertex.

The four point Green's function is thus:

(see Bailin and Love p.63).

As in the free theory it is easier to work with the connected Green's functions which are generated by $W[J]$ defined in Eq.6.3.9 as

$$Z[J] \equiv e^{iW[J]} \tag{6.4.19}$$

The claim is that the *connected* Green's functions $G(x_1, x_2, \cdots, x_n)$, are given by

$$i^n G(x_1, x_2, \cdots, x_n) = \left[i \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right]_{J(x)=0} \tag{6.4.20}$$

where we substantiated this claim in the free field theory. Let us now substantiate this claim in the interacting theory, at least to lowest order in λ.

From the above definition we have

$$iW[J] = \ln Z[J] \tag{6.4.21}$$

Using the approximation $\ln(1+x) \approx x$ we have from Eq.6.4.10, to lowest order in λ,

$$\begin{aligned} iW[J] \approx \ln(Z_0[J]) - \frac{i\lambda}{4!} \int d^4x 3(i\Delta_F(0))^2 \\ +6\frac{i\lambda}{4!} \int d^4x \int d^4y_1 d^4y_2 i\Delta_F(y_1 - x) i\Delta_F(x - x) i\Delta_F(x - y_2) J(y_1) J(y_2) \\ -\frac{i\lambda}{4!} \int d^4x \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 i\Delta_F(y_1 - x) i\Delta_F(y_2 - x) i\Delta_F(y_3 - x) i\Delta_F(y_4 - x) \\ \times J(y_1) J(y_2) J(y_3) J(y_4) \end{aligned} \tag{6.4.22}$$

Now the first term on the rhs is just the free result obtained in Eq.6.3.11,

$$\ln(Z_0[J]) = iW_0[J] = \left(\frac{-i}{2} \int d^4x d^4y J(x) \Delta_F(x - y) J(y) \right) \tag{6.4.23}$$

The additional terms on the rhs are the order λ corrections. The result in Eq.6.4.20 then leads to connected Green's functions which just correspond to the only the subset of diagrams in the Green's functions which are connected. Explicitly we find the connected two point Green's function to be just equal to the two point Green's function in Eq.6.4.17

$$G(x_1, x_2) = \mathcal{G}(x_1, x_2) = \mathcal{G}_0(x_1, x_2) - \frac{i}{2}\lambda \int d^4x i\Delta_F(x_1 - x) i\Delta_F(x - x) i\Delta_F(x - x_2) \tag{6.4.24}$$

The connected four point Green's function, on the other hand, is just equal to the single connected diagram of the four point Green's function, corresponding to the last term in Eq.6.4.18,

$$G(x_1, x_2, x_3, x_4) = -i\lambda \int d^4x i\Delta_F(x_1 - x) i\Delta_F(x_2 - x) i\Delta_F(x_3 - x) i\Delta_F(x_4 - x) \tag{6.4.25}$$

corresponding to just the last of the diagrams listed previously.

Here we note that the factor of 4! we put into the definition of the coupling in the interaction has been cancelled by the 4! ways in which the four external points x_1, x_2, x_3, x_4 are identified with the four sources at points $J(y_1), J(y_2), J(y_3), J(y_4)$ in the expansion of the generating functional in Eq.6.4.10. If we had not had the foresight to do this then the Feynman rule would have had a factor of 4! instead which is more inconvenient.

Note that the connected four point Green's function is zero in the free field theory.

6.5 The $2 \to 2$ Scattering Amplitude from LSZ

We first recall the LSZ reduction formula for the case of $2 \to 2$ scattering in Eq.3.6.11,

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= (\frac{i}{\sqrt{Z}})^2(\frac{-i}{\sqrt{Z}})^2 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{-i(p_1.x_1+p_2.x_2-q_1.y_1-q_2.y_2)}$$

$$\times(\partial^2_{x_1}+m^2)(\partial^2_{x_2}+m^2)(\partial^2_{y_1}+m^2)(\partial^2_{y_2}+m^2)\mathcal{G}(x_1,x_2,y_1,y_2) \qquad (6.5.1)$$

The four point Green's function to lowest order in λ is given in Eq.6.4.18, repeated below,

$$\mathcal{G}(x_1,x_2,y_1,y_2) = i\Delta_F(x_1-x_2)i\Delta_F(y_1-y_2)$$

$$+i\Delta_F(x_1-y_1)i\Delta_F(x_2-y_2) + i\Delta_F(x_1-y_2)i\Delta_F(x_2-y_1)$$

$$-\frac{i}{2}i\Delta_F(x_1-x_2)\lambda\int d^4x i\Delta_F(y_1-x)i\Delta_F(x-x)i\Delta_F(x-y_2)$$

$$-\frac{i}{2}i\Delta_F(y_1-y_2)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-x_2)$$

$$-\frac{i}{2}i\Delta_F(x_1-y_1)\lambda\int d^4x i\Delta_F(x_2-x)i\Delta_F(x-x)i\Delta_F(x-y_2)$$

$$-\frac{i}{2}i\Delta_F(x_2-y_2)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-y_1)$$

$$-\frac{i}{2}i\Delta_F(x_1-y_2)\lambda\int d^4x i\Delta_F(x_2-x)i\Delta_F(x-x)i\Delta_F(x-y_1)$$

$$-\frac{i}{2}i\Delta_F(x_2-y_1)\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x-x)i\Delta_F(x-y_2)$$

$$-i\lambda\int d^4x i\Delta_F(x_1-x)i\Delta_F(x_2-x)i\Delta_F(y_1-x)i\Delta_F(y_2-x) \qquad (6.5.2)$$

We now insert these Green's functions into the LSZ formula and use the result in Eq.6.2.9

$$\left[(\partial^2+m^2)\right]\Delta_F(x,z) = -\delta(x-z) \qquad (6.5.3)$$

to evaluate the S-matrix element.

The first three terms in the Green's function arise from the free Green's function. Two applications of Eq.6.5.3 leave terms like,

$$(\partial_{x_2}^2 + m^2)(\partial_{y_2}^2 + m^2)\delta^4(x_1 - x_2)\delta^4(y_1 - y_2)$$

and then we use results like

$$e^{-ip_2.x_2}\partial_{x_2}^2\delta^4(x_1 - x_2) = \delta^4(x_1 - x_2)\partial_{x_2}^2 e^{-ip_2.x_2}$$

which leads to a terms like

$$(p_2^2 - m^2)(q_2^2 - m^2)$$

which vanish on the mass shell. The next six terms also vanish by a similar argument. The only non-zero contribution comes from the last term, which corresponds to the connected diagram. Let us evaluate this explicitly. We have:

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= (\frac{i}{\sqrt{Z}})^2(\frac{-i}{\sqrt{Z}})^2 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{-i(p_1.x_1 + p_2.x_2 - q_1.y_1 - q_2.y_2)}$$

$$\times(\partial_{x_1}^2 + m^2)(\partial_{x_2}^2 + m^2)(\partial_{y_1}^2 + m^2)(\partial_{y_2}^2 + m^2)$$

$$-i\lambda \int d^4x i\Delta_F(x_1 - x) i\Delta_F(x_2 - x) i\Delta_F(y_1 - x) i\Delta_F(y_2 - x)$$

We easily find

$$< \mathbf{q}_1, \mathbf{q}_2\ out|\mathbf{p}_1, \mathbf{p}_2\ in >= (-i\lambda)(2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2) \tag{6.5.4}$$

The S-matrix element in Eq.6.5.4 conforms to the general structure in Eq.3.4.24. Thus we can identify the "scattering amplitude" or "matrix element" in this case as

$$i\mathcal{M}_f i = -i\lambda$$

which is just equal to the Feynman rule for the vertex introduced earlier.

Note that so far Feynman diagrams have been introduced to represent the position space n point Green's functions. Similar diagrams can clearly also be used to represent the momentum space scattering amplitude $i\mathcal{M}_f i$, with the following changes:

0. Only connected diagrams contribute.

1. The external legs are now labelled by their momentum which is on the mass shell.

2. The rule for the vertex is $-i\lambda$.

3. Four-momenta are conserved at each vertex.

4. Internal propagator lines are given by Eq.6.2.12 for $\Delta_F(k)$.

5. Integrate over each internal loop momentum k with weight $d^4k/(2\pi)^4$.

In the present example the Feynman diagram consists of the single diagram involving the vertex, but with the legs labelled by momentum.

6.6 Appendix: Proof of Eq.6.4.6

Here we prove

$$Z[J] = exp\left[i\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right]Z_0[J]$$

Our starting point is

$$Z[J] = \int \mathcal{D}\phi \exp\left(i\int d^4x\left[\mathcal{L}_0 + \mathcal{L}_I(\phi) + J\phi\right]\right)$$

$$= \int \mathcal{D}\phi\left(\exp i\int d^4x\mathcal{L}_I(\phi)\right)\left(\exp i\int d^4y(\mathcal{L}_0 + J\phi)\right)$$

We expand the first exponential

$$\left(\exp i\int d^4x\mathcal{L}_I(\phi)\right) = 1 + i\int d^4x\mathcal{L}_I(\phi) + \frac{i^2}{2!}\int d^4xd^4y\mathcal{L}_I(\phi(x))\mathcal{L}_I(\phi(y)) + \cdots$$

Now since

$$i\phi(x)\exp i\int d^4z(\mathcal{L}_0 + J\phi) = \frac{\delta}{\delta J(x)}\exp i\int d^4z(\mathcal{L}_0 + J\phi)$$

it follows that

$$\left(\exp i\int d^4x\mathcal{L}_I(\phi)\right)\left(\exp i\int d^4z(\mathcal{L}_0 + J\phi)\right)$$

$$= \int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\left(\exp i\int d^4z(\mathcal{L}_0 + J\phi)\right)$$

Since the operator

$$\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)$$

is independent of $\phi(x)$ it can be taken outside the functional integral and we have

$$Z[J] = \left(\exp\left[i\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right]\right)Z_0[J]$$

where

$$Z_0[J] = \int \mathcal{D}\phi\exp\left(i\int d^4x\left[\mathcal{L}_0 + J\phi\right]\right)$$

which completes the proof.

6.7 Problems Set 6

1. In the case of the free theory calculate the four point Green's function in Eq.6.3.8

$$\mathcal{G}_0(x_1, x_2, x_3, x_4) = i^2[\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4)$$
$$+\Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)]$$

by applying functional differentiation four times to the generating functional.

2. Consider an interacting theory based on the KG Lagrangian for a real scalar field but with an interaction

$$\mathcal{L}_{int} = -\frac{\lambda}{3!}\phi^3$$

Calculate the three point Green's functions to order λ. Justify the use of the combinatorial factor 3! by identifying the Feynman rule for the vertex.

3. Draw the Feynman diagrams for the connected four particle Green's functions for the ϕ^3 theory in Qu.2 above, to order λ^2. Using the Feynman rules, write down expressions for the connected four particle Green's functions to order λ^2. Using the LSZ formula calculate the $2 \to 2$ S-matrix element for $\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4)$ from the connected four particle Green's functions to order λ^2. Draw the Feynman diagrams which represent the scattering amplitude to this order, labelling all four-momenta.

Chapter 7

The Dirac Equation

7.1 Relativistic Wave Equations: Reprise

Recall the Schrodinger equation

$$H\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{7.1.1}$$

where H is the Hamiltonian (i.e. the energy operator). In relativity the square of the Hamiltonian is given by

$$H^2 = \mathbf{P}^2 + m^2 \tag{7.1.2}$$

Naively the relativistic Schrodinger equation looks like

$$\sqrt{\mathbf{P}^2 + m^2}\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{7.1.3}$$

but this is difficult to interpret because of the square root. There are two ways forward:

(1) Work with H^2. By iterating the Schrodinger equation we have

$$H^2\phi(t) = -\frac{\partial^2\phi(t)}{\partial t^2} \tag{7.1.4}$$

which is known as the Klein-Gordon (KG) equation. In this case the wavefunction describes spinless bosons.

(2) Invent a new Hamiltonian H_D which is linear in momentum, and whose square is equal to H^2 given above, $H_D^2 = \mathbf{P}^2 + m^2$. In this case we have

$$H_D\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{7.1.5}$$

which is known as the Dirac equation, with H_D being the Dirac Hamiltonian. In this case the wavefunction describes spin one half fermions, as we shall see.

So far we have been concerned with spin zero particles described by the KG equation. Now it is time to discuss the spin 1/2 particles from which all the atoms in the universe are made.

7.2 The Dirac Equation

Dirac wanted an equation first order in time derivatives and Lorentz covariant, so it had to be first order in spatial derivatives too. His starting point was to assume a Hamiltonian of the form,

$$H_D = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \beta m \tag{7.2.1}$$

where P_i are the three components of the momentum operator $\mathbf{P}$, and α_i and β are some "unknown quantities", which as will be seen below cannot simply be commuting numbers. When the requirement that the $H_D^2 = \mathbf{P}^2 + m^2$ is imposed, this implies that α_i and β must be interpreted as 4×4 matrices, as we shall discuss. The first step is to write the momentum operators explicitly in terms of their differential operators, using Eq.**??**, then the Dirac equation 7.1.5 becomes, using the Dirac Hamiltonian in Eq.7.2.1,

$$i\,\frac{\partial \psi}{\partial t} = (-i\,\alpha{\cdot}\nabla + \beta\mathbf{m})\psi \tag{7.2.2}$$

which is the position space Dirac equation. Remember that in field theory, the Dirac equation is the equation of motion for the field operator describing spin 1/2 fermions. In order for this equation to be Lorentz covariant, it will turn out that ψ cannot be a scalar under Lorentz transformations. In fact this will be precisely how the equation turns out to describe spin 1/2 particles. We will return to this below.

If ψ is to describe a free particle it is natural that it should satisfy the Klein-Gordon equation so that it has the correct energy-momentum relation. This requirement imposes relationships among the α and β. To see these, apply the operator on each side of equation (7.2.2) twice, i.e. iterate the equation,

$$-\frac{\partial^2 \psi}{\partial t^2} = [-\alpha^i \alpha^j \nabla^i \nabla^j - i\,(\beta\alpha^i + \alpha^i \beta) m \nabla^i + \beta^2 m^2]\psi$$

with an implicit sum over i and j from 1 to 3. The Klein-Gordon equation by comparison is

$$-\frac{\partial^2 \psi}{\partial t^2} = [-\nabla^i \nabla^i + m^2]\psi \tag{7.2.3}$$

If we do not assume that the α^i and β commute then the KG will clearly be satisfied if

$$\begin{aligned} \alpha_i \alpha_j + \alpha_j \alpha_i &= 2\delta_{ij} \\ \beta\alpha_i + \alpha_i \beta &= 0 \\ \beta^2 &= 1 \end{aligned} \tag{7.2.4}$$

for $i, j = 1, 2, 3$. It is clear that the α_i and β cannot be ordinary numbers, but it is natural to give them a realisation as matrices. In this case, ψ must be a multi-component *spinor* on which these matrices act.

▷ **Exercise 7.2.1**
Prove that any matrices α and β satisfying equation (7.2.4) are traceless with eigenvalues ± 1. Hence argue that they must be even dimensional.

In two dimensions a natural set of matrices for the α would be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2.5}$$

However, there is no other independent 2×2 matrix with the right properties for β, so the smallest dimension for which the Dirac matrices can be realised is four. One choice is the *Dirac representation*

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2.6}$$

Note that each entry above denotes a two-by-two block and that the 1 denotes the 2×2 identity matrix.

There is a theorem due to Pauli which states that all sets of matrices obeying the relations in (7.2.4) are equivalent. Since the Hermitian conjugates $\alpha^\dagger$ and $\beta^\dagger$ clearly obey the relations, you can, by a change of basis if necessary, assume that α and β are Hermitian. All the common choices of basis have this property. Furthermore, we would like α_i and β to be Hermitian so that the Dirac Hamiltonian (7.4.1) is Hermitian.

▷ **Exercise 7.2.2**
Derive the continuity equation $\partial_\mu J^\mu = 0$ for the Dirac equation with

$$\rho = J^0 = \psi^\dagger(x)\psi(x), \qquad \mathbf{J} = \psi^\dagger(x)\alpha\psi(\mathbf{x}). \tag{7.2.7}$$

We will see in section 7.8 that $(\rho, \mathbf{J})$ does indeed transform as a four-vector.

7.3 Free Particle Solutions I: Interpretation

We look for plane wave solutions of the form

$$\psi = \begin{pmatrix} \chi(\mathbf{p}) \\ \phi(\mathbf{p}) \end{pmatrix} e^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \tag{7.3.1}$$

where $\phi(\mathbf{p})$ and $\chi(\mathbf{p})$ are two-component spinors which depend on momentum $\mathbf{p}$ but are independent of $\mathbf{x}$. Using the Dirac representation of the matrices, and inserting the trial solution into the Dirac equation gives the pair of simultaneous equations

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}, \tag{7.3.2}$$

There are two simple cases for which Eq.7.3.2 can readily be solved, namely

(1) $\mathbf{p} = 0$, $m \neq 0$ corresponding physically to an electron in its rest frame.

(2) $m = 0$, $\mathbf{p} \neq 0$ corresponding physically to a massless neutrino.

For case (1), an electron in its rest frame, the equations 7.3.2 decouple and become simply,

$$E\chi = m\chi, \qquad E\phi = -m\phi \tag{7.3.3}$$

so that in this case we see that χ corresponds to solutions with $E = m$, while ϕ corresponds to solutions with $E = -m$: negative energy solutions!

These negative energy solutions persist for an electron with $\mathbf{p} \neq 0$ for which the solutions to Eq.7.3.2 are readily seen to be

$$\phi = \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}+\mathbf{m}}\,\chi, \qquad \chi = \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}-\mathbf{m}}\,\phi. \tag{7.3.4}$$

Thus the general positive energy solutions with $E = +|\sqrt{\mathbf{p}^2 + m^2}|$ are:

$$\psi(x) = \begin{pmatrix} \chi \\ \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}+\mathbf{m}}\chi \end{pmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}, \tag{7.3.5}$$

while the general negative energy solutions with $E = -|\sqrt{\mathbf{p}^2 + m^2}|$ are:

$$\psi(x) = \begin{pmatrix} \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}-\mathbf{m}}\phi \\ \phi \end{pmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}, \tag{7.3.6}$$

for arbitrary constant ϕ and χ. Clearly when $\mathbf{p} = 0$ these solutions reduce to the positive and negative energy solutions discussed previously. Now, since $E^2 = \mathbf{p}^2 + m^2$ by construction, we find, just as we did for the Klein-Gordon equation (**??**), that there exist positive and negative energy solutions given by equations (7.3.5) and (7.3.6) respectively. Once again, the existence of negative energy solutions vitiates the interpretation of ψ as a wavefunction.

Dirac interpreted the negative energy solutions by postulating the existence of a "sea" of negative energy states. The vacuum or ground state has all the negative energy states full. An additional electron must now occupy a positive energy state since the Pauli exclusion principle forbids it from falling into one of the filled negative energy states. By promoting one of these negative energy states to a positive energy one, by supplying energy, you create a pair: a positive energy electron and a hole in the negative energy sea corresponding to a positive energy positron. This was a radical new idea, and brought pair creation and antiparticles into physics. Positrons were discovered in cosmic rays by Carl Anderson in 1932.

The problem with Dirac's hole theory is that it doesn't work for bosons, such as particles governed by the Klein Gordon equation, for example. Such particles have no exclusion principle to stop them falling into the negative energy states, releasing their energy. We need a new interpretation and turn to Feynman for our answer.

According to Feynman and quantum field theory, we should interpret the emission (absorption) of a negative energy particle with momentum p^μ as the absorption (emission) of a positive energy antiparticle with momentum $-p^\mu$. So, in Figure 7.3.1, for example, an electron–positron pair is created at point A.

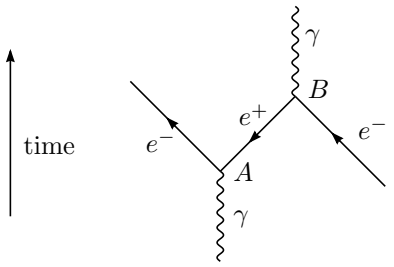


Figure 7.3.1 Feynman interpretation of a process in which a negative energy electron is absorbed. Time increases moving upwards.

The positron propagates to point B where it is annihilated by another electron.

Thus Feynman tells us to keep both types of free particle solution. One is to be used for particles and the other for the accompanying antiparticles. Let's return to our spinor solutions and write them in a conventional form. Take the positive energy solution of equation (7.3.5) and write,

$$\sqrt{E+m}\begin{pmatrix}\chi_r\\ \frac{\sigma\cdot\mathbf{p}}{\mathbf{E}+\mathbf{m}}\chi_r\end{pmatrix}e^{-ip\cdot x}\equiv u_p^r e^{-ip\cdot x}. \tag{7.3.7}$$

For the former negative energy solution of equation (7.3.6), change the sign of the energy, $E\to -E$, and the three-momentum, $\mathbf{p}\to -\mathbf{p}$, to obtain,

$$\sqrt{E+m}\begin{pmatrix}\frac{\sigma\cdot\mathbf{p}}{\mathbf{E}+\mathbf{m}}\chi_r\\ \chi_r\end{pmatrix}e^{ip\cdot x}\equiv v_p^r e^{ip\cdot x}. \tag{7.3.8}$$

In these two solutions E is now (and for the rest of the course) always positive and given by $E=(\mathbf{p}^2+m^2)^{1/2}$. The subscript r takes the values $1,2$, with

$$\chi_1=\begin{pmatrix}1\\0\end{pmatrix},\qquad \chi_2=\begin{pmatrix}0\\1\end{pmatrix}. \tag{7.3.9}$$

For the simple case $\mathbf{p}=0$ we may interpret χ_1 as the spin-up state and χ_2 as the spin-down state. Thus for $\mathbf{p}=0$ the 4-component wavefunction has a very simple interpretation: the first two components describe electrons with spin-up and spin-down, while the second two components describe positrons with spin-up and spin-down. Thus we understand on physical grounds why the wavefunction had to have four components. The general case $\mathbf{p}\neq 0$ is slightly more involved and is considered in the next section.

At this point I would like to introduce another notation, and define

$$\omega_p\equiv\sqrt{\mathbf{p}^2+m^2}, \tag{7.3.10}$$

so that, ω_p is the energy (positive) of a particle or anti-particle with three-momentum $\mathbf{p}$ (I write the subscript p instead of $\mathbf{p}$, but you should remember it really means the three-momentum). I will tend to use E or ω_p interchangeably.

The u-spinor solutions will correspond to particles and the v-spinor solutions to antiparticles. The role of the two χ's will become clear in the following section, where it will be shown that the two choices of r are spin labels. Note that each spinor solution depends on the three-momentum $\mathbf{p}$, so it is implicit that $p^0 = \omega_p$. In the expansion of the Dirac quantum field operator in terms of plane waves,

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3\, 2\omega_p} \sum_{r=1,2} \left[\hat{b}(p,r)u_p^r e^{-ip\cdot x} + \hat{d}^\dagger(p,r)v_p^r e^{ip\cdot x}\right] \tag{7.3.11}$$

the operator $\hat{b}$ annihilates a fermion of momentum $(\omega_p, \mathbf{p})$ and spin r, whilst $\hat{d}^\dagger$ creates an antifermion of momentum $(\omega_p, \mathbf{p})$ and spin r. The Hermitian conjugate Dirac field contains operators which do the opposite. This discussion should be clearer after your quantum field theory lectures.

The vacuum state $|0\rangle$ is defined by,

$$b(p,r)\,|0\rangle = d(p,r)\,|0\rangle = 0, \tag{7.3.12}$$

for every momentum $p = (\omega_p, \mathbf{p})$ and spin label r. This ensures the interpretation above: particles are created by the "daggered" operators and destroyed by the undaggered ones.

7.4 Free Particle Solutions II: Spin

Now it's time to justify the statements we have been making that the Dirac equation describes spin-1/2 particles. The Dirac Hamiltonian in momentum space is given in Eq.7.2.1 as

$$H_D = \alpha\cdot\mathbf{P} + \beta\mathbf{m} \tag{7.4.1}$$

and the orbital angular momentum operator is

$$\mathbf{L} = \mathbf{R} \times \mathbf{P}.$$

Normally you have to worry about operator ordering ambiguities when going from classical objects to quantum mechanical ones. For the components of $\mathbf{L}$ the problem does not arise — why not?

Evaluating the commutator of $\mathbf{L}$ with H_D,

$$\begin{aligned} [\mathbf{L}, H_D] &= [\mathbf{R}\times\mathbf{P}, \alpha\cdot\mathbf{P}] \\ &= [\mathbf{R}, \alpha\cdot\mathbf{P}]\times\mathbf{P} \\ &= i\alpha\times\mathbf{P}, \end{aligned} \tag{7.4.2}$$

we see that the orbital angular momentum is not conserved (otherwise the commutator would be zero). We'd like to find a *total* angular momentum $\mathbf{J}$ which *is* conserved, by adding an additional operator $\mathbf{S}$ to $\mathbf{L}$,

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \quad [\mathbf{J}, H_D] = 0 \tag{7.4.3}$$

To this end, consider the three matrices,

$$\mathbf{\Sigma} \equiv \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\alpha. \tag{7.4.4}$$

where the first equivalence is merely a definition of $\mathbf{\Sigma}$ and the last equality can readily be verified. The $\mathbf{\Sigma}/2$ have the correct commutation relations to represent angular momentum, since the Pauli matrices do, and their commutators with α and β are,

$$[\mathbf{\Sigma}, \beta] = 0, \qquad [\Sigma_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k. \tag{7.4.5}$$

▷ **Exercise 7.4.1**
Verify the commutation relations in equation (7.4.5).

From the relations in (7.4.5) we find that

$$[\mathbf{\Sigma}, H_D] = -2i\alpha \times \mathbf{P}.$$

Comparing this with the commutator of $\mathbf{L}$ with H_D in equation (7.4.2), you readily see that

$$\left[\mathbf{L} + \tfrac{1}{2}\mathbf{\Sigma}, H_D\right] = 0,$$

and we can identify

$$\mathbf{S} = \frac{1}{2}\mathbf{\Sigma}.$$

as the additional quantity which when added to $\mathbf{L}$ in Eq.7.4.3 yields a conserved total angular momentum $\mathbf{J}$. We interpret $\mathbf{S}$ as an angular momentum *intrinsic* to the particle. Now

$$\mathbf{S}^2 = \frac{1}{4}\begin{pmatrix} \sigma\cdot\sigma & 0 \\ 0 & \sigma\cdot\sigma \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and recalling that the eigenvalue of $\mathbf{J}^2$ for spin j is $j(j+1)$, we conclude that $\mathbf{S}$ represents spin-1/2 and the solutions of the Dirac equation have spin-1/2 as promised.

We worked in the Dirac representation of the matrices for convenience, but the result is of course independent of the representation.

Now consider the u-spinor solutions u_p^r of equation (7.3.7). Choose $\mathbf{p} = (0, 0, p_z)$ and write

$$u_\uparrow = u^1_{p_z} = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix}, \qquad u_\downarrow = u^2_{p_z} = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}. \tag{7.4.6}$$

It is easy to see that,

$$S_z u_\uparrow = \frac{1}{2}u_\uparrow, \qquad S_z u_\downarrow = -\frac{1}{2}u_\downarrow.$$

So, these two spinors represent spin up and spin down along the z-axis respectively. For the v-spinors, with the same choice for $\mathbf{p}$, write,

$$v_\downarrow = v^1_{p_z} = \begin{pmatrix} \sqrt{E-m} \\ 0 \\ \sqrt{E+m} \\ 0 \end{pmatrix}, \qquad v_\uparrow = v^2_{p_z} = \begin{pmatrix} 0 \\ -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} \end{pmatrix}, \tag{7.4.7}$$

where now,

$$S_z v_\downarrow = \frac{1}{2} v_\downarrow, \qquad S_z v_\uparrow = -\frac{1}{2} v_\uparrow.$$

This apparently perverse choice of up and down for the v's is because, as you see in equation (7.3.11) for the quantum Dirac field, $u_\uparrow$ multiplies an annihilation operator which *destroys* a particle with momentum p_z and spin up, whereas $v_\downarrow$ multiplies an operator which *creates* an antiparticle with momentum p_z and spin up.

7.5 Normalisation, Gamma Matrices

We have included a normalisation factor $\sqrt{E+m}$ in our spinors. With this factor,

$$u_p^{r\dagger} u_p^s = v_p^{r\dagger} v_p^s = 2\omega_p \delta^{rs}. \tag{7.5.1}$$

This corresponds to the standard relativistic normalisation of $2\omega_p$ particles per unit volume. It also means that $u^\dagger u$ transforms like the time component of a 4-vector under Lorentz transformations as we will see in section 7.8.

▷ **Exercise 7.5.1**
Check the normalisation condition for the spinors in equation (7.5.1).

I will now introduce (yet) more standard notation. Define the *gamma matrices*,

$$\gamma^0 = \beta, \qquad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha}. \tag{7.5.2}$$

In the Dirac representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{7.5.3}$$

In terms of these, the relations between the α and β in equation (7.2.4) can be written compactly as,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{7.5.4}$$

Combinations like $a_\mu \gamma^\mu$ occur frequently and are conventionally written as,

$$\not{a} = a_\mu \gamma^\mu = a^\mu \gamma_\mu,$$

pronounced "a slash." Note that γ^μ is not, despite appearances, a 4-vector — it just denotes a set of four matrices. However, the notation is deliberately

suggestive, for when combined with Dirac fields you can construct quantities which transform like vectors and other Lorentz tensors (see the next section).

Let's close this section by observing that using the gamma matrices the Dirac equation (7.2.2) becomes

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{7.5.5}$$

or in momentum space,

$$(p\!\!\!/ - m)\psi = 0. \tag{7.5.6}$$

The spinors u and v satisfy

$$\begin{array}{rcl}(p\!\!\!/ - m)u_p^r &=& 0\\ (p\!\!\!/ + m)v_p^r &=& 0\end{array} \tag{7.5.7}$$

▷ **Exercise 7.5.2**
Derive the momentum space equations satisfied by u_p^r and v_p^r.

7.6 Lorentz Covariance

We want the Dirac equation (7.5.5) to preserve its form under Lorentz transformations (LT's). Let $\Lambda^\mu{}_\nu$ represent an LT,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{7.6.1}$$

A familiar example of a LT is a boost along the z-axis, for which

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix},$$

with as usual $\beta = v$ (in units of c) and $\gamma = (1-\beta^2)^{-1/2}$. LT's can be thought of as generalised rotations.

The requirement is,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \quad \longrightarrow \quad (i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0,$$

where $\partial_\mu = \Lambda^\sigma{}_\mu\partial'_\sigma$. This last equality follows because

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\sigma}{\partial x^\mu}\frac{\partial}{\partial x'^\sigma} = \Lambda^\sigma{}_\mu\frac{\partial}{\partial x'^\sigma}$$

where Eq.7.6.1 has been used in the last step. We know that 4-vectors get their components mixed up by LT's, so we expect that the components of ψ might get mixed up also,

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x') \tag{7.6.2}$$

where $S(\Lambda)$ is a 4×4 matrix acting on the spinor index of ψ. Note that the argument $\Lambda^{-1}x'$ is just a fancy way of writing x, so each component of $\psi(x)$ is transformed into a linear combination of components of $\psi(x)$.

It is helpful to recall that for a vector field, the corresponding transformation is

$$A^\mu(x) \to A'^\mu(x')$$

where $x' = \Lambda x$. This makes sense physically if one thinks of space rotations of a vector field. For example the wind arrows on a weather map of England are an example of a vector field: at each point on the map there is associated an arrow. Consider the wind direction at a particular point on the map, say Abingdon. If the map of England is rotated, then one would expect on physical grounds that the wind vector at Abingdon always point in the same physical direction and have the same length. In order to achieve this, both the vector itself must rotate, and the point to which it is attached (Abingdon) must be correctly identified after the rotation. Thus the vector at the point x' (corresponding to Abingdon in the rotated frame) is equal to the vector at the point x (corresponding to Abingdon in the unrotated frame), but rotated so as to keep the physical sense of the vector the same in the rotated frame (so that the wind always blows towards Oxford, say, in the two frames). Thus having correctly identified the same point in the two frames all we need to do is rotate the vector:

$$A'^\mu(x') = \Lambda^\mu_\nu A^\mu(x).$$

A similar thing also happens in the case of the 4-component spinor field above, except that we do not (yet) know how the components of the wavefunction themselves must transform, i.e. we do not know S.

To determine S we rewrite the Dirac equation in terms of the primed variables (just a mathematical substitution),

$$(i\gamma^\mu \Lambda^\sigma{}_\mu \partial'_\sigma - m)\psi(\Lambda^{-1}x') = 0. \tag{7.6.3}$$

Some new matrices can be defined, $\gamma'^\sigma \equiv \gamma^\mu \Lambda^\sigma{}_\mu$ which satisfy the same anti-commutation relations as the γ^μ's in equation (7.5.4),

$$\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}. \tag{7.6.4}$$

▷ **Exercise 7.6.1**
Check relation (7.6.4).

Now we invoke the theorem (Pauli's theorem) which states that any two representations of the gamma matrices are equivalent. This means that there is a matrix $S(\Lambda)$ such that

$$\gamma'^\mu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda). \tag{7.6.5}$$

This allows us to rewrite equation (7.6.3) as

$$(i\gamma^\mu \partial'_\mu - m)S(\Lambda)\psi(\Lambda^{-1}x') = 0,$$

or using Eq.7.6.2,

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0, \tag{7.6.6}$$

so that the Dirac equation does indeed preserve its form in the primed frame.

To construct S explicitly we must solve Eq.7.6.5, which may be written as,

$$\gamma^\mu \Lambda^\sigma{}_\mu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda). \tag{7.6.7}$$

For an infinitesimal LT, it can be verified that,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \epsilon(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu) \tag{7.6.8}$$

where ϵ is an infinitesimal parameter and ρ and σ are fixed. Since this expression is antisymmetric in ρ and σ there are six choices for the pair (ρ, σ) corresponding to three rotations and three boosts.

For example a boost along the z-axis corresponds to $\rho = 0$, $\sigma = 3$, since in this case,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \epsilon(g^{0\mu}\delta^3{}_\nu - g^{3\mu}\delta^0{}_\nu)$$
$$= \begin{pmatrix} 1 & 0 & 0 & -\epsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon & 0 & 0 & 1 \end{pmatrix},$$

which can be identified with the previous example with $\beta = -\epsilon$ and $\gamma = 1$ in the low velocity limit.

Writing,

$$S(\Lambda) = 1 + i\epsilon s^{\rho\sigma} \tag{7.6.9}$$

where $s^{\rho\sigma}$ is a matrix to be determined for each choice of ρ and σ, we find that equation (7.6.5) for γ' is satisfied by,

$$s^{\rho\sigma} = \frac{i}{4}\left[\gamma^\rho, \gamma^\sigma\right] \equiv \frac{1}{2}\,\sigma^{\rho\sigma}. \tag{7.6.10}$$

Here, I have taken the opportunity to define the matrix $\sigma^{\rho\sigma}$. Thus S is given explicitly in terms of gamma matrices, for any LT specified by ρ, σ and ϵ.

▷ **Exercise 7.6.2**
Verify that equation (7.6.5) relating γ' and γ is satisfied by $s^{\rho\sigma}$ defined through equations (7.6.9) and (7.6.10).

We have thus determined how ψ transforms under LT's. To find quantities which are Lorentz invariant, or transform as vectors or tensors, we need to introduce the Pauli and Dirac adjoints. The Pauli adjoint $\overline{\psi}$ of a *spinor* ψ is defined by

$$\overline{\psi} \equiv \psi^\dagger\gamma^0 = \psi^\dagger\beta. \tag{7.6.11}$$

The Dirac adjoint of a *matrix* A is defined by

$$(\overline{\psi}A\phi)^* = \overline{\phi}\,\overline{A}\psi. \tag{7.6.12}$$

For Hermitian γ^0 it is easy to show that

$$\overline{A} = \gamma^0 A^\dagger \gamma^0. \tag{7.6.13}$$

Some properties of the Pauli and Dirac adjoints are:

$$\begin{aligned} \overline{(\lambda A + \mu B)} &= \lambda^* \overline{A} + \mu^* \overline{B}, \\ \overline{AB} &= \overline{B}\,\overline{A}, \\ \overline{A\psi} &= \overline{\psi}\,\overline{A}. \end{aligned}$$

With these definitions, $\overline{\psi}$ transforms as follows under LT's:

$$\overline{\psi} \to \overline{\psi}' = \overline{\psi} S^{-1}(\Lambda) \tag{7.6.14}$$

▷ **Exercise 7.6.3**

(1) Verify that $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$. This says that $\overline{\gamma}^\mu = \gamma^\mu$.

(2) Using (7.6.9) and (7.6.10) verify that $\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)$, i.e. $\overline{S} = S^{-1}$. So S is not unitary in general, although it *is* unitary for rotations (when ρ and σ are spatial indices). This is because the rotations are in the unitary $O(3)$ subgroup of the nonunitary Lorentz group. Here you show the result for an infinitesimal LT, but it is true for finite LT's.

(3) Show that $\overline{\psi}$ satisfies the equation

$$\overline{\psi}\,(-i\overleftarrow{\partial\!\!\!/} - m) = 0$$

where the arrow over $\partial\!\!\!/$ implies the derivative acts on $\overline{\psi}$.

(4) Hence prove that $\overline{\psi}$ transforms as in equation (7.6.14).

Note that result (2) of the problem above can be rewritten as $\overline{S}(\Lambda) = S^{-1}(\Lambda)$, and equation (7.6.5) for the similarity transformation of γ^μ to γ'^μ takes the form,

$$\overline{S}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu. \tag{7.6.15}$$

Combining the transformation properties of ψ and $\overline{\psi}$ in equations (7.6.2) and (7.6.14) we see that the bilinear $\overline{\psi}\psi$ is Lorentz invariant. In section 7.8 we'll consider the transformation properties of general bilinears.

Let me close this section by recasting the spinor normalisation equations (7.5.1) in terms of "Dirac inner products." The conditions become,

$$\begin{aligned} \overline{u}^r_p u^s_p &= 2m\delta^{rs} \\ \overline{u}^r_p v^s_p &= 0 \qquad = \quad \overline{v}^r_p u^s_p \\ \overline{v}^r_p v^s_p &= -2m\delta^{rs} \end{aligned} \tag{7.6.16}$$

▷ **Exercise 7.6.4**
Verify the normalisation properties in the above equations (7.6.16).

7.7 Parity

In the next section we are going to construct quantities bilinear in ψ and $\overline{\psi}$, and classify them according to their transformation properties under LT's. We normally use LT's which are in the connected Lorentz Group, $SO(3,1)$, meaning they can be obtained by a continuous deformation of the identity transformation. Indeed in the last section we considered LT's very close to the identity in equation (7.6.8). The full Lorentz group has four components generated by combining the $SO(3,1)$ transformations with the discrete operations of parity or space inversion, P, and time reversal, T,

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

LT's satisfy $\Lambda^T g \Lambda = g$ (see the preschool problems), so taking determinants shows that $\det \Lambda = \pm 1$. LT's in $SO(3,1)$ have determinant 1, since the identity does, but the P and T operations have determinant -1.

Let's now find the action of parity on the Dirac wavefunction and determine the wavefunction ψ_P in the parity-reversed system. According to the discussion of the previous section, and using the result of equation (7.6.15), we need to find a matrix S satisfying

$$\overline{S}\gamma^0 S = \gamma^0, \qquad \overline{S}\gamma^i S = -\gamma^i.$$

It's not hard to see that $S = \overline{S} = \gamma^0$ is an acceptable solution, from which it follows that the wavefunction ψ_P is

$$\psi_P(t, \mathbf{x}) = \gamma^0 \psi(t, -\mathbf{x}). \tag{7.7.1}$$

In fact you could multiply γ^0 by a phase and still have an acceptable definition for the parity transformation.

In the nonrelativistic limit, the wavefunction ψ approaches an eigenstate of parity. Since

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the u-spinors and v-spinors at rest have opposite eigenvalues, corresponding to particle and antiparticle having opposite *intrinsic* parities.

7.8 Bilinear Covariants

Now, as promised, we will construct and classify the bilinears. To begin, observe that by forming products of the gamma matrices it is possible to construct 16 linearly independent quantities. In equation (7.6.10) we have defined

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu],$$

and now it is convenient to define

$$\gamma_5 \equiv \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{7.8.1}$$

with the properties,

$$\gamma_5^\dagger = \gamma_5, \qquad \{\gamma_5, \gamma^\mu\} = 0.$$

Then the set of 16 matrices

$$\Gamma : \{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$$

form a basis for gamma matrix products.

Using the transformations of ψ and $\overline{\psi}$ from equations (7.6.2) and (7.6.14), together with the similarity transformation of γ^μ in equation (7.6.15), construct the 16 fermion bilinears and their transformation properties as follows:

$$\begin{array}{rcll} \overline{\psi}\psi & \to & \overline{\psi}\psi & \text{S scalar} \\ \overline{\psi}\gamma_5\psi & \to & \det(\Lambda)\,\overline{\psi}\gamma_5\psi & \text{P pseudoscalar} \\ \overline{\psi}\gamma^\mu\psi & \to & \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi & \text{V vector} \\ \overline{\psi}\gamma^\mu\gamma_5\psi & \to & \det(\Lambda)\,\Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\gamma_5\psi & \text{A axial vector} \\ \overline{\psi}\sigma^{\mu\nu}\psi & \to & \Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma\overline{\psi}\sigma^{\lambda\sigma}\psi & \text{T tensor} \end{array} \tag{7.8.2}$$

▷ **Exercise 7.8.1**
Verify the transformation properties of the bilinears in equation (7.8.2).

Observe that $\overline{\psi}\gamma^\mu\psi = (\rho, \mathbf{J})$ is just the current we found earlier in equation (7.2.7). Classically ρ is positive definite, but for the quantum Dirac field you find that the space integral of ρ is the charge operator, which counts the number of electrons minus the number of positrons,

$$Q \sim \int d^3x\, \psi^\dagger\psi \sim \int d^3p\, [b^\dagger b - d^\dagger d\,].$$

The continuity equation $\partial_\mu J^\mu = 0$ expresses conservation of electric charge.

7.9 Charge Conjugation

There is one more discrete invariance of the Dirac equation in addition to parity. It is charge conjugation, which takes you from particle to antiparticle and vice versa. For scalar fields the symmetry is just complex conjugation, but in order for the charge conjugate Dirac field to remain a solution of the Dirac equation, you have to mix its components as well:

$$\psi \to \psi_C = C\overline{\psi}^{\,T}.$$

Here $\overline{\psi}^{\,T} = \gamma^{0T}\psi^*$ and C is a matrix satisfying the condition

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu.$$

In the Dirac representation,

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}.$$

I refer you to textbooks such as [1] for details.

When Dirac wrote down his equation everybody thought parity and charge conjugation were exact symmetries of nature, so invariance under these transformations was essential. Now we know that neither of them, nor the combination CP, are respected by the standard electroweak model.

7.10 Neutrinos

In the particle data book [2] you will find only upper limits for the masses of the three neutrinos, and in the standard model they are massless. Let's look therefore at solutions of the Dirac equation with $m = 0$. From Eq.7.3.2 we have in this case

$$E\phi = \sigma\cdot\mathbf{p}\,\chi, \qquad E\chi = \sigma\cdot\mathbf{p}\,\phi. \tag{7.10.1}$$

These equations can easily be decoupled by taking the linear combinations and defining in a suggestive way the two component spinors ν_L and ν_R,

$$\nu_R \equiv \chi + \phi, \quad \nu_L \equiv \chi - \phi \tag{7.10.2}$$

which leads to

$$E\nu_R = \sigma\cdot\mathbf{p}\,\nu_R, \qquad E\nu_L = -\sigma\cdot\mathbf{p}\,\nu_L. \tag{7.10.3}$$

Since $E = |\mathbf{p}|$ for massless particles, these equations may be written,

$$\frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}\nu_L = -\nu_L, \qquad \frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}\nu_R = \nu_R \tag{7.10.4}$$

Since $\frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}$ is identified as the helicity operator (i.e. the spin operator projected in the direction of motion of the momentum of the particle) we see that the ν_L corresponds to solutions with negative helicity, while ν_R corresponds to solutions with positive helicity. In other words ν_L descibes a left-handed neutrino while ν_R describes a right-handed neutrino – and each type of neutrino is descibed by a two-component spinor.

The two-component spinors describing neutrinos transform very simply under LT's,

$$\nu_L \to e^{\frac{i}{2}\sigma.(\theta - \mathbf{i}\phi)}\nu_L \tag{7.10.5}$$

$$\nu_R \to e^{\frac{i}{2}\sigma.(\theta + \mathbf{i}\phi)}\nu_R \tag{7.10.6}$$

where $\theta = \mathbf{n}\theta$ corresponding to space rotations through an angle θ about the unit $\mathbf{n}$ axis, and $\phi = \mathbf{v}\phi$ corresponding to Lorentz boosts along the unit vector $\mathbf{v}$ with a speed $v = \tanh\phi$. Under parity transformations they become transformed into each other,

$$\nu_L \leftrightarrow \nu_R \tag{7.10.7}$$

so a theory which involves only ν_L without ν_R (such as the standard model) manifestly violates parity.

Although massless neutrinos can be described very simply using two component spinors as above, they may also be incorporated into the four-component formalism as follows. From equation (7.2.2) we have, in momentum space,

$$|\mathbf{p}|\psi = \alpha{\cdot}\mathbf{p}\,\psi.$$

For such a solution,

$$\gamma_5\psi = \gamma_5\frac{\alpha{\cdot}\mathbf{p}}{|\mathbf{p}|}\psi = 2\frac{\mathbf{S}{\cdot}\mathbf{p}}{|\mathbf{p}|}\psi,$$

using the spin operator $\mathbf{S} = \frac{1}{2}\mathbf{\Sigma} = \frac{1}{2}\gamma_5\alpha$, with $\mathbf{\Sigma}$ defined in equation (7.4.4). But $\mathbf{S}{\cdot}\mathbf{p}/|\mathbf{p}|$ is the projection of spin onto the direction of motion, known as the *helicity*, and is equal to $\pm 1/2$. Thus $(1{+}\gamma_5)/2$ projects out the neutrino with helicity $1/2$ (right handed) and $(1{-}\gamma_5)/2$ projects out the neutrino with helicity $-1/2$ (left handed),

$$\frac{(1{+}\gamma_5)}{2}\psi \equiv \psi_R, \quad \frac{(1{-}\gamma_5)}{2}\psi \equiv \psi_L, \tag{7.10.8}$$

which defines the four-component spinors ψ_R and ψ_L.

To date, only left handed neutrinos have been observed, and only left handed neutrinos appear in the standard model. Since

$$\gamma^0\frac{1}{2}(1{-}\gamma_5)\psi = \frac{1}{2}(1{+}\gamma_5)\gamma^0\psi,$$

any theory involving only left handed neutrinos necessarily violates parity - as we saw before in the two-component formalism.

Finally note that in the Dirac representation which we have been using,

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{7.10.9}$$

and the relation between the two-component and four-component formalisms is via the change of variables in Eq.7.10.2. However there exists a representation in which this change of variables is done automatically and the (massless) Dirac equation falls apart into the two two-component equations discussed above. In this chiral representation,

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.10.10}$$

and hence,

$$\frac{(1{+}\gamma_5)}{2}\psi = \begin{pmatrix} 0 \\ \nu_R \end{pmatrix}, \quad \frac{(1{-}\gamma_5)}{2}\psi = \begin{pmatrix} \nu_L \\ 0 \end{pmatrix}. \tag{7.10.11}$$

where we have identified ν_R and ν_L as the two-component spinors discussed previously. These results are also applicable to the electron in the approximation that its mass is neglected, by the simple transcription $\nu_R \to e_R$, $\nu_L \to e_L$.

The standard model (and the minimal supersymmetric standard model) contains only left handed massless neutrinos, and neutrino mass terms are forbidden by gauge symmetry, at least given the limited number of fields present in the standard model. If extra fields (e.g. right handed neutrinos) are added then neutrino masses become possible. If neutrino oscillations are confirmed as the solution to the solar neutrino problem, or are discovered in laboratory experiments, then such a modification would become a necessity.

Chapter 8

The Free Dirac Field

8.1 Canonical Quantisation

Dirac Field Theory is defined to be the theory whose field equations correspond to the Dirac equation. We regard the two Dirac fields $\psi(x)$ and $\overline{\psi}(x)$ as being dynamically independent fields and postulate the Dirac Lagrangian density:

$$\mathcal{L} = \overline{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \tag{8.1.1}$$

The Euler-Lagrange equation analagous to Eq.2.2.10

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} - \frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0 \tag{8.1.2}$$

leads to the Dirac equation.

The canonical momentum from Eq.2.2.5 is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}(x)} = i\psi^\dagger(x) \tag{8.1.3}$$

The Hamiltonian density is analagous to Eq.2.2.7

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger i\frac{\partial \psi}{\partial t} \tag{8.1.4}$$

which is not positive definite. The general solution to the Dirac equation may be expanded in terms of plane waves analagously to Eq.2.3.19

$$\psi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}\sum_{\alpha=1,2}[b_\alpha(\mathbf{k})u^\alpha(\mathbf{k})e^{-ik.x} + d^\dagger_\alpha(\mathbf{k})v^\alpha(\mathbf{k})e^{ik.x}] \tag{8.1.5}$$

$$\overline{\psi}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}\sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})\overline{u}^\alpha(\mathbf{k})e^{ik.x} + d_\alpha(\mathbf{k})\overline{v}^\alpha(\mathbf{k})e^{-ik.x}] \tag{8.1.6}$$

The total Hamiltonian is

$$H = \int d^3x\mathcal{H} \tag{8.1.7}$$

After some algebra we find

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}k_0 \sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})b_\alpha(\mathbf{k}) - d_\alpha(\mathbf{k})d^\dagger_\alpha(\mathbf{k})] \tag{8.1.8}$$

So far no commutation relations have been assumed, and H could quite easily be negative, unlike the Hamiltonian in the case of the charged scalars for example which was positive definite as seen in Eq.2.3.37. In order to give a positive definite Hamiltonian we require the creation and annihilation operators to satisfy *anticommutation* relations, first proposed by Wigner:

$$\left\{b_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}')\right\} = (2\pi)^3\frac{k_0}{m}\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\alpha\alpha'} \tag{8.1.9}$$

$$\left\{d_\alpha(\mathbf{k}), d^\dagger_{\alpha'}(\mathbf{k}')\right\} = (2\pi)^3\frac{k_0}{m}\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\alpha\alpha'} \tag{8.1.10}$$

$$\{b_\alpha(\mathbf{k}), b_{\alpha'}(\mathbf{k}')\} = 0 \tag{8.1.11}$$

$$\left\{b^\dagger_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0 \tag{8.1.12}$$

$$\{d_\alpha(\mathbf{k}), d_{\alpha'}(\mathbf{k}')\} = 0 \tag{8.1.13}$$

$$\left\{d^\dagger_\alpha(\mathbf{k}), d^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0 \tag{8.1.14}$$

The Hamiltonian is then defined as the normal ordered version of Eq.8.1.8 *but with a change of sign for each interchange of operator*

$$H = \int d^3x : \psi^\dagger i\frac{\partial\psi}{\partial t} : \tag{8.1.15}$$

which results in

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}k_0 \sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})b_\alpha(\mathbf{k}) + d_\alpha(\mathbf{k})d^\dagger_\alpha(\mathbf{k})] \tag{8.1.16}$$

which is now postive definite.

Anticommutation implies Fermi statistics for example:

$$\left\{b^\dagger_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0$$

$$\Rightarrow b^\dagger_\alpha(\mathbf{k})b^\dagger_\alpha(\mathbf{k}) = 0$$

$$\Rightarrow b^\dagger_\alpha(\mathbf{k})b^\dagger_\alpha(\mathbf{k})|0>= 0$$

so that two quanta in the same state are not allowed (Pauli exclusion principle).

The charge operator is the analogue of Eq.2.3.36

$$Q = \int d^3\mathbf{x} : j_0(x) := \int d^3\mathbf{x} : \psi^\dagger i\partial\psi :$$

$$Q = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}k_0 \sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})b_\alpha(\mathbf{k}) - d^\dagger_\alpha(\mathbf{k})d_\alpha(\mathbf{k})] \tag{8.1.17}$$

which shows that $b^\dagger$ creates fermions while $d^\dagger$ creates antifermions of opposite charge.

Finally the equal time commutation relations are (after some algebra):

$$\left\{\psi_i(\mathbf{x},t),\psi_j^\dagger(\mathbf{x}',t)\right\}=\delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij} \tag{8.1.18}$$

$$\{\psi_i(\mathbf{x},t),\psi_j(\mathbf{x}',t)\}=0 \tag{8.1.19}$$

$$\left\{\psi_i^\dagger(\mathbf{x},t),\psi_j^\dagger(\mathbf{x}',t)\right\}=0 \tag{8.1.20}$$

In fact at all times we have:

$$\{\psi(x),\psi(y)\}=0 \tag{8.1.21}$$

8.2 Path Integral Quantisation

We have seen that Green's functions in quantum field theory may be generated from generating functionals, which are functional integrals over classical field functions. In the case of scalar field theory the classical field functions are commuting c-numbers. However in the case of spinor field theory the classical field functions cannot be regarded as commuting c-numbers. The problem is that the classical Hamiltonian is also given by Eq.8.1.8 and is similarly non-positive definite. Thus it would appear that classical spinor field theory does not make physical sense. Thus spinor fields are essentially non-classical, and this poses a problem for the path integral approach where functional integrals over classical fields will occur.

In chapter 2 we discussed the free Klein-Gordon field and showed how it could be regarded as a continuum of harmonic oscillators, one at each spacetime point, and each coupled to their nearest neighbours in a Lorentz invariant way. The path integral results in chapter 4 may be taken to apply to each of these little oscillators. In chapter 6 we generalised the path integral results from a single coordinate $q(t)$ to many coordinates $q_i(t)$, one for each oscillator. We then considered the continuum limit,

$$q_i(t)\rightarrow\phi(x,y,z,t) \tag{8.2.1}$$

just as we did to arrive at canonical field theory. Thus the Heisenberg operators $\hat{q}(t)$ in Eq.6.1.3 become Heisenberg fields $\hat{\phi}(x,y,z,t)$ whose eigenstates are given by

$$\hat{\phi}(x,y,z,t)|\phi(x,y,z,t)>=\phi(x,y,z,t)|\phi(x,y,z,t)> \tag{8.2.2}$$

and the generating functional involves the eigenvalue of the field $\phi(x,y,z,t)$ rather than the field operator.

Now we want to do the same thing for the spinor field theory, and the generating functional will involve the eigenvalue field on the right hand side of the equation:

$$\hat{\psi}(x,y,z,t)|\psi(x,y,z,t)>=\psi(x,y,z,t)|\psi(x,y,z,t)> \tag{8.2.3}$$

However if we apply the same field operator twice we encounter a problem:

$$\hat{\psi}^2(x,y,z,t)|\psi(x,y,z,t)>=\psi^2(x,y,z,t)|\psi(x,y,z,t)> \tag{8.2.4}$$

The problem is that the lhs is zero by the anticommutatation property assumed for the fermion field operator, but the rhs is apparently non-zero. In order to overcome this problem we must no longer regard the field eigenvalue $\psi(x,y,z,t)$ as being a commuting c-number, but instead regard it as an *anticommuting Grassmann variable* which obeys the anticommutation relations:

$$\{\psi(x),\psi(y)\}=0 \tag{8.2.5}$$

Then the product of two identical field eigenvalues is also zero and the previous problem is overcome.

What exactly are Grassmann variables? Well they are well defined mathematical objects which multiply operators and vectors just as if they were c-numbers, but have the property that they anticommute amongst themselves. They are discussed further in the Appendix. Since anticommuting field operators implies Grassmann field eigenvalues, the classical spinor field must be regarded as Grassmannian. Thus the classical Hamiltonian (i.e. the energy) becomes neither positive nor negative but rather as function of Grassmann variables, and this solves its positivity problem.

Clearly we must quantise the Dirac fields using functional methods based on spinor fields corresponding to Grassmann variables. By analogy with Eq.6.1.11 the generating functional for the free Dirac field is:

$$Z_0[\eta,\overline{\eta}]\propto\int\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{i(\int d^4x(\mathcal{L}+\overline{\eta}\psi+\overline{\psi}\eta))} \tag{8.2.6}$$

where we have introduced a source term $\overline{\eta}(x)$ for $\psi(x)$ and another one $\eta(x)$ for $\overline{\psi}(x)$, and both the sources and the fields are Grassmann variables. The Lagrangian is just the free Dirac Lagrangian introduced earlier.

By analogy with Eq.6.1.13 the Green's functions may be defined by the VEV of the Heisenberg field operators

$$\mathcal{G}(x_1,\cdots,x_n;y_1,\cdots,y_n)\equiv<0|T(\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)\overline{\hat{\psi}}(y_1)\cdots\overline{\hat{\psi}}(y_n))|0> \tag{8.2.7}$$

where the time ordering puts earlier times to the right as usual but in doing so we must introduce a minus sign every time two fermion fields are anticommuted. Again by analogy with Eq.6.1.14 the Green's functions are given from the generating functional as,

$$i^n\mathcal{G}(x_1,\cdots,x_n;y_1,\cdots,y_n)=$$

$$\left[\frac{\delta^{2n}Z_0[\eta,\overline{\eta}]}{\delta\overline{\eta}(x_1)\cdots\delta\overline{\eta}(x_n)\delta\eta(y_1)\cdots\delta\eta(y_n)}\right]_{\eta=0,\overline{\eta}=0} \tag{8.2.8}$$

8.3 The Feynman Propagator for the Dirac Field

We are familiar with the Feynman propagator of the scalar field as being the VEV of the time ordered product of two Heisenberg fields in the free field theory (or of two interaction picture fields in the interacting field theory which amounts to the same thing). In the language of Green's functions, the Feynman propagator is just the two point Green's function in the free field theory. Previously we calculated the Feynman propagator in two ways: by using canonical methods in section 3.5, and using functional methods in section 6.2. Here we shall use the functional approach, following the steps analagous to section 6.2.

We first consider the free Dirac field Lagrangian density

$$\mathcal{L} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{8.3.1}$$

The path integral expression for the generating functional is explictly:

$$Z_0[\eta,\overline{\eta}] \propto \int \mathcal{D}\psi\mathcal{D}\overline{\psi}$$

$$\exp\left(i\int d^4x\left[\overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) + \overline{\eta}\psi + \overline{\psi}\eta\right]\right)$$

As in the scalar case, in order to identify the Feynman propagator, we wish to perform the functional integral over the fields, and as in the previous case it is possible to do so for the free field theory. We first recall the Grassmannian version of the most important integral in the world from the Appendix:

$$Z_0' = \int\prod_{i=1}^n da_i e^{-\sum_{i,j=1}^n a_i K_{ij} a_j + \sum_{k=1}^n \eta_k a_k} \tag{8.3.2}$$

$$Z_0' = \sqrt{det(K)}e^{-\frac{1}{4}\sum_{i,j=1}^n \eta_i (K^{-1})_{ij}\eta_j} \tag{8.3.3}$$

The continuum version of this is:

$$Z_0[\eta] = \int \mathcal{D}\psi\exp(-\frac{1}{2}\int d^4xd^4x'\left[\psi(x')A(x',x)\psi(x)\right] + \int d^4x\eta(x)\psi(x))$$

$$Z_0[\eta] = \sqrt{det(A)}\exp(-\frac{1}{2}\int d^4xd^4x'\left[\eta(x')A^{-1}(x',x)\eta(x)\right]$$

This applies to real Grassmannian functions. For complex Grassmannian functions we need the following generalisation:

$$Z_0[\eta,\eta^*] = \int \mathcal{D}\psi\mathcal{D}\psi^*$$

$$\exp\left(i\int d^4xd^4x'\left[\psi^*(x')A(x',x)\psi(x)\right] + i\int d^4x\eta^*(x)\psi(x) + \psi^*(x)\eta(x)\right)$$

$$Z_0[\eta,\eta^*] = \sqrt{det(iA)}\exp(-i\int d^4xd^4x'\left[\eta^*(x')A^{-1}(x',x)\eta(x)\right] \tag{8.3.4}$$

In order to perform the functional integrations we first write the generating functional as:

$$Z_0[\eta,\overline{\eta}] \propto \int \mathcal{D}\psi\mathcal{D}\overline{\psi}$$
$$\exp(i\int d^4xd^4x'\{\overline{\psi}(x')\left[\delta^4(x-x')(i\gamma^\mu\partial^x_\mu - m)\right]\psi(x)$$
$$+i\int d^4x\overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x)\})$$

We identify the operator in square brackets as $A(x',x)$,

$$A(x',x) \equiv \left[\delta^4(x-x')(i\gamma^\mu\partial^x_\mu - m)\right] \tag{8.3.5}$$

and using Eq.8.3.4, replacing ψ^* by $\overline{\psi} = \psi^\dagger\gamma^0$, and similarly for η we find

$$Z_0[\eta,\overline{\eta}] \propto \sqrt{det(iA)}\exp(-i\int d^4xd^4x'\,[\overline{\eta}(x')S_F(x'-x)\eta(x)] \tag{8.3.6}$$

where

$$S_F(x'-x) = A^{-1}(x',x) \tag{8.3.7}$$

is the Feynman propagator for the Dirac field, as is readily verified by constructing the two point Green's function from Eq.8.3.6.

It only remains to explicitly find an expression for the Feynman propagator. From above we see that

$$S_F(x'-x) = \left[\delta^4(x-x')(i\gamma^\mu\partial^x_\mu - m)\right]^{-1} \tag{8.3.8}$$

Clearly we have

$$\int d^4y[S_F(x,y)]^{-1}S_F(y,z) = \delta(x-z)$$

Hence from its definition we have

$$\int d^4y\left[\delta^4(x-y)(i\gamma^\mu\partial^x_\mu - m)\right]S_F(y,z) = \delta(x-z)$$

Integrating over y, we find the equation which must be satisfied by $S_F(x,z)$,

$$(i\gamma^\mu\partial^x_\mu - m)S_F(x,z) = \delta(x-z) \tag{8.3.9}$$

The solution to this equation is formally

$$S_F(x,z) = \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}+m}{k^2-m^2}e^{-ik.(x-z)}$$

as may be verified by substitution. As in the scalar propagator case in section 6.2 the result is ambiguous due to the poles, and as in that case the resolution to the problem is to remember that the generating functional is strictly only defined off the real axis. The analysis is identical to that in section 6.2 and the result is:

$$S_F(x-z) = \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}+m}{k^2-m^2+i\epsilon}e^{-ik.(x-z)} \tag{8.3.10}$$

The Fourier transform of the Feynman propagator is thus

$$S_F(k) = \frac{\not{k}+m}{k^2-m^2+i\epsilon} \tag{8.3.11}$$

8.4 Appendix: Grassmann Variables

To begin with consider two Grassmann variables a, b which satisfy the anticommutation relations:

$$\{a, a\} = 0, \quad \{a, b\} = 0, \quad \{b, b\} = 0 \tag{8.4.1}$$

They are not operators; they do not act on any vectors in a vector space. However they do obey an algebra: the Grassmann algebra indicated above. We can think of them as "operators which do not act on anything" if we like (although please don't say this in the presence of a mathematician unless you want to see him or her turn blue.)

Without loss of generality any function of the two Grassmann variables may be written as:

$$f(a, b) = f_0 + f_1 a + \tilde{f}_1 b + f_2 ab \tag{8.4.2}$$

where the f's are non-Grassmannian c-numbers. The *left* derivatives of the function are:

$$\frac{\partial f}{\partial a} = f_1 + f_2 b \tag{8.4.3}$$

$$\frac{\partial f}{\partial b} = \tilde{f}_1 - f_2 a \tag{8.4.4}$$

so called because the infinitesimal denominator is really an inverse Grassmann variable which acts on the *left* in this case. Unless otherwise stated all our derivatives will be *left* derivatives. We find:

$$\frac{\partial^2 f}{\partial a \partial b} = -\frac{\partial^2 f}{\partial b \partial a} = -f_2 \tag{8.4.5}$$

We may also define integration with respect to Grassmann variables. The simplest Grassmann integral is zero:

$$\int da = 0 \tag{8.4.6}$$

(to prove this first show that $(\int da)^2 = -(\int da)^2$). The next simplest Grassmann integral is equal to unity:

$$\int daa = 1 \tag{8.4.7}$$

(in fact since $a^2 = 0$ the above integral serves as a sort of normalisation of the Grassmann variables.) Eqs.8.4.6 and 8.4.7 show that Grassmann integration has the same effect as differentiation.

Using Eq.8.4.2 we find that

$$\int dadb f(a, b) = \frac{\partial^2 f}{\partial a \partial b} \tag{8.4.8}$$

An interesting result is:

$$\int dadbe^{-ba} = \int dadb(1-ba) = 1 \tag{8.4.9}$$

Finally recall the most important integral in the world from Eq.5.6.10:

$$Z_0 = \int_{-\infty}^{\infty} \prod_{i=1}^{n} dq_i e^{-\sum_{i,j=1}^{n} q_i K_{ij} q_j + \sum_{k=1}^{n} J_k q_k} \tag{8.4.10}$$

This is an n dimensional integral and K_{ij} is an $n \times n$ symmetric matrix, while J_k is an n component column vector. Amazingly the above integral can be done for all invertible K_{ij}:

$$Z_0 = \frac{\pi^{n/2}}{\sqrt{det(K)}} e^{+\frac{1}{4}\sum_{i,j=1}^{n} J_i (K^{-1})_{ij} J_j} \tag{8.4.11}$$

Now we consider the analagous integral but involving Grassmann variables $a_1, \cdots a_n$ and Grassmann variables $\eta_1, \cdots \eta_n$,

$$Z_0 = \int \prod_{i=1}^{n} da_i e^{-\sum_{i,j=1}^{n} a_i K_{ij} a_j + \sum_{k=1}^{n} \eta_k a_k} \tag{8.4.12}$$

and we assume that K_{ij} is now *antisymmetric* rather than symmetric otherwise we would get a zero result. For a similar reason we must assume that n is even. In this case the result is:

$$Z_0 = \sqrt{det(K)} e^{-\frac{1}{4}\sum_{i,j=1}^{n} \eta_i (K^{-1})_{ij} \eta_j} \tag{8.4.13}$$

The proof of this is found in Bailin and Love 8.1.

Chapter 9

The Free Electromagnetic Field

9.1 The Classical Electromagnetic Field

The four Maxwell equations are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

It is straightforward to show that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

In covariant form,

$$\partial_\mu j^\mu = 0$$

where $j^\mu = (c\rho, \mathbf{j})$.

It is convenient (and even essential) to introduce scalar and vector potentials ϕ and $\mathbf{A}$ by defining

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t.$$

whence two of the Maxwell equations become automatic.

Recall the gauge invariance of electrodynamics which says that $\mathbf{E}$ and $\mathbf{B}$ are unchanged when

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda \quad \text{and} \quad \phi \to \phi - \frac{\partial \Lambda}{\partial t}$$

for any scalar function Λ. Gauge invariance corresponds to a lack of uniqueness of the scalar and vector potentials. This lack of uniqueness can be reduced by imposing a further condition on the scalar and vector potentials, for example

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

Assuming that ϕ and $\mathbf{A}$ can be combined into a four vector

$$A^\mu = (\phi/c, \mathbf{A})$$

this can be written as

$$\partial_\mu A^\mu = 0 \tag{9.1.1}$$

which is known as the *Lorentz gauge* condition. Gauge invariance in four-vector notation is just:

$$A^\mu \to A^\mu + \partial_\mu \Lambda \tag{9.1.2}$$

Note that even the imposition of the Lorentz gauge condition does not completely fix the vector potential; it merely restricts the function Λ to satisfy

$$\partial^2 \Lambda = 0 \tag{9.1.3}$$

With the Lorentz gauge condition Maxwell's equations are equivalent to

$$\partial^2 A^\mu = \mu_0 j^\mu$$

The tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ clearly has six independent components, and can be written:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

It is straightforward to show that,

$$F_{\mu\nu}F^{\mu\nu} = -2\left(\frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2\right)$$

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = -\frac{8}{c}\mathbf{E}\cdot\mathbf{B}$$

where

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of 0123} \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise} \end{cases}$$

This gives the relativistic invariants which can be constructed from $\mathbf{E}$ and $\mathbf{B}$.

It is easy to see that in any gauge the Maxwell equations can be written,

$$\partial_\mu F^{\mu\nu} = j^\nu$$

The Maxwell equations, in this compact form, can be reproduced by the following Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \tag{9.1.4}$$

via the Euler-Lagrange equations for each of the four A_μ fields separately.

In Lorentz gauge the Lagrangian density has the more general form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.1.5}$$

where ξ is a free parameter. The EL equations then imply

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi}\partial^\nu(\partial_\mu A^\mu) = j^\nu \tag{9.1.6}$$

which reduce to Maxwell's equations in Lorentz gauge. The extra term in the Lagrangian density $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ thus has no effect on physics in Lorentz gauge. In fact it is possible to turn the argument around and use this term to fix the gauge to be Lorentz gauge by imposing current conservation instead of obtaining it as a consequence of Maxwell's equations. If one adds the extra term to the Lagrangian and imposes current conservation then Eq.9.1.6 implies immediately the Lorentz gauge condition by the antisymmetry of $F^{\mu\nu}$. For this reason the extra term is referred to as a *gauge fixing term* and ξ is a Lagrange multiplier. The choice $\xi = 1$ is known as Feynman gauge although it is within the framework of the Lorentz gauge.

As usual we can expand the field $A_\mu(x)$ in its Fourier components

$$A_\mu(x)(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a_\mu(k)e^{-ik.x} + a^*_\mu(k)e^{ik.x}] \tag{9.1.7}$$

where $\omega = k_0 = |\mathbf{k}|$. The Lorentz gauge condition implies

$$k.a(k) = 0 \tag{9.1.8}$$

This implies that

$$a_0(k) = \hat{\mathbf{k}}.\mathbf{a}(k)$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. Thus the time component of a_μ equals the longitudinal component $\mathbf{a}.\mathbf{k}$. Of course only the transverse components are physical (since the $\mathbf{E}$ and $\mathbf{B}$ fields are always orthogonal to the three momentum) and it can be shown that the contribution to the Hamiltonian from the time component and longitudinal component cancel against each other. In fact it is possible to completely specify the gauge by requiring that

$$a_0(k) = \hat{\mathbf{k}}.\mathbf{a}(k) = 0$$

which is called Coulomb gauge. In Coulomb gauge we can write

$$a_\mu(k) = \sum_{\lambda=1,2} a^\lambda(k)\epsilon^\lambda_\mu(k)$$

where $\epsilon^\lambda_\mu(k)$ are two orthonormal spacelike vectors in the plane transverse to $\mathbf{k}$.

In a general Lorentz gauge we can write:

$$a_\mu(k) = \sum_{\lambda=0,1,2,3} a^\lambda(k)\epsilon^\lambda_\mu(k)$$

where now $\epsilon^\lambda_\mu(k)$ are arbitrary unit four-vectors. Suppose that $\mathbf{k}$ is along the third axis, $k = (\omega, 0, 0, \omega)$ then we can define the basis vectors as:

$$\epsilon^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{9.1.9}$$

so that we call $\lambda = 1, 2$ the physical transverse polarisations, $\lambda = 0$ the unphysical timelike polarisation and $\lambda = 3$ the unphysical longitudinal polarisation. Clearly,

$$k.\epsilon^{1,2} = 0$$

and

$$\epsilon^\lambda.\epsilon^{\lambda'} = g^{\lambda\lambda'}$$

which is in fact a basis independent result, although we shall always work in this basis.

9.2 Canonical Quantisation

We shall now quantise the free e.m. theory ($j^\mu = 0$). To quantise the theory canonically we introduce the canonical momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \tag{9.2.1}$$

and impose the equal time covariant canonical commutation relations

$$[A_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{x}', t)] = i g_{\mu\nu}\delta^3(\mathbf{x} - \mathbf{x}') \tag{9.2.2}$$

$$[A_\mu(\mathbf{x}, t), A_\nu(\mathbf{x}', t)] = 0 \tag{9.2.3}$$

$$[\pi_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{x}', t)] = 0 \tag{9.2.4}$$

Now if the Lagrangian were simply

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.2.5}$$

then we would find that

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

which would imply that π^0 always commutes with A^0, which loses us both covariance and quantum mechanics at a stroke!

We clearly need a π^0 that does not vanish. In order to do this we need to change the Lagrangian without changing the physics. But we have learned how to do this in Lorentz gauge which corresponds to the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.2.6}$$

and the field equations:

$$\partial^2 A_\mu - (1 - \frac{1}{\xi})\partial_\mu(\partial_\nu A^\nu) = 0 \tag{9.2.7}$$

Henceforth for simplicity we shall take $\xi = 1$ which is called Feynman gauge (a sub-class of Lorentz gauge).

At first sight this doesn't help us because we find

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu$$

which apparently vanishes in Lorentz gauge. However we shall only assume that matrix elements of $\partial_\mu A^\mu$ vanish rather than imposing the operator condition that it vanish.

In Feynman gauge we have the field equations:

$$\partial^2 A_\mu = 0 \tag{9.2.8}$$

and we can once again expand the A_μ field in plane wave solutions similar to the previous section:

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} \sum_{\lambda=0}^{3} [\epsilon_\mu^\lambda(k) a^\lambda(k) e^{-ik.x} + \epsilon_\mu^{\lambda *}(k) a^\lambda(k)^\dagger e^{ik.x}] \tag{9.2.9}$$

Here $\epsilon_\mu^\lambda(k)$ are the set of four linearly independent vectors defined in Eq.9.1.9, but now we regard a and its hermitian conjugate as operators whose commutation relations readily follow from Eq.9.2.2

$$[a^\lambda(k), a^{\lambda'}(k')^\dagger] = -g^{\lambda\lambda'} 2k_0 (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \tag{9.2.10}$$

For longitudinal and transverse photons quantisation proceeds in the usual way. But for timelike photons with $\lambda = \lambda' = 0$ we have a negative quantity on the rhs which gives problems. This leads to timelike photons with negative norm. However it is possible to overcome these problems using the Gupta-Bleuler formalism. However at this point we prefer to abandon the canonical approach and move on to the path integral approach which has its own problems.

9.3 Path Integral Quantisation

We have seen that the freedom to make gauge transformations means that the A^μ fields are not uniquely specified, and this causes problems with the theory in the canonical formalism. It should be no surprise that these problems persist in the path integral approach.

The generating functional in this case is

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x(\mathcal{L}+J^\mu A_\mu\right)} \tag{9.3.1}$$

where $\mathcal{L}$ is the Lagrangian for the free photon field which we might naively take to be

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.3.2}$$

(since we have already found problems with this form in the canonical formalism it really is naive to expect it to work here). The field equations in this case are as in Eq.9.1.6

$$\partial_\mu F^{\mu\nu} = 0 \tag{9.3.3}$$

which can be written as

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\mu = 0 \tag{9.3.4}$$

After partial integration and discarding surface terms we can write the generating functional as

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x \frac{1}{2}A^\mu[g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu]A^\nu+J^\mu A_\mu\right)} \tag{9.3.5}$$

By now we know that the photon propagator $D_{\mu\nu}$ is going to be the inverse of the operator in square brackets, and it will satisfy the equation:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)D^{\nu\lambda}(x-y) = \delta^\lambda_\mu\delta^4(x-y) \tag{9.3.6}$$

If we multiply this equation by ∂^μ we get zero multiplying $D^{\nu\lambda}(x-y)$ on the lhs and something non-zero on thhe rhs, which would seem to imply that $D^{\nu\lambda}(x-y)$ is infinite. In fact the problem is that the operator in square brackets does not have an inverse! To show this all we need to do is show that it has a zero eigenvalue, and this can easily be done:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\partial^\mu\Omega = 0$$

for any function Ω.

From the point of view of the path integral the problem is that the functional integral is taken over all A_μ including those related by a gauge transformation, leading to an infinite overcounting in the calculation of the generating functional, and hence an infinite overcounting for the Green's functions which are obtained from it by functional differentiation. To cure this problem we

need to fix a particular gauge, and we do this by imposing the Lorentz gauge condition:

$$\partial_\mu A^\mu = 0 \tag{9.3.7}$$

Recall the Lagrangian with gauge fixing term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.3.8}$$

and the field equations:

$$\partial^2 A_\mu - (1 - \frac{1}{\xi})\partial_\mu(\partial_\nu A^\nu) = 0 \tag{9.3.9}$$

After partial integration and discarding surface terms we can now write the generating functional as

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x \frac{1}{2}A^\mu[g_{\mu\nu}\partial^2 + (\frac{1}{\xi}-1)\partial_\mu\partial_\nu]A^\nu + J^\mu A_\mu\right)} \tag{9.3.10}$$

and the operator in square brackets now has an inverse given by

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} - \frac{\left[g_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\epsilon} e^{-ik.(x-y)} \tag{9.3.11}$$

The Fourier transform of the Feynman propagator is thus

$$D_{\mu\nu}(k) = -\frac{\left[g_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\epsilon} \tag{9.3.12}$$

Amongst this class of gauge choices two common choices are Feynman gauge ($\xi = 1$) and Landau gauge ($\xi = 0$).

Chapter 10

Quantum Electrodynamics

10.1 Feynman Rules of QED

QED involves the interaction of electrons and photons where the interaction corresponds to the Lagrangian

$$\mathcal{L}_{int} = -e\overline{\psi}\gamma^{\mu}A_{\mu}\psi. \tag{10.1.1}$$

Such an interaction may be introduced by the concept of "minimal substitution" familiar from classical electrodynamics. The momentum and energy become:

$$\mathbf{p} \to \mathbf{p} - e\mathbf{A}$$

$$E \to E - e\phi$$

or in four vector notation, the four momentum becomes:

$$p^{\mu} \to p^{\mu} - eA^{\mu}$$

Applying this classical concept of minimal substitution to the Dirac equation gives:

$$(i\not{D} - m)\psi = 0 \tag{10.1.2}$$

where we have introduced the covariant derivative notation

$$D_{\mu} \equiv \partial_{\mu} + ieA_{\mu}$$

The QED Lagrangian describing electrons, photons and their interactions is then given by,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_{\mu}A^{\mu})^2 + \overline{\psi}(i\not{D} - m)\psi. \tag{10.1.3}$$

Here, $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ is the electromagnetic covariant derivative, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and $(\partial \cdot A)^2/2$ is the gauge fixing term for Feynman gauge.

The QED Lagrangian is invariant under a symmetry called *gauge symmetry*, which consists of the simultaneous gauge transformations of the photon field:

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{10.1.4}$$

and a phase transformation on the electron field

$$\psi \to e^{-ie\Lambda}\psi \tag{10.1.5}$$

The point is that the value of the phase transformation given by the same gauge function $\Lambda(x)$ as controls the photon gauge transformation. It is important to emphasise that $\Lambda(x)$ is a function of x so that the action of a derivative on $e^{-ie\Lambda}\psi$ will yield two terms by the product rule. However the simultaneous gauge transformation of the photon field means that the covariant derivative of ψ transforms like ψ itself under the combined gauge transformations above:

$$D_\mu \psi \to e^{-ie\Lambda} D_\mu \psi \tag{10.1.6}$$

Thus the QED Lagrangian is invariant under the simultaneous transformations above, referred to collectively as a gauge transformation.

In this chapter we are going to get some practice calculating cross sections and decay rates in QED. The starting point is the set of Feynman rules in Table 10.1.1 derived from the QED Lagrangian above. The fermion propagator is (up to factors of i) the inverse of the operator, $\not{p} - m$, which appears in the quadratic term in the fermion fields, as we saw in section 8.3. The derivation of the photon propagator, along with the need for gauge fixing, was discussed in section 9.3. The external line factors are easily derived by considering simple matrix elements in the operator formalism, where they are left behind from the expansions of fields in terms of annihilation and creation operators, after the operators have all been (anti-)commuted until they annihilate the vacuum. One could consider for example the process $\gamma \to e^+e^-$ in complete analogy to the example in section 3.4, working in the interaction picture. In path integral language the natural objects to compute are Green functions, vacuum expectation values of time ordered products of fields: it takes a little more work to convert them to transition amplitudes and see the external line factors appear, but the proceedure is exactly analagous to $\lambda\phi^4$ theory in section 6.4.

The spinor indices in the Feynman rules are such that matrix multiplication is performed in the opposite order to that defining the flow of fermion number. The arrow on the fermion line itself denotes the fermion number flow, *not* the direction of the momentum associated with the line: I will try always to indicate the momentum flow separately as in Table 10.1.1. This will become clear in the examples which follow. We have already met the Dirac spinors u and v. I will say more about the photon polarisation vector ϵ when we need to use it.

10.2 Electron–Muon Scattering

To lowest order in the electromagnetic coupling, just one diagram contributes to this process. It is shown in Figure 10.2.1. The amplitude obtained from this diagram is

$$i\mathcal{M}_{fi} = (-ie)\,\overline{u}(p_c)\gamma^\mu u(p_a)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(-ie)\,\overline{u}(p_d)\gamma_\nu u(p_b). \tag{10.2.1}$$

Note that I have changed my notation for the spinors: now I label their momentum as an argument instead of as a subscript, and I drop the spin label unless I need to use it. In constructing this amplitude we have followed the fermion lines backwards with respect to fermion flow when working out the order of matrix multiplication.

The cross-section involves the squared modulus of the amplitude, which is

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}\,L^{\mu\nu}_{(e)}L_{(\mu)\,\mu\nu},$$

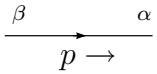
every ...	draw ...	write ...
nal photon line		$\frac{-ig^{\mu\nu}}{q^2+i\epsilon}$
nal fermion line		$\frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i}$
ex		$-ie\gamma^\mu_{\alpha\beta}$
going electron		$\overline{u}^s_p$
ming electron		u^s_p
going positron		v^s_p
ming positron		$\overline{v}^s_p$
going photon		$\epsilon^{*\mu}$
ming photon		ϵ^μ
tach a directed momentum to every internal line		
nserve momentum at every vertex		

Table 10.1.1 Feynman rules for QED. μ, ν are Lorentz indices and α, β are spinor indices.

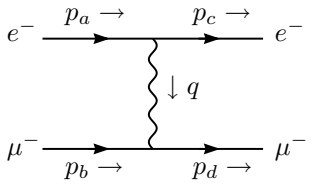


Figure 10.2.1 Lowest order Feynman diagram for electron–muon scattering.

where the subscripts e and μ refer to the electron and muon respectively and,

$$L^{\mu\nu}_{(e)} = \overline{u}(p_c)\gamma^\mu u(p_a)\overline{u}(p_a)\gamma^\nu u(p_c),$$

with a similar expression for $L^{\mu\nu}_{(\mu)}$.

▷ **Exercise 10.2.1**
Verify the expression for $|\mathcal{M}_{fi}|^2$.

Usually we have an unpolarised beam and target and do not measure the polarisation of the outgoing particles. Thus we calculate the squared amplitudes for each possible spin combination, then average over initial spin states and sum over final spin states. Note that we square and then sum since the different possibilities are in principle distinguishable. In contrast, if several Feynman diagrams contribute to the same process, you have to sum the amplitudes first. We will see examples of this below.

The spin sums are made easy by the following results (I temporarily restore spin labels on spinors):

$$\begin{aligned} \sum_r u^r(p)\,\overline{u}^r(p) &= \not{p} + m \\ \sum_r v^r(p)\,\overline{v}^r(p) &= \not{p} - m \end{aligned} \tag{10.2.2}$$

▷ **Exercise 10.2.2**
Derive the spin sum relations in equation (10.2.2).

Using the spin sums we find,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{4q^4}\ \text{tr}\left(\gamma^\mu(\not{p}_a + m_e)\gamma^\nu(\not{p}_c + m_e)\right)\text{tr}\left(\gamma_\mu(\not{p}_b + m_\mu)\gamma_\nu(\not{p}_d + m_\mu)\right).$$

Since all calculations of cross sections or decay rates in QED require the evaluation of traces of products of gamma matrices, you will generally find a table of "trace theorems" in any quantum field theory textbook [1]. All these theorems can be derived from the fundamental anticommutation relations of the gamma matrices in equation (7.5.4) together with the invariance of the trace under a cyclic change of its arguments. For now it suffices to use,

$$\begin{aligned} \text{tr}(\not{a}\not{b}) &= 4\,a{\cdot}b \\ \text{tr}(\not{a}\not{b}\not{c}\not{d}) &= 4(a{\cdot}b\,c{\cdot}d - a{\cdot}c\,b{\cdot}d + a{\cdot}d\,b{\cdot}c) \\ \text{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 0 \quad \text{for } n \text{ odd} \end{aligned} \tag{10.2.3}$$

Figure 10.3.1 Lowest order Feynman diagrams for electron–electron scattering.

▷ **Exercise 10.2.3**
Derive the trace results in equation (10.2.3)

Using these results, and expressing the answer in terms of the Mandelstam variables of equation (4.5.1), we find,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = \frac{2e^4}{t^2}\Big(s^2+u^2-4(m_e^2+m_\mu^2)(s+u)+6(m_e^2+m_\mu^2)^2\Big).$$

This can now be used in the $2 \rightarrow 2$ cross section formula (4.8.3) to give, in the high energy limit, $s, u \gg m_e^2, m_\mu^2$,

$$\frac{d\sigma}{d\Omega^*} = \frac{e^4}{32\pi^2 s}\,\frac{s^2+u^2}{t^2}. \tag{10.2.4}$$

for the differential cross section in the centre of mass frame.

▷ **Exercise 10.2.4**
Derive the result for the electron–muon scattering cross section in equation (10.2.4).

Other calculations of cross sections or decay rates will follow the same steps we have used above. You draw the diagrams, write down the amplitude, square it and evaluate the traces (if you are using spin sum/averages). There are one or two more wrinkles to be aware of, which we will meet below.

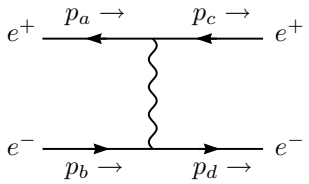
10.3 Electron–Electron Scattering

Since the two scattered particles are now identical, you can't just replace m_μ by m_e in the calculation we did above. If you look at the diagram of Figure 10.2.1 (with the muons replaced by electrons) you will see that the outgoing legs can be labelled in two ways. Hence we get the two diagrams of Figure 10.3.1.

The two diagrams give the amplitudes,

$$\begin{aligned} i\mathcal{M}_1 &= \frac{ie^2}{t}\overline{u}(p_c)\gamma^\mu u(p_a)\overline{u}(p_d)\gamma_\mu u(p_b), \\ i\mathcal{M}_2 &= -\frac{ie^2}{u}\overline{u}(p_d)\gamma^\mu u(p_a)\overline{u}(p_c)\gamma_\mu u(p_b). \end{aligned}$$

Notice the additional minus sign in the second amplitude, which comes from the anticommuting nature of fermion fields. You should accept as part of the



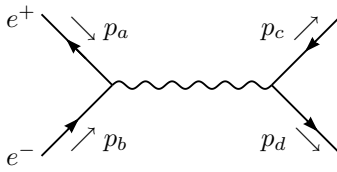


Figure 10.4.1 Lowest order Feynman diagrams for electron–positron scattering in QED.

Feynman rules for QED that when diagrams differ by an interchange of two fermion lines, a relative minus sign must be included. This is important because

$$|\mathcal{M}_{fi}|^2 = |\mathcal{M}_1 + \mathcal{M}_2|^2,$$

so the interference term will have the wrong sign if you don't include the extra sign difference between the two diagrams.

10.4 Electron–Positron Annihilation

10.4.1 $e^+e^- \to e^+e^-$

For this process the two diagrams are shown in Figure 10.4.1, with the one on the right known as the annihilation diagram. They are just what you get from the diagrams for electron–electron scattering in Figure 10.3.1 if you twist round the fermion lines. The fact that the diagrams are related this way implies a relation between the amplitudes. The interchange of incoming particles/antiparticles with outgoing antiparticles/particles is called *crossing*. This is a case where the general results of crossing symmetry can be applied, and our diagrammatic calculations give an explicit realisation. Theorists spent a great deal of time studying such general properties of amplitudes in the 1960's when quantum field theory was unfashionable.

10.4.2 $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to$ hadrons

If electrons and positrons collide and produce muon–antimuon or quark-antiquark pairs, then the annihilation diagram is the only one which contributes. At sufficiently high energies that the quark masses can be neglected, this immediately gives the lowest order QED prediction for the ratio of the annihilation cross section into hadrons to that into $\mu^+\mu^-$,

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_f Q_f^2, \tag{10.4.1}$$

where the sum is over quark flavours f and Q_f is the quark's charge in units of e. The 3 comes from the existence of three colours for each flavour of quark. Historically this was important: you could look for a step in the value of R as

your e^+e^- collider's CM energy rose through a threshold for producing a new quark flavour. If you didn't know about colour, the height of the step would seem too large. Incidentally, another place the number of colours enters is in the decay of a π^0 to two photons. There is a factor of 3 in the amplitude from summing over colours, without which the predicted decay rate would be one ninth of its real size.

At the energies used today at LEP, of course, you have to remember the diagram with a Z replacing the photon. We will say some more about this later.

▷ **Exercise 10.4.1**
Show that the cross-section for $e^+e^- \to \mu^+\mu^-$ is equal to $4\pi\alpha^2/(3s)$, neglecting the lepton masses.

10.5 Compton Scattering

The diagrams which need to be evaluated to compute the Compton cross section for $\gamma e \to \gamma e$ are shown in Figure 10.5.1. For unpolarised initial and/or final states, the cross section calculation involves terms of the form

$$\sum_\lambda \epsilon^{*\mu}_\lambda(k)\, \epsilon^\nu_\lambda(k), \tag{10.5.1}$$

where λ represents the polarisation of the photon of momentum k. Since the photon is massless, the sum is over the two transverse polarisation states, and must vanish when contracted with k_μ or k_ν. In addition, however, since the photon is coupled to the electromagnetic current $J^\mu = \overline{\psi}\gamma^\mu\psi$ of equation (7.2.7), any term in the polarisation sum (10.5.1) proportional to k^μ or k^ν does not contribute to the cross section. This is because the current is conserved, $\partial_\mu J^\mu = 0$, so in momentum space $k_\mu J^\mu = 0$. The upshot is that in calculations you can use,

$$\sum_\lambda \epsilon^{*\mu}_\lambda(k)\, \epsilon^\nu_\lambda(k) = -g^{\mu\nu}, \tag{10.5.2}$$

since the remaining terms on the right hand side do not contribute.

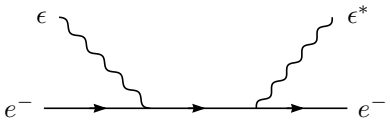
The more precise statement is that

$$\sum_\lambda \epsilon^{*\mu}_\lambda(k) a_\mu\, \epsilon^\nu_\lambda(k) b_\nu = -a.b, \tag{10.5.3}$$

provided that a_μ and b_μ are conserved currents, that is $k.a(k) = k.b(k) = 0$. For example if

$$a_\nu = \overline{u}(p_1)\gamma_\nu u(p_3)$$

and $k = p_1 - p_3$ then by using the Dirac equation it follows that $k.a = 0$. Current conservation is also related to gauge invariance of the amplitude. For instance if we worked in a general gauge other than Feynman gauge then any diagram



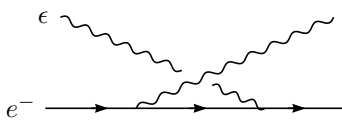


Figure 10.5.1 Feynman diagrams for Compton scattering.

with an internal photon line would contain an extra term in the numerator of the propagator of the form:

$$(\xi - 1)\frac{k^\mu k^\nu}{k^2}$$

However current conservation guarantees that such a term vanishes leading to an amplitude which does not depend upon the value of ξ and hence is independent of the choice of gauge, or gauge invariant. Gauge invariance is a powerful requirement on the theory, and must be maintained to all orders in perturbation theory. However current conservation by itself is insufficient to guarantee gauge invariance beyond the tree-level. The general conditions which guarantee gauge invariance to all orders are called Ward Identities, but this is a topic for Quantum Field Theory II.

Bibliography

[1] I J R Aitchison and A J G Hey *Gauge theories in particle physics, 2nd ed* Adam Hilger 1989
J Bjorken and S Drell *Relativistic Quantum Mechanics* McGraw-Hill 1964
F Mandl and G Shaw *Quantum Field Theory* Wiley 1984
L Ryder *Quantum Field Theory* Cambridge University Press 1985

[2] Particle Data Group, Physical Review **D50** (1994) 1173

[3] J Schwinger, Physical Review **73** (1948) 416

[4] H A Bethe, Physical Review **72** (1947) 339