MATHEMATICS REVISION NOTES:

1 Circles and Spheres

Circumference of a circle of radius R

$$C = 2\pi R$$

Area of a circle of radius R

$$A = \pi R^2$$

Surface Area of a sphere of radius R

$$A = 4\pi R^2$$

Volume of a sphere of radius R

$$V = \frac{4}{3}\pi R^3$$

2 Trigonometry

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

$$\frac{d}{d\theta}\tan\theta = \sec^2\theta = \frac{1}{\cos^2\theta}$$

$$\frac{d}{d\theta}\cot\theta = -\csc^2\theta = -\frac{1}{\sin^2\theta}$$

3 Integration

$$\int_0^a x\sin x dx = \sin a - a\cos a$$

4 Spherical Polar Coordinates

A point whose cartesian coordinates are (x, y, z) can be described in terms of spherical polar coordinates (r, θ, ϕ) where

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta, \end{aligned}$$

or inverting, we have

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

5 Volume Integrals

In Cartesian Coordinates (x, y, z)

$$dV = dx\, dy\, dz$$

In spherical polar coordinates (r, θ, ϕ)

$$dV = r^2\, dr\, \sin\theta d\theta d\phi.$$

6 Differential Equations

1.
$$\frac{dy}{dx} = -ay$$

   Solution

$$y = y(0)\exp(-ax)$$

2.
$$\frac{dy}{dx} = ce^{-bx} - ay$$

   Particular solution:
   Search for a solution of the form

$$y = Ae^{-bx}.$$

Inserting this into the differential equation gives

$$-b\,A\,e^{-bx} \;=\; ce^{-bx} - A\,a\,e^{-bx},$$

and matching coefficients we get

$$A \;=\; \frac{c}{(a-b)}$$

To this we may add the general solution to

$$\frac{dy}{dx} \;=\; -ay,$$

which is

$$B\,e^{-ax},$$

so that the general solution is

$$y \;=\; \frac{c}{(a-b)}e^{-bx} \;+\; Be^{-ax}.$$

If we have the initial condition that $y = 0$ when $x = 0$ we can determine the arbitrary constant B and arrive at the solution

$$y \;=\; \frac{c}{(a-b)}\left(e^{-bx} \;-\; e^{-ax}\right).$$

3.

$$\frac{dy}{dx} \;=\; b \;-\; a\,y$$

We may rewrite this as

$$\frac{dy}{(b-ay)} \;=\; dx$$

Integrating both sides gives

$$-\frac{1}{a}\ln\left(\frac{b-ay}{B}\right) \;=\; x,$$

where B is an arbitrary constant of integration. This can be rearranged to

$$\ln\left(\frac{b-ay}{B}\right) \;=\; -ax$$

Taking the exponential of both asides and rearranging terms we have

$$y \;=\; \frac{Be^{-ax}+b}{a}.$$

If $y = 0$ when $x = 0$ then we can set $B = -b$

4.

$$x\frac{dy}{dx} \;=\; -by^2$$

We rewrite this as

$$\frac{dy}{-y^2} \;=\; b\frac{dx}{x}$$

Integrating both sides and adding the constant of integration, we get

$$\frac{1}{y} \;=\; b\ln(x) \;+\; C$$

Taking the inverse

$$y \;=\; \frac{1}{(C + b\ln(x))}$$

C is determined from the boundary condition, $y = y_0$ when $x = x_0$ - we can write the solution as

$$y \;=\; \frac{1}{1/y_0 - \ln(x_0) + b\ln(x)}.$$

This may be rewritten in a simpler form as

$$y \;=\; \frac{y_0}{1 + by_0\ln(x/x_0)}$$

7 Complex Numbers

$$z \;=\; x \;+\; i\,y \;=\; |z|\,e^{i\delta},$$

where

$$|z| \;=\; \sqrt{x^2 + y^2}, \qquad \tan\delta \;=\; \frac{y}{x}$$

$$w \;=\; \frac{1}{z} \;=\; \frac{1}{x \;+\; i\,y} \;=\; \frac{x \;-\; i\,y}{x^2 + y^2} \;=\; |w|\,e^{-i\delta},$$

where

$$|w| \;=\; \frac{1}{\sqrt{x^2 + y^2}}.$$