

Chapter 3

Nuclear Size and Shape

The unit of nuclear length is called the “fermi”, (fm)

$$1 fm = 10^{-15} m.$$

There are deviations from the Rutherford scattering formula when the energy of the incident α -particle becomes too large, so that the distance of closest approach is of order a few fermi's.

The reason for this is that the Rutherford scattering formula was derived assuming that the nucleus was a point particle. In reality it has a finite size with a radius R of order $10^{-15} m$.

The nucleus therefore has a charge distribution, $\rho(\mathbf{r})$. In terms of quantum mechanics we have

$$\rho(\mathbf{r}) = Ze|\Psi(\mathbf{r})|^2,$$

where Z is the atomic number and is equal to the number of protons in the nucleus, and Ψ is the wave-function for one of these protons. ($|\Psi(\mathbf{r})|^2$ is therefore the probability density for one proton). Nuclear ‘radius’ is not really a very precise term - it is the extent over which the electric charge distribution of the proton, and therefore its wavefunction, is not too small, although in principle the wave-function extends throughout all space.

It is difficult to produce α -particles with sufficient energy to probe the charge distribution of the nucleus, so we use high energy electrons instead.

For electrons the projectile charge z is replaced by 1 in the Rutherford scattering formula. There is one further change which is due to the fact that these electrons are moving relativistically with a velocity v close to c . This correction was first calculated by Mott and we have

$$\frac{d\sigma}{d\Omega}|_{\text{Mott}} = \frac{d\sigma}{d\Omega}|_{\text{Rutherford}} \left(1 - \frac{v^2}{c^2} \sin^2 \left(\frac{\theta}{2} \right) \right)$$

We account for the charge distribution of the nucleus by writing the differential cross-

section as

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{\text{Mott}} |F(q^2)|^2. \quad (3.0.1)$$

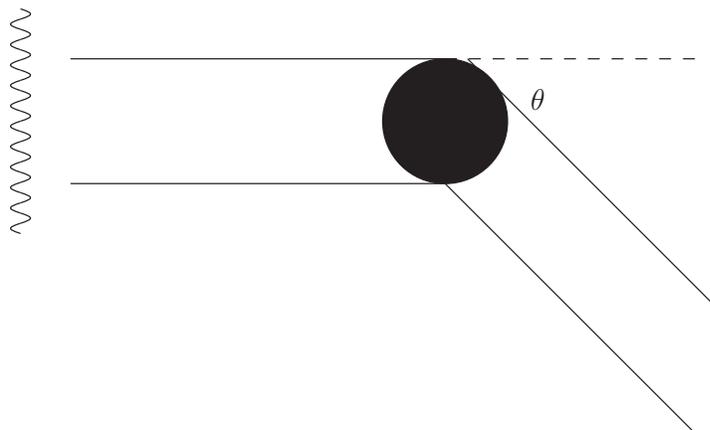
The correction factor $F(q^2)$ is called the “electric form-factor” and \mathbf{q} is the momentum transferred by the electron in the scattering and its magnitude is related to the scattering angle by (see the Appendix to the previous section)

$$q = 2p \sin\left(\frac{\theta}{2}\right),$$

where p is the momentum of the incident electron.

To understand the structure of the electric form-factor we need to recall that the electron has a de Broglie wavelength $\lambda = h/p$, and when this wavelength is of the order of the nuclear ‘radius’ we get a diffraction pattern.

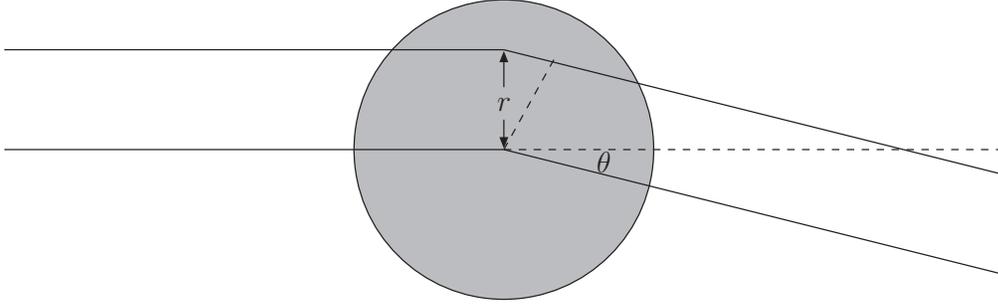
As a simple example suppose that the nucleus were a solid sphere of radius R with an infinite potential inside the sphere and zero potential outside, so that the electron cannot penetrate the sphere.



The wave that passes over the nucleus travels a distance $2R \sin \theta$ further than the wave that passes below the nucleus. If this difference is equal to $\lambda/2, 3\lambda/2 \dots$ then we get destructive interference. At these angles the differential cross-section vanishes.

The real case is a little more complicated than that. A proper quantum mechanical treatment (which is exactly analogous to diffraction in optics) shows that the electric form-factor is actually the Fourier transform of the charge distribution. For a spherically symmetric charge distribution this leads to

$$F(q^2) = \frac{4\pi\hbar}{Ze q} \int r \rho(r) \sin\left(\frac{qr}{\hbar}\right) dr. \quad (3.0.2)$$



Qualitatively, the reason for this is that the part of the wavefront that passes through the nucleus at a distance r from the centre and is scattered through an angle θ travels a further distance than the part of the wave that passes through the centre, by an amount proportional to r and therefore suffers a phase change (relative to the part of the wave passing through the centre). This phase change also depends on the scattering angle θ and is equal to qr/\hbar . This means that different parts of the wavefront suffer a different phase change (just as in optical diffraction) - these different amplitudes are summed to get the total amplitude at some scattering angle θ and this gives rise to the diffraction pattern. The contribution to the amplitude from the part of the wavefront which passes at a distance r from the centre of the nucleus is proportional to the charge density, $\rho(r)$, at r . The total scattering amplitude is therefore the sum of the amplitudes from all these different parts, which is what the integral in eq.(3.0.2) means.

Thus we see that a study of the diffractive scattering of electrons from a nucleus can give us information about the charge distribution inside the nucleus.

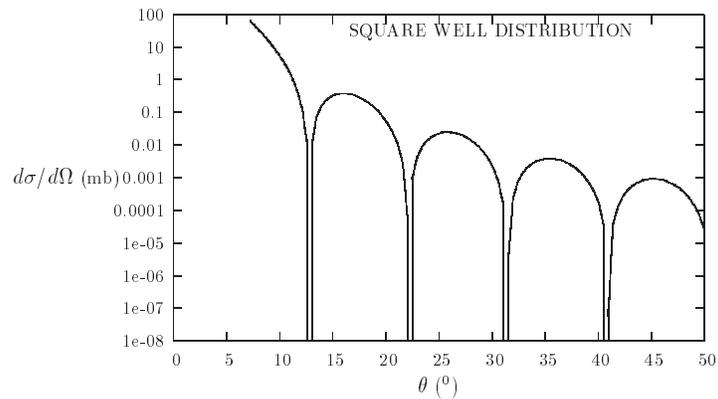
For example, if we assume that the charge distribution is a constant for $r < a$ and zero outside

$$\begin{aligned}\rho(r) &= \frac{3Ze}{4\pi R^3}, \quad r < R \\ &= 0 \quad r > R,\end{aligned}$$

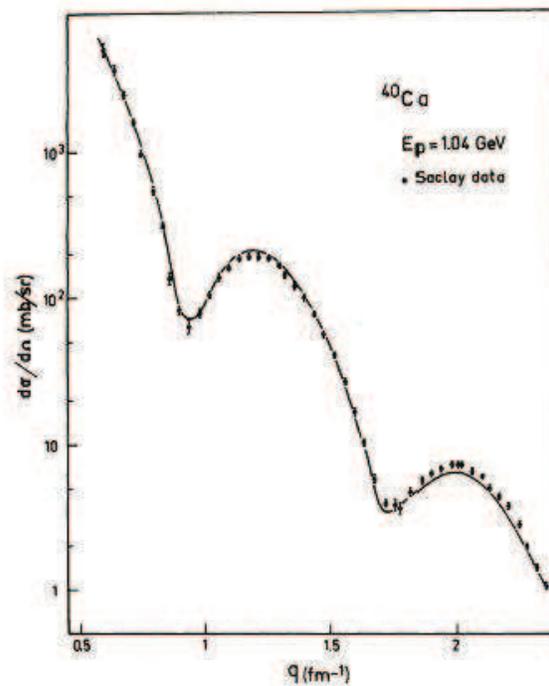
the integral in the Fourier transform eq.(3.0.2) can be done analytically via integrating by parts to give

$$F(q^2) = 3 \left(\frac{\hbar}{qR} \right)^3 \left(\sin(qR/\hbar) - \frac{qR}{\hbar} \cos(qR/\hbar) \right).$$

Feeding this back into eq.(3.0.1) for the diffractive differential cross-section we get



This is not quite what is observed in experiment which is more like this example of scattering of electrons of energy 1.04 GeV against a Ca nucleus



We see that although there are oscillations in the differential cross-section, it never actually vanishes. The reason for the discrepancy is that the square-well model for the charge distribution is unrealistic. The charge distribution rapidly becomes small as r exceeds a few fermi's, but never goes to zero.

A rough (to within about 30%) estimate of the nuclear radius R can be obtained from the

first minimum of the diffraction pattern and assuming that this occurs when

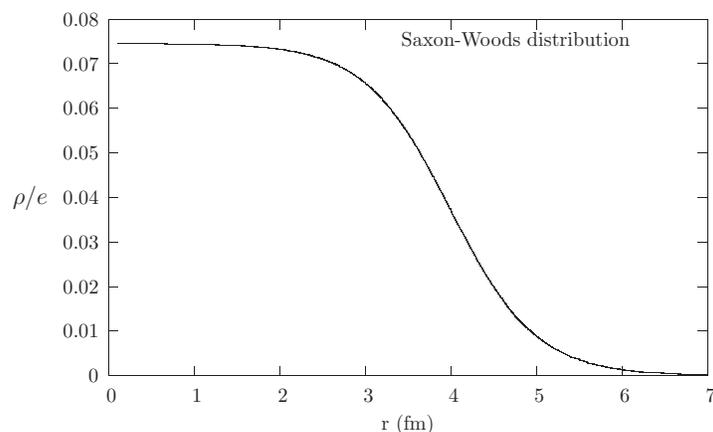
$$\frac{qR}{\hbar} \approx \pi$$

In the above case this occurs at $q/\hbar \approx 1\text{fm}^{-1}$ (the x -scale is given in fm^{-1} which means it is really q/\hbar), giving an approximate nuclear radius of about 3 fm.

A more realistic charge distribution is the Saxon-Woods model for which

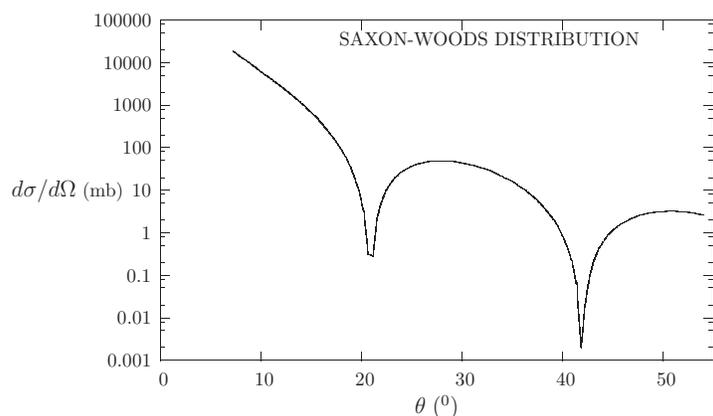
$$\rho(r) \propto \frac{1}{1 + \exp((r - R)/\delta)},$$

which looks like



We can interpret R as the nuclear ‘radius’ and δ as the ‘surface depth’ - it measures the range in r over which the charge distribution changes from the order of its value at the centre to much smaller than this value.

This leads to a differential cross-section which looks like



This has dips but no zeros and is much more similar in shape to the experimental results.

In fact, the Saxon-Woods model fits data from most nuclei rather well with empirical values for R and δ depending on the atomic mass number, A (the total number of protons

and neutrons in the nucleus):

$$R = (1.18A^{1/3} - 0.48) \text{ fm}$$

$$\delta = 0.4 - 0.5 \text{ fm for } A > 40$$

The first term in the expression for R is easily understandable as one would expect the volume occupied by a nucleus to be proportional to A , so that the radius is proportional to $A^{1/3}$.

3.1 Electric Quadrupole Moments

So far, we have assumed that the charge distribution is spherically symmetric. If that were the case we would have

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \frac{1}{3} \langle r^2 \rangle,$$

where

$$\langle x^2 \rangle = \frac{1}{Ze} \int x^2 \rho(\mathbf{r}) d^3\mathbf{r}$$

etc.

However, for many nuclei this is not the case and they possess an “electric quadrupole moment” defined (with respect to an axis z) as

$$Q = \int (3z^2 - r^2) \rho(\mathbf{r}) d^3\mathbf{r}$$

The Q/e has dimensions of area and is therefore usually quoted in barns.

Nuclei that possess an electric quadrupole moment have a shape which is an oblate spheroid for $Q < 0$ and a prolate spheroid for $Q > 0$.



On the other hand, the electric dipole moment, which is a vector defined by

$$\mathbf{d} = \int \mathbf{r} \rho(\mathbf{r}) d^3\mathbf{r},$$

is almost zero. The reason for this is that to a very good approximation, the wavefunction of a proton in a nucleus is a parity eigenstate, i.e.

$$\Psi(\mathbf{r}) = \pm \Psi(-\mathbf{r})$$

which implies

$$\rho(\mathbf{r}) = \rho(-\mathbf{r}),$$

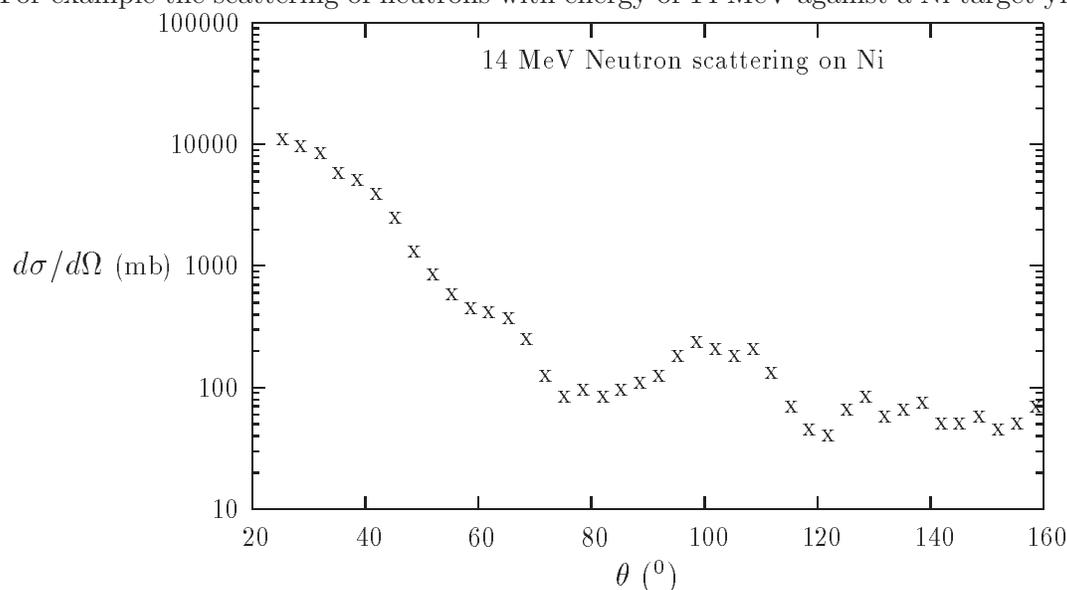
so that the electric dipole moment vanishes by symmetric integration.

3.2 Strong Force Distribution

The protons and neutrons inside a nucleus are held together by a strong nuclear force. This has to be strong enough to overcome the Coulomb repulsion between the protons, but unlike the Coulomb force, it extends only over a short range of a few fermi's.

Electron diffractive scattering is used to examine the distribution of electric charge (i.e. the protons) within the nucleus. Similar experiments are performed using high energy neutrons in order to probe the distribution of the strong force, i.e the distribution of all “nucleons” (neutrons and protons). In this case the form factor $F(q)$ is not the electric form-factor but the form-factor associated with the strong force.

For example the scattering of neutrons with energy of 14 MeV against a Ni target yields:



The Saxon-Woods model is also useful for the analyses of these data and yields a nuclear radius (for large A), given by

$$R = 1.2A^{1/3} \text{ fm}$$

and

$$\delta = 0.75 \text{ fm.}$$

We see that the strong force extends over approximately the same region as the nuclear charge, and that the ‘volume’ of the nucleus is proportional to the number of nucleons.

