NOTES ON FOURIER TRANSFORMS:

1 Fourier Series

Consider a function $f(x)$ defined on the domain $-L/2 \leq x \leq L/2$.

According to Fourier's theorem we may write this as

$$f(x) \;=\; \sum_{n=1}^{\infty} a_n \sin\left(\frac{2n\pi x}{L}\right) + \sum_{n=0}^{\infty} b_n \cos\left(\frac{2n\pi x}{L}\right)$$

The coefficents a_n and b_n are given by

$$a_n \;=\; \frac{1}{2L} \int_{-L/2}^{L/2} f(x) \sin\left(\frac{2n\pi x}{L}\right) dx$$

$$b_0 \;=\; \frac{1}{L} \int_{-L/2}^{L/2} f(x)\, dx$$

$$b_n \;=\; \frac{1}{L} \int_{-L/2}^{L/2} f(x) \cos\left(\frac{2n\pi x}{L}\right) dx, \quad (n > 0)$$

This can be seen from the integrals

$$\int_{-L/2}^{L/2} \cos\left(\frac{2n\pi x}{L}\right) \cos\left(\frac{2m\pi x}{L}\right) dx \;=\; \frac{L}{2}\delta_{mn}$$

$$\int_{-L/2}^{L/2} \sin\left(\frac{2n\pi x}{L}\right) \sin\left(\frac{2m\pi x}{L}\right) dx \;=\; \frac{L}{2}\delta_{mn}$$

$$\int_{-L/2}^{L/2} \sin\left(\frac{2n\pi x}{L}\right) \cos\left(\frac{2m\pi x}{L}\right) dx \;=\; 0$$

Introducing the complex coefficent

$$A_n \;=\; b_n \;-\; i\, a_n$$

and recalling that

$$\cos\left(\frac{2n\pi x}{L}\right) + i \sin\left(\frac{2n\pi x}{L}\right) \;=\; \exp\left(i\frac{2n\pi x}{L}\right),$$

we may rewrite this as

$$f(x) \;=\; \sum_{n=0}^{\infty} A_n \exp\left(i\frac{2n\pi x}{L}\right),$$

where

$$A_n \;=\; \int_{-L/2}^{L/2} f(x)\exp\left(-i\frac{2n\pi x}{L}\right)\, dx$$

This can be seen from the integral

$$\int_{-L/2}^{L/2} f(x)\exp\left(i\frac{2n\pi x}{L}\right)\exp\left(-i\frac{2m\pi x}{L}\right)\, dx \;=\; L\delta_{mn}$$

2 Fourier Transforms

Now we take $L \;\rightarrow \infty$ so that $f(x)$ is defined everywhere in x.

The intervals $\frac{n}{L}$ go to zero, so we replace $\frac{2\pi n}{L}$ by the continuous variable k, the sum over n in the Fourier series is replaced by an integral over k, and the coefficients A_n are replaced by a (complex) function $A(k)$.

Thus we have

$$f(x) \;=\; \int_{-\infty}^{\infty} A(k)e^{ikx}dk$$

where

$$A(k) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

This can be seen from the definition of the Dirac delta-functio

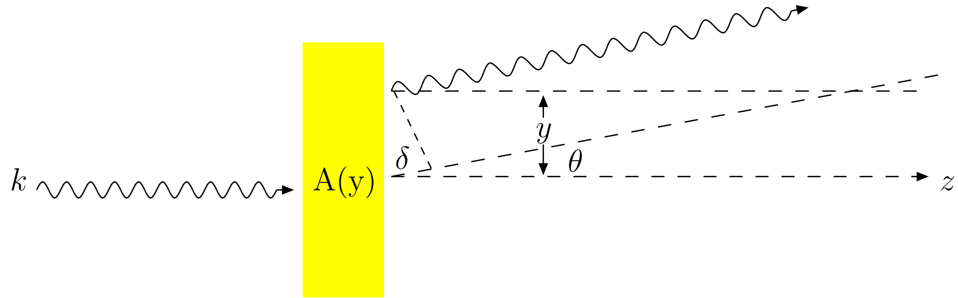
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}dx \;=\; \delta(k-k')$$

and

$$\int_{\infty}^{\infty} A(k')\delta(k-k')\, dk' \;=\; A(k)$$

3 Application Fraunhoffer Diffraction

Consider a photon of wavenumber k ($=2\pi/\lambda$), moving in the z-direction and incident upon a diffracting device which attenuates the amplitude at transverse distance y form the z-axis, by a factor $A(y)$, situated at $z = 0$.



The wave that is emitted from a distance y from the centre of the diffracting object at an angle θ to the z-axis travels a shorter distance than the wave emitted from the centre ($y = 0$) by an amount

$$\delta \;=\; y \sin\theta$$

Therefore the phase difference is

$$ky \sin\theta$$

and the amplitude for this wave is $A(y)$.

Now we sum over all the waves form all values of y, to obtain the diffraction amplitude

$$\mathcal{A}_{diff.}(\theta) \;=\; \int A(y) e^{iky\sin\theta}\, dy$$

We can write

$$q \;=\; k \sin\theta,$$

where q is the magnitude of the (vectorial) diffeence between the incoming wave-vector and the outgoing wave-vector (at angle θ), to get

$$\mathcal{A}_{diff.}(\theta) \;=\; \int A(y) e^{iqy}\, dy$$

Thus we see that the diffraction amplitude is the Fourier transform of the attentionation function of the diffracting object.

Example: The diffracting object is a slit of width d, so that

$$A(y) \;=\; 1, \;\; \left(-\frac{d}{2} \;\leq\; y \;\leq\; \frac{d}{2}\right)$$

$$A(y) \;=\; 0, \;\; \left(y \;<\; -\frac{d}{2}, \text{ or } y \;>\; \frac{d}{2}\right)$$

In this case we have

$$\mathcal{A}_{diff.}(\theta) \;=\; \int_{-d/2}^{d/2} e^{iqy}\, dy \;=\; 2i\frac{\sin\left(\frac{qd}{2}\right)}{q} \;=\; 2i\frac{\sin\left(\frac{k\sin\theta d}{2}\right)}{k\sin\theta}$$

This is the single slit Fraunhoffer diffraction amplitude.